PROCEEDINGS OF SCIENCE

How to exploit canonical bases for $gg \to H$ at NNNLO

Alexander Hasselhuhn[∗†]
Karlsruhe Institute of Technology
E-mail: alexander.hasselhuhn@kit.edu

I report on the analytic calculation of the channel $qq' \to H$ contributing to the Higgs production cross section in gluon fusion at N^{3}LO. The focus is set on the computation of the master integrals using a canonical basis. It is shown how the structure of a canonical basis helps in the computation of the boundary values. Finally the iterated integrals involving square root letters are discussed which emerge in the result.

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∗Speaker.
†A footnote may follow.

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1. Introduction

Since the discovery of the Higgs boson [1, 2] a major challenge of the LHC experiments is the precise determination of the Higgs properties. An important ingredient for corresponding analyses is the Higgs production cross section in gluon fusion.

The leading order (LO) contributions to $\sigma(pp \to H+X)$ [3, 4, 5, 6] have been known for a long time already. Also the next-to-leading order (NLO) QCD [7, 8], electroweak [9] and mixed QCD-electroweak [10] corrections were obtained. At next-to-next-to-leading order (NNLO) the contributions have been computed in the limit of an infinitely heavy top quark [11, 12, 13, 14]. At the same order also the finite top quark mass corrections have been studied [15, 16, 17, 18, 19, 20, 21], which amount to 1%.

For a calculation of Higgs production at N^{3}LO the effective Higgs-gluon coupling is needed at the same accuracy, which was achieved in a four loop calculation [22, 23, 24]. Furthermore the $\mathscr{O}(\varepsilon)$ contributions to the NNLO master integrals [21, 25] as well as the LO, NLO and NNLO partonic cross sections to higher orders in the dimensional regulator are known [26, 27]. The necessary convolutions of partonic cross sections with splitting functions from collinear factorization were obtained in [26, 28, 27]. The dependence on the factorization and renormalization scales is known to N^{3}LO [27].

Finally, using asymptotic expansions to very high orders in the soft limit, the complete N^{3}LO contribution to $gg \to H$ was obtained [29, 30] reaching an accuracy which is high enough for phenomenological application. Nevertheless a second computation is needed to confirm the result. Here I want to present a step in this direction, which is the analytic computation of the contribution with two quarks of different flavor in the initial state. In particular, I will detail two technical aspects: some advantages of using a canonical basis of master integrals for the computation, as well as the properties of the iterated integrals appearing in the result of the calculation.

2. An exact computation of the qq'-channel

The effective theory of QCD with an infinite top quark mass is given by the lagrangian density

$$\mathscr{L}_{Y,\text{eff}} = -\frac{H}{4v} C_1 (G_{\mu\nu} G^{\mu\nu}) + \mathscr{L}^{(5)}_{QCD}\,. \tag{2.1}$$

where the Lagrangian of five flavor QCD is denoted by $\mathscr{L}^{(5)}_{QCD}$, H is the Higgs field, v its vacuum expectation value and C_1 is the matching coefficient between the full and the effective theory. $G_{\mu\nu}$ is the gluonic field strength tensor.

In the effective theory the cross section for $gg \to H$ depends on one kinematical quantity

$$x = \frac{m_n}{s} \tag{2.2}$$

only, where m_h is the Higgs mass and s is the partonic center of mass energy.

In this paper we consider the process $qq' \to H$, which contributes to the N^{3}LO corrections to Higgs production in the gluon fusion channel. The diagrams are organized in 17 Topologies, and are characterized by two quark lines of different flavor passing from the initial state through to the final state. Thus the final state contains at least two real partons in addition to the Higgs boson. As a consequence, one has to deal with real-real-virtual contributions (3-particle phase space integrals) as well as triple real contributions (4-particle phase space integrals).

In order to deal with the computation of the phase space integrals we makes use of the relation between phase space integrals and loop integrals [13] as given by the Cutkosky rules. This correspondence makes it possible to apply methods for loop calculations to the problem, such as integration by parts relations, or more precisely the reduction to master integrals. Details to the IBP reduction are given in [31].

The reduction to master integrals also supplies one with the possibility of deriving a coupled system of differential equations

$$\partial_x \tilde{f}(x,\varepsilon) = \tilde{A}(x,\varepsilon)\tilde{f}(x,\varepsilon), \tag{2.3}$$

for the master integrals. Solving such a system is straight forward if a change of basis

$$f(x,\varepsilon) = B(x,\varepsilon)\tilde{f}(x,\varepsilon), \tag{2.4}$$

can be found such that the differential equation assumes the shape (cf. [32])

$$\partial_x f(x,\varepsilon) = \varepsilon A(x) f(x,\varepsilon). \tag{2.5}$$

Here the vector of master integrals $\tilde{f}$ is mapped onto the canonical master integrals f via the matrix $B(x,\varepsilon)$. The matrices $A(x,\varepsilon)$ and $A(x)$ define the respective differential equations.

For canonical differential equations of the topologies of the $qq' \to H$ channel one finds the general form

$$A(x) = \frac{a}{1-x} + \frac{b}{1+x} + \frac{c}{x} + \frac{d}{1+4x} + \frac{e}{x\sqrt{1+4x}}\,. \tag{2.6}$$

where d and e only contribute to the topology in fig. 1.

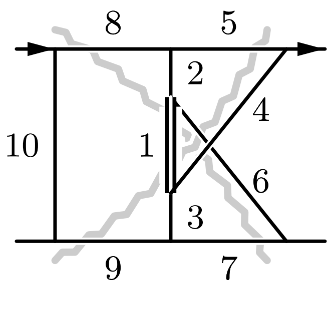

Figure 1: The topology BT3 generates iterated integrals with square root letters.

The solution of the differential equation can then be constructed order by order in ε, introducing appropriate integration constants. The integration constants are fixed from direct computations of the master integrals in the limit $x \to 1$. For the computation of these boundary values we refer to [31].

3. Relations among the boundary values

Apart from the construction of the x-dependence to finite order in ε, the canonical differential equation also supplies one with the resummation of the leading logarithms $\log(1-x)$ in the limit $x \to 1$. As was observed in [33], the knowledge of the expansion regions contributing to this limit together with the resummed solution in the same limit already has the power to imply relations among the boundary values.

The reasoning starts with the observation that a differential equation

$$\partial_x f = \frac{\varepsilon a}{1-x} f \tag{3.1}$$

can be solved to all orders in ε by a matrix exponential

$$f(x,\varepsilon) = R\,T(x,\varepsilon) f_R \quad \text{with} \quad T(x,\varepsilon) := (1-x)^{-\varepsilon\Lambda} \exp\Big\{-(J-\Lambda)\varepsilon \log(1-x)\Big\}\,, \tag{3.2}$$

where $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ is the diagonal of the Jordan normal form J of a. Furthermore R relates the two matrices with each other, i.e. $a = RJR^{-1}$. The vector f_R is independent of x.

With this in mind, we write the canonical differential equation as

$$\partial_x f(x,\varepsilon) = \left[\frac{\varepsilon a}{1-x} + \varepsilon B(x)\right] f(x,\varepsilon), \tag{3.3}$$

where $B(x)$ is regular in the limit $x \to 1$. Transforming f via $g := T^{-1}(x,\varepsilon)R^{-1}f$, one finds the differential equation

$$\partial_x g(x,\varepsilon) = \varepsilon T^{-1} R^{-1} B R T g \tag{3.4}$$

which shows, that g only receives $\log(1-x)$-contributions starting from $O((1-x)^1)$.

As was pointed out in [33], each entry in g is now connected with a unique factor $(1-x)^{-\varepsilon\lambda_i}$. On the other hand, expansion by regions associates with each region a factor of the same kind. If factors of this type occur in Λ, which don't occur in the asymptotic expansion of the according master integrals, the corresponding initial values of g vanish, which implies relations among the integration constants for the canonical master.

Another source of relations originates in the interplay of the basis change from the master integrals of the reduction to the canonical masters with the matching of the boundary values. The soft expansions of the master integrals have to be transformed into the canonical basis before the matching of the integration constants can be performed. The solved and matched canonical masters are then mapped back onto the reduction basis. If this is done with generic expressions for the soft expansions, one observes that some matched master integrals, when expanded, are completely determined from expansion coefficients of other master integrals.

4. Iterated integrals with square root letters

All integral families, except for the one depicted in fig. 1 evaluate to harmonic polylogarithms. The iterated integrals which leave the class of HPLs contain an irrational letter with square roots, and occur with maximum weight 5. Iterated integrals with square root letters have been studied before [34, 35, 36, 37]. Methods for algebraic manipulations and other properties of these objects are available in the Mathematica package `HarmonicSums` [38, 39, 40], which was used in the present calculation.

Since the integration constants are determined from the soft expansions of the master integrals, we need the soft expansion of the iterated integrals with square roots. If the matching is performed in the canonical basis, however, only the leading terms are needed which are of orders $(1-x)^0$ and $\log(1-x)^i$. This simplifies the expansion procedure for the iterated integrals. Note, that an iterated integral $H_{\vec{w}}(x)$ is finite in the limit $x \to 1$ if there is at most one letter in the alphabet that is divergent in that limit and this letter does not occur in the leftmost position in $\vec{w}$. Here the only divergent index is $1/(1-x)$. So the procedure for determining the leading terms takes the form

1. Remove the index $1/(1-x)$ away from the leftmost position, making all divergencies explicit in terms of factors $\log(1-x)^i$.

2. Evaluate all remaining finite iterated integrals at 1.

3. Reduce the constants to a basis.

This procedure is independent of the actual set of letters which are finite in $x \to 1$, so in particular it works if (finite) square root letters are present.

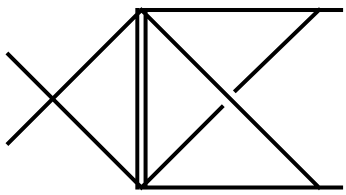

Figure 2: Common subtopology of all the graphs in BT3 which generate square root letters.

The square root letters only occur in the topology BT3, cf. fig. 1. Out of the 24 master integrals of this topology which are needed, only six canonical master integrals show the square root letter. All others can be represented in terms of harmonic polylogarithms. The master integrals in which square root letters occur can be classified as having the common subtopology drawn in Fig. 2. The contributing integrals with this property are

$$
\begin{aligned}
&\mathrm{BT3}(1,0,0,1,1,1,1,1,1,0,0,0),\\
&\mathrm{BT3}(1,0,-1,1,1,1,1,1,1,0,0,0),\\
&\mathrm{BT3}(1,0,0,1,1,1,1,1,1,1,0,0),\\
&\mathrm{BT3}(1,1,1,1,1,1,1,1,1,0,0,0),\\
&\mathrm{BT3}(1,1,1,1,1,1,1,1,1,0,-1,0),\\
&\mathrm{BT3}(1,1,1,1,1,1,1,1,1,-1,0,0)\,.
\end{aligned}
\tag{4.1}
$$

Since the square root letter is combined only with $1/x$ and $1/(1+x)$ in the final result, it is possible to transform the alphabet into a rational alphabet with the transformation $x \to (1-x')/x'^2$. This however introduces the cyclotomic letters $1/(1-x'-x'^2)$ and $x'/(1-x'-x'^2)$ (for the definition and properties of cyclotomic polylogarithms, the reader is referred to [41]). The cyclotomic polynomials can be factorized over the complex numbers, leading to a representation of the functions in terms of multiple polylogs. This leads to a complex representation in the three letter alphabet $\{1,0,(-1)^{1/3}\}$. Here is an example for such a relation:

$$\begin{aligned}
H_{0,-1,s4}(x) = & -8H_1\left(\frac{2x-\sqrt{4x+1}+1}{2x}\right)\mathrm{Re}\left[H_{1,e_3}\left(\frac{2x-\sqrt{4x+1}+1}{2x}\right)\right] \\
& +4\,\mathrm{Re}\left[H_{0,e_3,1}\left(\frac{2x-\sqrt{4x+1}+1}{2x}\right)\right]+16\,\mathrm{Re}\left[H_{1,1,e_3}\left(\frac{2x-\sqrt{4x+1}+1}{2x}\right)\right] \\
& -4H_{0,1,1}\left(\frac{2x-\sqrt{4x+1}+1}{2x}\right)+\frac{4}{3}{H_1}^3\left(\frac{2x-\sqrt{4x+1}+1}{2x}\right),
\end{aligned}$$

where the index $s4$ refers to the letter $f_{s4}=\frac{1}{x}\left(\frac{1}{\sqrt{1+4x}}-1\right)$.

5. Conclusions

I reported on the computation of contributions to Higgs production in gluon fusion at N^{3}LO from diagrams with two quarks of different flavor in the initial state. The computation is performed in the limit of an infinite top quark mass. Integration by parts relations were used for a reduction to master integrals. The master integrals were determined using differential equations and a canonical basis.

This paper focused on the one hand on technical advantages in the use of a basis of canonical master integrals. In particular it was used to derive additional relations for the boundary conditions. A second focus was set onto the structure of the result. Here an interesting feature is the occurrence of iterated integrals with square root letters, which can be related to multiple polylogarithms involving the sixth root of unity.

The analytic results were published in [31].

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