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Fermion bags, topology and index theorems

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The fermion bag formulation helps to extend the concepts of topology and index theorem associated with non-Abelian gauge theories to simple lattice fermion field theories. Using this extension we can argue that fermion masses can arise either through the traditional mechanism where some lattice symmetry of the action that forbids fermion mass terms is explicitly, anomalously, or spontaneously broken, or through a non-traditional mechanism where all lattice symmetries continue to be preserved. We provide examples of simple fermion lattice field theories for each of these scenarios of fermion mass generation.

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1. Introduction

Concepts of topology and index theorem play an important role in the physics of fermion mass generation in non-Abelian gauge theories like QCD [1, 2]. These ideas explain how massless fermions can acquire a mass through either explicit, anomalous or spontaneous symmetry breaking. In the case of anomalous symmetry breaking, the symmetry of the action that protects fermions from becoming massive is explicitly broken through a term hidden in the measure and is related to the topological charge density of the gauge field. An equivalent way to understand this effect is that the chiral condensate which is forbidden by the symmetry, acquires a non-zero expectation value due to fermion zero modes that are guaranteed by the index theorem. QCD with a single quark flavor ($N_f = 1$) is an example where this mechanism of fermion mass generation occurs. In the case where the symmetry is spontaneously broken, the chiral condensate is formed due to a non-zero density of small eigenvalues. Instanton gas models suggest that this density arises when exact topological zero modes of opposite chiralities mix to produce the necessary density of approximate zero modes [3]. QCD with two flavors ($N_f = 2$) of massless fermions is an example of this phenomena.

In this talk we argue that the above ideas have analogies even in simple fermion lattice field theories with staggered fermions when the problem is formulated in the fermion bag approach [4, 5]. The subtle arguments of the continuum can be made explicit in these simpler examples. They provide "toy model" explanations of the more complex phenomena that occurs in QCD. The lattice fermion models also teach us about a non-traditional mechanism of fermion mass generation that occurs without spontaneous symmetry breaking [6, 7].

2. Topology and Index Theorem

In order to discuss the analogies mentioned above we first review the concepts of topology and index theorem within non-Abelian gauge theories using a formal discussion in the continuum. We then show that the formal arguments can be made concrete within simple lattice fermion field theories using the fermion bag approach. The Euclidean partition function of QCD in the continuum can be formally written as

$$Z = \int [dA]\, \mathrm{e}^{-S(A)} \int [d\overline{\psi}\, d\psi]\, \mathrm{e}^{-\,\overline{\psi}\, D(A)\, \psi} \tag{2.1}$$

where $A_\mu(x)$ is the non-Abelian matrix valued gauge field, $\overline{\psi}(x)$ and $\psi(x)$ are quark fields, $S(A)$ is the classical gauge action and $D(A) = \gamma_\mu\, (\partial_\mu - iA_\mu)$ is the massless Dirac operator that anticommutes with the usual fifth Dirac matrix γ_5,

$$\gamma_5 D(A) \;=\; -\,D(A)\gamma_5. \tag{2.2}$$

We assume space-time has a finite volume with a geometry that allows us to define a topological charge Q for every gauge field configuration. The index theorem then relates the index of the Dirac operator $D(A)$ with Q as follows. We first note that the zero modes of $D(A)$ can be chosen to be simultaneous eigenstates of γ_5, i.e.,

$$D(A)|z_a^\pm\rangle \;=\; 0, \quad \gamma_5|z_a^\pm\rangle = \pm|z_a^\pm\rangle, \tag{2.3}$$

where a labels the distinct zero modes. If n^+ and n^- are the number of positive and negative eigenstates of γ_5 in this zero mode subspace, then the index of the Dirac operator is defined as $(n^+ - n^-)$. The index theorem then states that $Q = (n^+ - n^-)$, which means $D(A)$ has at least $|Q|$ exact zero modes.

In the fermion bag formulation of simple fermion lattice field theories with staggered fermions, one can write the Euclidean partition function as

$$Z \;=\; \sum_B \mathrm{e}^{-S(B)} \int [d\overline{\psi}\; d\psi]\; \mathrm{e}^{\,-\;\overline{\psi}\;W(B)\;\psi} \tag{2.4}$$

where B refers to a fermion bag configuration and $S(B)$ its classical action. The fermion dynamics is described by the fermion bag matrix $W(B)$ that depends on the configuration B. Thus, the fermion bag configuration B is analogous to the gauge field A, $W(B)$ is analogous to the Dirac operator $D(A)$ and $S(B)$ is analogous to $S(A)$. We will show through concrete examples that it is possible to define a topological charge Q for each fermion bag configuration so that there is an index theorem in analogy with non-Abelian gauge theories that relates Q with the zero modes of $W(B)$. In fact in analogy with QCD, we can define a matrix Ξ_B that plays the role of γ_5 in the sense that it anticommutes with $W(B)$:

$$\Xi_B W(B) \;=\; -\,W(B)\Xi_B. \tag{2.5}$$

Thus, the zero modes of $W(B)$ will be simultaneous eigenstates of the matrix Ξ_B such that we can define n_+ and n_- to be the positive and negative eigenstates of Ξ_B in the zero mode subspace. The index theorem then as usual states that $Q = (n^+ - n^-)$. This means that $W(B)$ has at least $|Q|$ exact zero modes just like in QCD.

Surprisingly, the simplest example to see the above analogy is free massive staggered fermions whose action is given by

$$S = \sum_{x,y} \overline{\psi}_x\;(M_{x,y} + m\delta_{x,y})\;\psi_y, \tag{2.6}$$

where $M_{x,y}$ is the usual free massless staggered fermion matrix given by

$$M_{x,y} \;=\; \frac{1}{2}\sum_{x,\alpha} \eta_{x,\alpha}\Big(\delta_{x+\alpha,y} - \delta_{x,y+\alpha}\Big). \tag{2.7}$$

Since M has non-zero matrix elements only between even and odd sites it is useful to think of it as an off-diagonal anti-symmetric block matrix. We can then define Ξ as a diagonal matrix that gives the parity of the sites (even sites have parity 1 and odd sites have parity -1). Thus,

$$M \;=\; \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix},\;\; \Xi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.8}$$

and $\Xi M = -M\Xi$. We choose space-time as a finite hypercubic lattice with even number of sites in each direction and assume anti-periodic boundary conditions to avoid zero modes in the massless limit.

To see how concepts of topology and index theorem emerge even within this simple model, we begin with the partition function of the model,

$$Z \;=\; \int [d\overline{\psi}\; d\psi]\; \mathrm{e}^{\,-\;\overline{\psi}\;M\;\psi} \mathrm{e}^{-m\overline{\psi}\psi}. \tag{2.9}$$

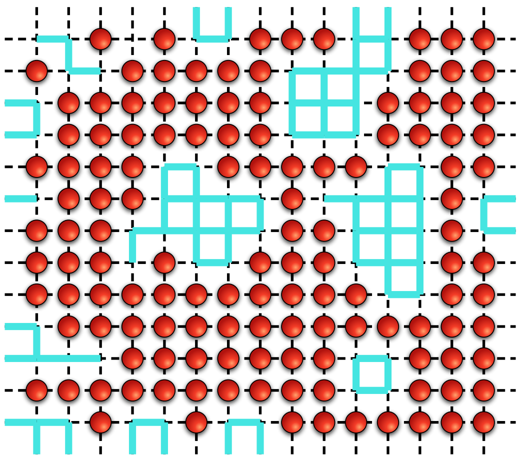

Figure 1: A fermion bag configuration with monomers represented by filled sites. Fermions move freely on sites that do not contain the monomers.

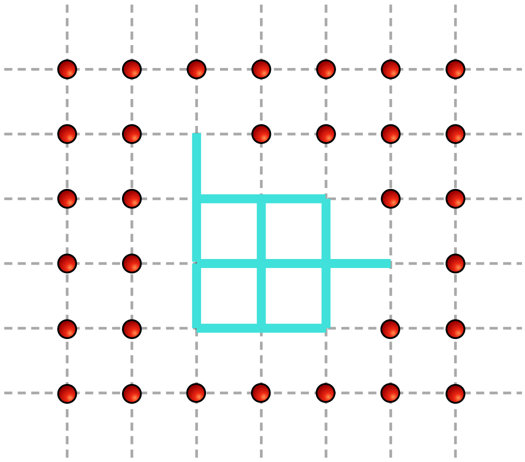

Figure 2: An example of a fermion bag configuration with topological charge $Q = \pm 1$. The fermion bag matrix $W(B)$ satisfies an index theorem with at least $|Q|$ zero modes.

and expand it in powers of m by writing

$$e^{-m\overline{\psi}_x\psi_x} \;=\; 1 - m\overline{\psi}_x\psi_x. \tag{2.10}$$

Distinguishing sites that contain the term $-m\overline{\psi}_x\psi_x$ (called monomer sites) from the sites that contain 1 we define a configuration of monomers and refer to it the fermion bag configuration B (see Fig. 1). Performing the Grassmann integration over each local monomer term,

$$\int d\overline{\psi}_x\, d\psi_x(-m\overline{\psi}_x\psi_x) \;=\; m, \tag{2.11}$$

we can write the partition function in the fermion bag formulation as

$$Z \;=\; \sum_B e^{k\log(m)} \int [d\overline{\psi}\, d\psi]\, e^{-\,\overline{\psi}\, W(B)\, \psi} \tag{2.12}$$

The matrix $W(B)$ is the same as the staggered matrix M but restricted to non-monomer sites and thus depends on B. It is easy to see that it anti-commutes with Ξ_B which is defined as the matrix Ξ but again restricted to these same non-monomer sites.[1] These matrices again take the block form

$$W(B) \;=\; \begin{pmatrix} 0 & C(B) \\ -C(B)^T & 0 \end{pmatrix}, \quad \Xi_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.13}$$

just like (2.8).

If a fermion bag configuration B contains n_e even and n_o odd non-monomer sites, then we define its topological charge as $Q = n_e - n_o$. Since $W(B)$ anti-commutes with Ξ_B, zero modes of

[1] Note that the size of $W(B)$ and Ξ_B depend on B. In the continuum discussion of non-Abelian gauge theories, the operators $D(A)$ and γ_5 are infinite dimensional but depend on A in a regulated theory.

$W(B)$ can be chosen to be simultaneous eigenstates of Ξ_B. If we label the zero modes as $|z_a^\pm\rangle$ as before then

$$W(B)|z_a^\pm\rangle \;=\; 0,\;\; \gamma_5|z_a^\pm\rangle = \pm|z_a^\pm\rangle. \tag{2.14}$$

As usual we define n^+ and n^- as the number of positive and negative eigenstates of Ξ_B in the zero mode subspace. Using the property that $W(B)$ is an anti-symmetric matrix and has the form given in (2.13) it is easy to argue that $W(B)$ will contain at least $|Q|$ zero modes and that $Q = n^+ - n^-$. An example of a $Q = 1$ fermion bag configuration is illustrated in Fig. 2.

3. Traditional Fermion Masses

Traditionally, fermion masses arise from fermion bilinear terms in the action. When some symmetry of the theory prevents such terms, fermions are expected to be massless. Fermion masses arise when this symmetry is explicitly, anomalously (through the measure), or spontaneously broken. It is well known that anomalous symmetry breaking can also be considered as an explicit form of symmetry breaking that is present in the full quantum action that includes the effects of the measure. This is even more striking in the above example when free massive staggered fermions are formulated in the fermion bag language. In this reformulation, the fermion bag matrix $W(B)$ satisfies all chiral symmetry properties of free staggered fermions and chiral symmetry breaking comes from the measure. The mechanism responsible for generating the mass now seems very similar to the one in $N_f = 1$ QCD where exact topological zero modes produce a non-zero chiral condensate even though it is forbidden due to the chiral symmetry of the action.

We can make all this mathematically explicit through the expression for the chiral condensate, which is given by

$$\langle\overline{\psi}\psi\rangle \;=\; -\frac{1}{ZV}\int [dA]\; \mathrm{e}^{-S(A)} \int [d\overline{\psi}d\psi]\; \mathrm{e}^{-\overline{\psi}D(A)\psi}\;\; \Big\{\int d^4x\;\overline{\psi}(x)\;\psi(x)\Big\}. \tag{3.1}$$

In a finite volume, the partition function gets contributions only from the $Q = 0$ sector, while the chiral condensate gets contributions from $|Q| = 1$ sectors. Assuming λ are the non-zero eigenvalues of $D(A)$ that depend on the gauge field A, we obtain the expressions

$$Z \;=\; \int [dA_{|Q|=0}]\; \mathrm{e}^{-S(A)}\Big\{\prod_\lambda \lambda\Big\}, \tag{3.2}$$

$$\langle\overline{\psi}\psi\rangle \;=\; \frac{1}{ZV}\int [dA_{|Q|=1}]\; \mathrm{e}^{-S(A)}\Big\{\prod_\lambda \lambda\Big\}. \tag{3.3}$$

Note that the zero modes of the Dirac operator are absent in (3.3) since the associated Grassmann variables enter through the condensate. It is easily seen that the chiral condensate in free staggered fermions can be formulated in exactly the same way in the fermion bag approach. This shows clearly that anomalous chiral symmetry breaking in $N_f = 1$ QCD cannot be distinguished from explicit symmetry breaking although the precise mass term in the effective action is difficult to compute.

Interestingly the case of $N_f = 2$ QCD is different, since now the chiral condensate vanishes exactly in a finite volume.[2] This can be traced back to the fact that the theory contains flavored

[2] A non-zero density of small eigenvalues cannot arise in a finite volume!

chiral symmetries and hence all zero modes come in pairs. Hence, the fermion determinant kills all contributions to the chiral condensate from non-zero Q sectors. But we know that the chiral condensate can still be non-zero if the flavored chiral symmetries break spontaneously. In a finite volume this can be understood through the computation of the chiral susceptibility defined through

$$\chi \;=\; \frac{1}{ZV}\int\,[dA]\;\mathrm{e}^{-S(A)}\int[d\overline{\psi}d\psi]\;\mathrm{e}^{-\overline{\psi}D(A)\psi}\;\;\Big\{\int d^4x\;\overline{\psi}(x)\;\psi(x)\Big\}^2. \tag{3.4}$$

Performing the Grassmann integration now leads to the expressions

$$Z \;=\; \int\,[dA_{|Q|=0}]\;\mathrm{e}^{-S(A)}\Big\{\prod_{\lambda'}\lambda'\Big\}^2, \tag{3.5}$$

$$\chi \;=\; \frac{1}{ZV}\left[\int\,[dA_{|Q|=0}]\;\mathrm{e}^{-S(A)}\Big(\sum_{\lambda}\frac{1}{|\lambda|^2}\Big)\;\Big\{\prod_{\lambda'}\lambda'\Big\}^2\;+\;\int\,[dA_{|Q|=1}]\;\mathrm{e}^{-S(A)}\Big\{\prod_{\lambda'}\lambda'\Big\}^2\right]. \tag{3.6}$$

When spontaneous symmetry breaking occurs, many small non-zero eigenvalues are expected to emerge at the scale $|\lambda|\Sigma V \sim 1$ where Σ is the chiral condensate. This then leads to the result $\chi \sim \Sigma^2 V$ as expected in the spontaneously broken phase.

4. Non-Traditional Fermion Masses

The partition function of a fermion model that has analogies with $N_f = 2$ QCD is given by the four-fermion model

$$S = \sum_{x,y,a=1,2}\overline{\psi}^a_x\,D_{x,y}\;\psi^a_y\;-\;U\;\sum_x \overline{\psi}^1_x\psi^1_x\overline{\psi}^2_x\psi^2_x, \tag{4.1}$$

where now we work with two flavors of staggered fermions. When this model is formulated in the fermion bag representation, expressions for the partition function and chiral susceptibilities are given by

$$Z \;=\; \sum_{B(Q=0)}\;\mathrm{e}^{k\log(U)}\Big\{\prod_{\lambda'}\lambda'\Big\}^2, \tag{4.2}$$

$$\chi \;=\; \frac{1}{ZV}\left[\sum_{B(Q=0)}\;\mathrm{e}^{k\log(U)}\Big(\sum_{\lambda}\frac{1}{|\lambda|^2}\Big)\;\Big\{\prod_{\lambda'}\lambda'\Big\}^2\;+\;\sum_{B(|Q|=1)}\;\mathrm{e}^{k\log(U)}\Big\{\prod_{\lambda'}\lambda'\Big\}^2\right], \tag{4.3}$$

in analogy with (3.5) and (3.6). In the above expressions λ are the non-zero eigenvalues of $W(B)$ and the fermion bag configurations are the same as illustrated in Fig. 1. Topology and index theorems also work out as discussed earlier. Thus, in principle one can expect a spontaneously broken phase like in $N_f = 2$ QCD. However, since four-fermion couplings are irrelevant perturbatively in three and higher dimensions, there is a perturbative massless phase in the model. If a spontaneously broken phase emerges it can only do so at larger couplings.

At sufficiently strong couplings the matrix $W(B)$ splits into disconnected blocks since space-time itself splits into disconnected bags. Hence the smallest eigenvalues of $W(B)$ are determined by the sizes of the biggest connected bags which do not scale with the space-time volume. In other words one cannot expect $\lambda\Sigma V \sim 1$ as required in the spontaneously broken phase and the fermion bilinear condensate must vanish. On the other hand fermions are still massive since fermionic correlation functions are also non-zero only within fermion bags. They decay exponentially although the chiral symmetry that preserves fermions from becoming massive at weak couplings remains unbroken. This non-traditional mechanism of fermion mass generation where the mass is not related to a fermion bilinear condensate was discovered long ago but has remained unappreciated [6, 7].

The above arguments show that there are at least two phases in the four-fermion model. Recently a direct second order phase transition between the symmetric massless phase and the symmetric massive phase was discovered in three space-time dimensions [8, 9]. Such phase transitions in four space-time dimensions, especially in the presence of non-Abelian gauge fields would be quite exciting. Unfortunately, recent studies suggest the presence of an intermediate phase in four space-time dimensions [10]. While this phase could be the usual spontaneously broken phase, recent studies suggest that one cannot rule out something more exotic [11]. Finally, in two dimensions the entire $U > 0$ is in a non-traditionally massive phase. The critical point at $U = 0$ is asymptotically free as expected in two dimensional four-fermion field theories [12]. Understanding the behavior of the low lying spectrum of $W(B)$ in the critical region in various dimensions would be very interesting.

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