

## Relaxation time of the fermions in the magnetic field (I) - the case for relativistic fermions -

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The chiral magnetic effect (CME) is the quantum anomaly related electric charge transport phenomenon along the external magnetic field, which appears in various systems possessing chiral fermions, such as the quark-gluon plasma, condensed matter physics and astrophysics.

The magnetic field dependence of the relaxation time is needed to compare the theory and experiments quantitatively. However, the model calculation of the relaxation time has been made by Argyres and Adams only for the non-relativistic fermion with the strong magnetic field limit[1]. Our study is computation of the relaxation time for the relativistic fermions in the magnetic field, extending the work by Argyres and Adams.

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## 1. Introduction

Chiral anomaly, which is known as Adler-Bell-Jackiw anomaly after the discovered in 1969, is an important concept in the field theory. The statement of the anomaly is that in the quantum field theory with fermions coupled with electromagnetic gauge field, the chiral fermion number is not conserved, and obeys the anomaly equation

$$\frac{dN_5}{dt} = \frac{e^2}{2\pi^2} \mathbf{E} \cdot \mathbf{B}. \quad (1.1)$$

In 1983 Nielsen and Ninomiya pointed out that condensed matter systems have an effect analogous to the mechanism of the chiral anomaly[2]. They considered a band structure with two Weyl nodes which possess the opposite chirality, applying the parallel electric and magnetic field. The fermion states are quantized by the magnetic field to form Landau levels, and the fermion gets drifted by the electric field within a given level. Then the equation for chiral fermion number completely matches with eq.(1.1). At the same time, drifted fermions get scattered back by impurities. The balance between the drifting and the scattering determines the magnitude of the electric current measured in observation. They predicted the enhancement of the magnetoconductivity proportional to  $\mathbf{E} \cdot \mathbf{B}$  caused by the chiral anomaly effect.

Their prediction had not been tested in the observation for 30 years, because the example of Weyl nodes had not been discovered in solids. The related recent interest of the topological band structure and Berry curvature change the situation. The interest of Weyl semimetals has begun with theoretical proposal by Wan et al [3], and now several examples are theoretically predicted and experimentally observed. There are several papers which report the phenomena related to the chiral anomaly such as unusual magnetoresistance in Weyl semimetals[4, 5, 6, 7, 8].

To compare these measurement and the theoretical prediction quantitatively, the magnetic field dependence of the relaxation time has to be known. The relaxation time for non-relativistic fermions in strong magnetic field limit was calculated in 1956[1]. However, since the fermion in Weyl semimetals considered here is massless fermion, the relaxation time should be for relativistic fermions. The purpose of our study is to estimate the relativistic effect to the relaxation time in strong magnetic field limit and zero temperature.

## 2. Basics of the transport theory

In this section, we derive the relation between the electric current and the relaxation time, reviewing the basics of the transport theory. The electric current density in  $z$  direction is given by

$$J = \frac{1}{V} \sum_{\mathbf{p}} (-e) \frac{\partial \varepsilon}{\partial p_z} f(\mathbf{p}) \times (\text{number of degeneracy points}), \quad (2.1)$$

where  $f$  is the distribution function of the fermions. Without the external field, electron states with the energy less than Fermi level energy  $\mu$  are filled. The distribution function in thermodynamical equilibrium is the Fermi distribution function:

$$f_0(\varepsilon) = \frac{1}{1 + \exp\left(\frac{\varepsilon(\mathbf{p}) - \mu}{k_B T}\right)}. \quad (2.2)$$

With the electric field  $E$  in  $z$  direction, the distribution function is slightly deviated:  $f = f_0 + \delta f$ . The electric field accelerates the electrons towards the same direction:  $dp_z/dt = eE$ . Therefore the imbalance of the distribution function by the drifting is

$$\left(\frac{\partial f}{\partial t}\right)_{\text{drift}} = \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{d\mathbf{p}}{dt} = eE \frac{df}{dp_z}. \quad (2.3)$$

At the same time, the accelerated electrons get scattered back. So we assume that the distribution function exponentially get back into the thermodynamical equilibrium in relaxation time  $\tau$ :  $\delta f \propto e^{-t/\tau}$ . Then the scattering effect is written as

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = -\frac{f - f_0}{\tau}. \quad (2.4)$$

In the lowest order in  $E$ , the solution of the steady state condition:  $(\partial f/\partial t)_{\text{drift}} + (\partial f/\partial t)_{\text{coll}} = 0$ , is

$$f = f_0 + eE\tau \frac{\partial f_0}{\partial p_z} + \mathcal{O}(E^2). \quad (2.5)$$

From eq.(2.1), we can derive the magnitude of the electric current if we know the relaxation time.

### 3. Relaxation time for relativistic fermions

In this section we calculate the relaxation time for relativistic fermions. In this report, we take Weyl representation for  $\gamma$ -matrices.

#### 3.1 Massive Dirac fermion in magnetic field

We consider the Dirac fermion with  $m \neq 0, \mu \neq 0$ , which obeys the Dirac equation:

$$[i\mathcal{D} - m + \mu\gamma^0] \psi(x) = 0. \quad (3.1)$$

Since we consider constant magnetic field in  $z$  direction, we set the gauge field;  $A^2 = Bx$  and the other components are zero. Multiplying  $\gamma^0$  to the Dirac equation (3.1) from left,

$$[i\partial_0 + \mu + i\gamma^0\gamma^j D_i - m\gamma^0] \psi(x) = 0. \quad (3.2)$$

To obtain the wave function  $\psi(x)$ , first we introduce an auxiliary function  $\Phi$  which satisfies

$$[i\partial_0 + \mu + i\gamma^0\gamma^j D_i - m\gamma^0] [i\partial_0 + \mu - i\gamma^0\gamma^j D_i + m\gamma^0] \Phi = 0. \quad (3.3)$$

Then we can obtain the solution to eq.(3.1) for  $\psi(x)$  as

$$\psi(x) = [i\partial_0 + \mu - i\gamma^0\gamma^j D_i + m\gamma^0] \Phi. \quad (3.4)$$

From eq.(3.3), taking the energy as  $\omega$  and the momenta in  $y, z$  directions as  $p_2, p_3$ , eigenfunction  $\Phi$  satisfies a harmonic oscillator type equation

$$\left[-\partial_1^2 + (eB)^2 \left(x - \frac{p_2}{eB}\right)^2 + p_3^2 + m^2 - eB\sigma^3\right] \Phi = (\omega + \mu)^2 \Phi. \quad (3.5)$$

Therefore, the energy levels are given by Landau levels

$$\omega_{n,\sigma^3}(p_3) = -\mu \pm \sqrt{2eB \left( n + \frac{1}{2} \right) + p_3^2 + m^2 - eB\sigma^3}. \quad (3.6)$$

Then we get

$$v_z \equiv \frac{\partial \varepsilon}{\partial p_z} = \frac{p_z}{\varepsilon}, \quad (3.7)$$

where  $\varepsilon = \omega + \mu$ . The eigenfunction  $\Phi$  is

$$\begin{aligned} \Phi_{n,\sigma^3,\gamma_5}(x, p_2, p_3) &= N_{n,\sigma^3,\gamma_5}(p_3) \exp(ip_2 y + ip_3 z) \sqrt[4]{\frac{eB}{\pi 2^{2n} n!}} \\ &\times \exp \left[ -\frac{1}{2} eB \left( x - \frac{p_2}{eB} \right)^2 \right] H_n \left( \sqrt{eB} x - \frac{p_2}{\sqrt{eB}} \right) \chi_{\sigma^3,\gamma_5}, \end{aligned} \quad (3.8)$$

where  $N_{n,\sigma^3,\gamma_5}(p_3)$  is unknown normalization constant,  $H_n \left( \sqrt{eB} x - \frac{p_2}{\sqrt{eB}} \right)$  are Hermite polynomials, and  $\chi_{\sigma^3,\gamma_5}$  is the eigen spinor of  $\sigma^3$  and  $\gamma_5$  given by

$$\chi_{+,L} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \chi_{-,L} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \chi_{+,R} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \chi_{-,R} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (3.9)$$

A simple calculation shows that the eigenfunction  $\Phi$  is normalized as

$$\int dx \Phi^\dagger \Phi = |N_{n,\sigma^3,\gamma_5}(p_3)|^2 \chi^\dagger \chi.$$

The wave function of the Dirac fermion is obtained as

$$\begin{aligned} \psi(x) &= \langle x | n, \sigma^3; p_y, p_z \rangle \\ &= [\omega_{n,\sigma^3}(p_z) + \mu - i\gamma^0 \gamma^i D_i + m\gamma^0] (\Phi_{n,\sigma^3,L}(x, p_y, p_z) + \Phi_{n,\sigma^3,R}(x, p_y, p_z)). \end{aligned} \quad (3.10)$$

The normalization constant  $N_{n,\sigma^3}(p_z)$  is determined by the normalization condition

$$\langle n', \sigma^3; p'_y, p'_z | n, \sigma^3; p_y, p_z \rangle = (2\pi)^2 \delta(p_y - p'_y) \delta(p_z - p'_z) \delta_{n,n'} \delta_{\sigma^3,\sigma^3'}. \quad (3.11)$$

In the strong magnetic limit  $eB \rightarrow \infty$  only the lowest energy band  $n = 0, \sigma^3 = +1$  contribute the scattering process. So we consider only the lowest energy band. After some calculation, we get the lowest energy wave function

$$\begin{aligned} \psi_{0,+}(\mathbf{x}, p_y, p_z) &= (\omega_{0,+}(p_z) + \mu - p_z + m) \Phi_{0,+,L}(x, p_y, p_z) \\ &+ (\omega_{0,+}(p_z) + \mu + p_z + m) \Phi_{0,+,R}(x, p_y, p_z). \end{aligned} \quad (3.12)$$

Then we notice

$$\begin{aligned} \langle p'_y, p'_z | p_y, p_z \rangle &= \int d^3x \langle p'_y, p'_z | x \rangle \langle x | p_y, p_z \rangle \\ &= 4(\omega(p_z) + \mu)(\omega(p_z) + \mu + m) |N(p_z)|^2 (2\pi)^2 \delta(p_y - p'_y) \delta(p_z - p'_z). \end{aligned}$$

From the normalization condition (3.11), the normalization constant is obtained as

$$|N_{0,+}(p_z)|^2 = \frac{1}{4(\omega(p_z) + \mu)(\omega(p_z) + \mu + m)}. \quad (3.13)$$

### 3.2 Relaxation time for scattering by charged impurities

Since we consider only the lowest energy band  $n = 0$ ,  $\sigma^3 = +1$ , let us omit the labels  $n, \sigma^3$  for short.

The variation of the distribution function by the scattering effect is expressed in terms of the transition probability in unit time  $W(p_y, p_z \rightarrow p'_y, p'_z)$  as

$$\left(\frac{\partial f}{\partial t}\right)_{\text{scatt}} = - \sum_{p'_y, p'_z} f(\mathbf{p})(1 - f(\mathbf{p}'))W(p_y, p_z \rightarrow p'_y, p'_z) + \sum_{p'_y, p'_z} f(\mathbf{p}')(1 - f(\mathbf{p}))W(p'_y, p'_z \rightarrow p_y, p_z).$$

In first order of  $\delta f(\mathbf{p})$ ,

$$\begin{aligned} \left(\frac{\partial f}{\partial t}\right)_{\text{scatt}} &= - \sum_{p'_y, p'_z} W(p_y, p_z \rightarrow p'_y, p'_z) [\delta f(\mathbf{p})(1 - f_0(\epsilon')) - f_0(\epsilon)\delta f(\mathbf{p}')] \\ &\quad + \sum_{p'_y, p'_z} W(p'_y, p'_z \rightarrow p_y, p_z) [\delta f(\mathbf{p}')(1 - f_0(\epsilon)) - f_0(\epsilon')\delta f(\mathbf{p})]. \end{aligned}$$

Due to the energy conservation,  $f_0(\epsilon) = f_0(\epsilon')$ , so we get

$$\left(\frac{\partial f}{\partial t}\right)_{\text{scatt}} = - \sum_{p'_y, p'_z} W(p_y, p_z \rightarrow p'_y, p'_z) [\delta f(\mathbf{p}) - \delta f(\mathbf{p}')].$$

Substituting  $\delta f(\mathbf{p}) = eE v_z \tau \frac{\partial f_0}{\partial \epsilon} + \mathcal{O}(E^2)$ , we get

$$\left(\frac{\partial f}{\partial t}\right)_{\text{scatt}} = -eE \frac{\partial f_0}{\partial \epsilon(\mathbf{p})} \sum_{p'_y, p'_z} W(p_y, p_z \rightarrow p'_y, p'_z) [\tau(\mathbf{p})v_z(\mathbf{p}) - \tau(\mathbf{p}')v_z(\mathbf{p}')].$$

in lowest order of  $E$ . From the eqs.(2.4,2.5), the left hand side of the above equation is

$$\left(\frac{\partial f}{\partial t}\right)_{\text{scatt}} = -eE v_z \frac{\partial f_0}{\partial \epsilon} + \mathcal{O}(E^2).$$

Then we get

$$v_z(\mathbf{p}) = \sum_{p'_y, p'_z} W(p_y, p_z \rightarrow p'_y, p'_z) [\tau(\mathbf{p})v_z(\mathbf{p}) - \tau(\mathbf{p}')v_z(\mathbf{p}')]. \quad (3.14)$$

The interaction between the fermions and the charged impurity is given by the screened Coulomb potential:

$$V(\mathbf{x}) = \left(\frac{4\pi e^2}{\kappa}\right) \frac{\exp(-|\mathbf{x}|/r_0)}{|\mathbf{x}|}, \quad (3.15)$$

where  $r_0$  is the screening length,  $\kappa$  is dielectric constant.

Transition rate in unit time is given by

$$W(p_y, p_z \rightarrow p'_y, p'_z) = 2\pi \sum_{\mathbf{R}} |\langle p'_y, p'_z | v(\mathbf{r} - \mathbf{R}) | p_y, p_z \rangle|^2 \delta(\omega(p'_z) - \omega(p_z)), \quad (3.16)$$

where  $\mathbf{R}$  stands for a position of the impurity. Fourier transform of the screened Coulomb potential is

$$V(\mathbf{r} - \mathbf{R}) = \frac{4\pi e^2}{\kappa} \frac{1}{V} \sum_{\mathbf{q}} \frac{e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{R})}}{q^2 + 1/r_s^2},$$

so we should calculate the matrix element  $\langle p'_y, p'_z | e^{i\mathbf{q}\cdot\mathbf{x}} | p_y, p_z \rangle$ . Assuming that the impurities are distributed uniformly with density  $N_I$ , we can obtain the transition rate in unit time

$$\begin{aligned}
W(p_y, p_z \rightarrow p'_y, p'_z) &= 2\pi\delta(\omega(p'_z) - \omega(p_z)) \left( \frac{4\pi e^2}{\kappa} \right)^2 \frac{N_I}{V} \\
&\times \frac{[(\omega(p'_z) + \mu + m)(\omega(p_z) + \mu + m) + p'_z p_z]^2}{4(\omega(p_z) + \mu)(\omega(p_z) + \mu + m)(\omega(p'_z) + \mu)(\omega(p'_z) + \mu + m)} \\
&\times \sum_{q_x} \frac{\exp[-\frac{1}{2eB}\{q_x^2 + (p_y - p'_y)^2\}]}{[q_x^2 + (p_y - p'_y)^2 + (p_z - p'_z)^2 + 1/r_s^2]^2}. \quad (3.17)
\end{aligned}$$

After some calculation, we find

$$\int dp'_y dq_x \frac{\exp[-\frac{1}{2eB}\{q_x^2 + (p_y - p'_y)^2\}]}{[q_x^2 + (p_y - p'_y)^2 + (p_z - p'_z)^2 + 1/r_s^2]^2} = \frac{\pi}{(p_z - p'_z)^2 + 1/r_s^2} I\left(\frac{1}{2eB}((p_z - p'_z)^2 + 1/r_s^2)\right).$$

The function  $I(\gamma)$  is defined as

$$I(\gamma) \equiv \int_0^\infty dx \frac{x e^{-x}}{x + \gamma} = 1 + \gamma e^\gamma \text{Ei}(-\gamma), \quad (3.18)$$

where  $\text{Ei}(\gamma)$  is the exponential integral. For energy conservation, initial state with  $z$  direction momentum  $p_z$  can be scattered only to final state with  $-p_z$ . Then eq.(3.14) becomes

$$\frac{p_z}{\varepsilon} = \left( \frac{4\pi e^2}{\kappa} \right)^2 \frac{N_I}{16\pi\varepsilon^2(p_z)(\varepsilon(p_z) + m)^2} \frac{[ (\varepsilon(p_z) + m)^2 - p_z^2 ]^2}{4p_z^2 + 1/r_s^2} I\left(\frac{4p_z^2 + 1/r_s^2}{2eB}\right) (\tau(p_z) + \tau(-p_z)). \quad (3.19)$$

The distribution function is normalized as

$$\sum_{\mathbf{p}} f(\mathbf{p}) = \sum_{\mathbf{p}} f_0(\mathbf{p}) = 1.$$

So the deviation from thermodynamical equilibrium satisfies

$$\sum_{\mathbf{p}} \delta f(\mathbf{p}) = 0. \quad (3.20)$$

At zero temperature

$$\delta f(\mathbf{p}) = eE\tau(\mathbf{p})\delta(\varepsilon - \mu) \frac{p_z}{\varepsilon}, \quad (3.21)$$

therefore we get

$$\sum_{p_x, p_y} (\tau(p_x, p_y, p_z) - \tau(p_x, p_y, -p_z)) = 0. \quad (3.22)$$

Then the relaxation time is

$$\frac{1}{\tau(p_z)} = \left( \frac{4\pi e^2}{\kappa} \right)^2 \frac{N_I}{8\pi p_z \varepsilon(p_z) (\varepsilon(p_z) + m)^2} \frac{[ (\varepsilon(p_z) + m)^2 - p_z^2 ]^2}{4p_z^2 + 1/r_s^2} I\left(\frac{4p_z^2 + 1/r_s^2}{2eB}\right). \quad (3.23)$$

Only the states with  $\varepsilon = \mu$  contribute the scattering, so we substitute  $p_z = \sqrt{\mu^2 - m^2}$ . Then

$$\frac{1}{\tau} = \frac{8\pi e^4 N_I}{\kappa^2} \frac{m^2}{\mu \sqrt{\mu^2 - m^2}} \frac{1}{4(\mu^2 - m^2) + 1/r_s^2} I\left(\frac{1}{2eB}(4(\mu^2 - m^2) + 1/r_s^2)\right). \quad (3.24)$$

This result matches with the result in [1] in non-relativistic limit.

#### 4. Summary and discussion

In this report, we have presented our study on relaxation time for the relativistic fermions scattered by the Coulomb impurities in strong magnetic field. We reviewed the basics of the transport theory. Calculating transition probability, we have obtained the relativistic expression for the relaxation time.

In strong magnetic field limit  $eB \rightarrow \infty$ ,  $I\left(\frac{1}{2eB}(4(\mu^2 - m^2) + 1/r_s^2)\right) \rightarrow 1$ , then

$$\frac{1}{\tau} \rightarrow \frac{8\pi e^4 N_I}{\kappa^2} \frac{m^2}{\mu \sqrt{\mu^2 - m^2}} \frac{1}{4(\mu^2 - m^2) + 1/r_s^2}. \quad (4.1)$$

Then we get the interpolating formula between the relativistic and non-relativistic relaxation time in strong magnetic field limit:

$$\frac{1}{\tau} \propto m^2 \quad (m/\mu \sim 0) \quad (4.2)$$

$$\frac{1}{\tau} \propto (\mu^2 - m^2)^{-1/2} \quad (m/\mu \sim 1). \quad (4.3)$$

In our study, we consider only the lowest band. When  $2eB < p_z^2 + m^2 - \mu^2$ , the first excited states come to contribute the scattering. So we should consider higher energy band states in that case. We considered zero temperature case where scattering by the acoustic phonon can be neglected. However in finite temperature case, we should calculate the effect of the phonon scattering.

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