Evaluating multi-loop Feynman integrals using differential equations: automatizing the transformation to a canonical basis

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In the past years the method of differential equations has been proven to be a powerful tool for the computation of multi-loop Feynman integrals. This method relies on the choice of a basis of master integrals in which the dependence on the dimensional regulator factorizes. We will present an algorithm, which automatizes the transformation to such a basis, starting from some basis that is, for instance, obtained by the usual integration-by-parts reduction techniques. Our algorithm only requires some mild assumptions about the reduction basis. In particular, it applies to problems with multiple scales and rational dependence on the regulator.

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1. Introduction

One of the bottlenecks in modern multi-loop calculations is the treatment of the occurring multi-loop Feynman integrals. The integration-by-parts reduction [1–4] to master integrals can be performed with a number of publicly available tools [5–9]. Although in practice often limited by computer resources, this step can be considered as conceptually solved. The same can not be said about the computation of the master-integrals themselves. One of the most widely used methods is the method of differential equations [3, 10, 11], which received a major improvement by the suggestion [12] to use a canonical basis of master integrals. Once the differential equation is expressed with respect to such a basis, the solution in terms of iterated integrals [13, 14] can be immediately obtained up to integration constants. This refined method of differential equations has been successfully applied in many calculations in recent years [12, 15–38]. In the univariate case, a complete algorithm [39] using difference equations is available, which does not rely on the existence of a canonical basis.

With the new approach the main difficulty is to find a canonical basis of master integrals. Several different strategies have been employed to construct a canonical basis [12, 16, 21, 25, 29, 40, 41]. Here we want to propose an algorithm that computes the transformation to a canonical basis. It is applicable to multi-scale problems with rational dependence on the regulator. It is first shown that the problem to find a transformation to a canonical basis can be mapped to the problem of finding a rational solution of a finite number of differential equations. Then we argue that this can be achieved with an ansatz that is a linear combination of functions from a special class of rational functions [42, 43].

2. Preliminaries

Let $\vec{f}(\varepsilon, \{x_j\})$ be the m dimensional vector of master integrals, which are functions of M dimensionless kinematic invariants $\{x_j\}$ and the dimensional regulator ε. The derivative of a master integral with respect to the kinematic invariants can always be written as a linear combination of master integrals from the same or lower sectors. Thus, upon differentiating with respect to all kinematic invariants, the following linear system of differential equations is obtained

$$\partial_i \vec{f}(\varepsilon, \{x_j\}) = a_i(\varepsilon, \{x_j\}) \vec{f}(\varepsilon, \{x_j\}), \quad i = 1, \ldots, M \tag{2.1}$$

with the $a_i(\varepsilon, \{x_j\})$ being $m \times m$ matrices of rational functions in the kinematic invariants $\{x_j\}$ and ε. In the more compact differential notation equation (2.1) can be written as

$$d\vec{f}(\varepsilon, \{x_j\}) = a(\varepsilon, \{x_j\}) \vec{f}(\varepsilon, \{x_j\}), \tag{2.2}$$

with

$$a(\varepsilon, \{x_j\}) = \sum_{i=1}^{M} a_i(\varepsilon, \{x_j\}) dx_i. \tag{2.3}$$

Transforming the basis of master integrals with an invertible transformation T,

$$\vec{f} = T(\varepsilon, \{x_j\}) \vec{f}', \tag{2.4}$$

as suggested in [12], leads to the following transformation law for $a(\varepsilon, \{x_j\})$:

$$a' = T^{-1}aT - T^{-1}\mathrm{d}T. \tag{2.5}$$

If $a(\varepsilon, \{x_j\})$ is exact, i.e. an $A(\varepsilon, \{x_j\})$ exist such that

$$\mathrm{d}A(\varepsilon, \{x_j\}) = a(\varepsilon, \{x_j\}) \tag{2.6}$$

and the dependence of $A(\varepsilon, \{x_j\})$ on the kinematic invariants is only logarithmic

$$A(\varepsilon, \{x_j\}) = \sum_{l=1}^{N} A_l(\varepsilon) \log(L_l(\{x_j\})), \tag{2.7}$$

$a(\varepsilon, \{x_j\})$ is called to be in *dlog-form*. Here $L_l(\{x_j\})$ denotes polynomials in the kinematic invariants and the A_l are $m \times m$ matrices which solely depend on ε. The set of polynomials

$$\mathscr{A} = \{L_1(\{x_j\}), \ldots, L_N(\{x_j\})\} \tag{2.8}$$

is commonly referred to as the *alphabet* of the differential equation. The individual polynomials are called the *letters* of the differential equation. In [12] it was observed that with a suitable change of the basis of master integrals it is often possible to arrive at a form in which the dependence on ε factorizes:

$$A(\varepsilon, \{x_j\}) = \varepsilon \sum_{l=1}^{N} \tilde{A}_l \log(L_l(\{x_j\})), \tag{2.9}$$

with $\tilde{A}_l$ being constant $m \times m$ matrices. In this form, which is called ε-*form*, it is particularly easy to solve the differential equation in terms of iterated integrals [13, 14].

3. General properties of the Transformation

It is useful to first look into general properties of the transformation law (2.5). Consider a transformation T, which transforms the differential equation into ε-form. Then an $\tilde{A}$ exists such that

$$\varepsilon \mathrm{d}\tilde{A}(\varepsilon, \{x_j\}) = a'(\varepsilon, \{x_j\}) \tag{3.1}$$

holds. In this case (2.5) can be written in the form

$$\varepsilon \mathrm{d}\tilde{A} = T^{-1}aT - T^{-1}\mathrm{d}T. \tag{3.2}$$

Taking the trace on both sides of (3.2) leads to

$$\varepsilon \mathrm{Tr}[\mathrm{d}\tilde{A}] = \mathrm{Tr}[a] - \mathrm{Tr}[T^{-1}\mathrm{d}T]. \tag{3.3}$$

Applying Jacobi's formula for the differential of determinants

$$\mathrm{d}\det(T) = \det(T)\mathrm{Tr}[T^{-1}\mathrm{d}T] \tag{3.4}$$

one obtains

$$\mathrm{d}\log(\det(T)) = \mathrm{Tr}[a] - \varepsilon \mathrm{Tr}[\mathrm{d}\tilde{A}]. \tag{3.5}$$

It follows that a necessary condition for the existence of an ε-form is that the form $\mathrm{Tr}[a]$ is exact:

$$\mathrm{Tr}[a] = \mathrm{d}\left(\varepsilon\mathrm{Tr}[\tilde{A}] + \log(\det(T))\right). \tag{3.6}$$

In fact, with (2.9) it is evident that $\mathrm{Tr}[a]$ has to be in dlog-form

$$\mathrm{Tr}[a] = \varepsilon\sum_{l=1}^{N}\mathrm{Tr}[\tilde{A}_l]\mathrm{d}\log(L_l(\varepsilon,\{x_j\})) + \mathrm{d}\log(\det(T)). \tag{3.7}$$

As the components of T are required to be rational in the invariants and ε, $\det(T)$ will also have this property. One may bring all summands of $\det(T)$ on a common denominator and factorize the numerator and denominator into irreducible polynomials in $K[\varepsilon,\{x_j\}]$. Here, $K[\varepsilon,\{x_j\}]$ denotes the ring of polynomials in the invariants and ε with coefficients in a field K. There is no need to specify the field at this point, for the present application one may have the real and complex numbers in mind. Thus, $\det(T)$ can be written as

$$\det T = C(\varepsilon)p_1(\{x_j\})^{e_1}\cdots p_K(\{x_j\})^{e_K}q_1(\varepsilon,\{x_j\})^{d_1}\cdots q_L(\varepsilon,\{x_j\})^{d_L}, \tag{3.8}$$

with $e_i \in \mathbb{Z}$ and $d_j \in \mathbb{Z}$. The irreducible factors, which only depend on the invariants, are labeled p and those, which depend on both ε and the invariants, q. The product of all factors, which only depend on ε is denoted by $C(\varepsilon)$. The factorization allows to rewrite (3.7)

$$\begin{aligned}\mathrm{Tr}[a] &= \varepsilon\sum_{l=1}^{N}\mathrm{Tr}[\tilde{A}_l]\mathrm{d}\log(L_l(\{x_j\})) + \sum_{i=1}^{K}e_i\mathrm{d}\log(p_i(\{x_j\})) \\ &+ \sum_{j=1}^{L}d_j\mathrm{d}\log(q_j(\varepsilon,\{x_j\})).\end{aligned} \tag{3.9}$$

This equation can be understood as a necessary condition on the form of $\mathrm{Tr}[a]$ for a rational transformation T to exist, which transforms the differential equation into ε-form. In particular, it implies

$$\mathrm{Tr}[a^{(k)}] = 0 \quad \forall\, k < 0, \tag{3.10}$$

where the $a^{(k)}$ denote the coefficients of the ε-expansion of $a(\varepsilon,\{x_j\})$. Furthermore, one observes that the coefficients of the dlog-terms stemming from $\det(T)$ are integers whereas the coefficients of the dlog-terms from $\mathrm{Tr}[\mathrm{d}\tilde{A}]$ are proportional to ε. This allows one to calculate the determinant of T up to a rational function $C(\varepsilon)$. Moreover, one can also read off the traces of the $\tilde{A}_l$ of the resulting ε-form. In practice, one can test whether $\mathrm{Tr}[a]$ can be written as follows

$$\mathrm{Tr}[a] = \varepsilon X(\{x_j\}) + Y(\varepsilon,\{x_j\}), \tag{3.11}$$

with X and Y denoting sums of dlog-terms. If this is not the case one may conclude that no rational transformation exists, which can transform $a(\varepsilon,\{x_j\})$ into ε-form. Otherwise one can extract

$$\det(T) = C(\varepsilon)\exp\left(\int_{\gamma}Y(\varepsilon,\{x_j\})\right), \tag{3.12}$$

$$\mathrm{Tr}[\mathrm{d}\tilde{A}] = X(\{x_j\}). \tag{3.13}$$

As will be argued later, both equations provide useful information for the determination of T. Note that for one-dimensional sectors (3.12) already fixes the transformation up to a rational function in ε. The choice of this function does not alter the resulting a'. Therefore, one may set for the undetermined $C(\varepsilon)$

$$C(\varepsilon) = 1, \tag{3.14}$$

which then completely fixes the transformation. The determinant provides valuable information for the computation of T for higher-dimensional sectors as well.

4. Expanding the Transformation

Every invertible transformation T, which transforms the differential equation into ε-form, has to satisfy (3.2) for some $d\tilde{A}$, which has to be determined as well. For invertible T, equation (3.2) can equivalently be written as

$$dT - aT + \varepsilon T d\tilde{A} = 0. \tag{4.1}$$

This form has the advantage of not containing the inverse of T. The strategy to find a solution of this equation is to expand T in ε and solve for its coefficients order by order. In general, the expansion of T may have infinitely many non-vanishing coefficients. This poses a problem for the algorithmic computation of these coefficients. In the following it will be shown how this problem can be circumvented.

The equation (4.1) is invariant under the multiplication of T by a rational function $g(\varepsilon)$. Any such rational function can be written as a product of some power of ε and a polynomial $\eta(\varepsilon)$ with non-vanishing constant coefficient

$$g(\varepsilon) = \varepsilon^\tau \eta(\varepsilon). \tag{4.2}$$

Demanding the expansion of T to start at order ε^0

$$T = \sum_{n=0}^{\infty} \varepsilon^n T^{(n)}, \quad T^{(0)} \neq 0, \tag{4.3}$$

only fixes the value of τ, but one is still free to choose some $\eta(\varepsilon)$. As $a(\varepsilon, \{x_j\})$ is required to be rational in both the invariants and ε, there exists a polynomial $h(\varepsilon, \{x_j\})$ such that $\hat{a} := ah$ has a finite Taylor expansion in ε

$$\hat{a} = \sum_{k=0}^{k_{\max}} \varepsilon^k \hat{a}^{(k)}, \quad k_{\max} < \infty. \tag{4.4}$$

In addition to that, h is required to be *minimal* in the sense that it shall have the smallest possible number of irreducible factors for which $\hat{a}$ has a finite ε-expansion. This fixes h up to a multiplicative constant, which is irrelevant here. Defining $\hat{T} := Th$ and rewriting equation (4.1) in terms of $\hat{T}$ yields

$$-\hat{T}dh + hd\hat{T} - \hat{a}\hat{T} + \varepsilon h\hat{T}d\tilde{A} = 0. \tag{4.5}$$

It can be shown that each solution T of (4.1) corresponds via $T = \hat{T}/h$ to a solution $\hat{T}$ of (4.5) with finite ε-expansion. This allows one to avoid computing infinitely many coefficients in the expansion of T. Instead, one may calculate $\hat{T}$ by expanding (4.5) in ε:

$$-\hat{T}dh + hd\hat{T} = \sum_{n=2l_{\min}}^{n_{\max}+l_{\max}} \varepsilon^n \sum_{k=l_{\min}}^{\min(l_{\max},n-l_{\min})} \left(-dh^{(k)}\hat{T}^{(n-k)} + h^{(k)}d\hat{T}^{(n-k)} \right), \tag{4.6}$$

$$\varepsilon h\hat{T}d\tilde{A} = \sum_{n=2l_{\min}}^{n_{\max}+l_{\max}} \varepsilon^{n+1} \sum_{k=l_{\min}}^{\min(l_{\max},n-l_{\min})} h^{(k)}\hat{T}^{(n-k)}d\tilde{A} \tag{4.7}$$

$$= \sum_{n=2l_{\min}+1}^{n_{\max}+l_{\max}+1} \varepsilon^n \sum_{k=l_{\min}}^{\min(l_{\max},n-l_{\min}-1)} h^{(k)}\hat{T}^{(n-k-1)}d\tilde{A}, \tag{4.8}$$

$$\hat{a}\hat{T} = \sum_{n=l_{\min}}^{n_{\max}+k_{\max}} \varepsilon^n \sum_{k=0}^{\min(k_{\max},n-l_{\min})} \hat{a}^{(k)}\hat{T}^{(n-k)}, \tag{4.9}$$

with

$$h = \sum_{l=l_{\min}}^{l_{\max}} h^{(l)}, \quad l_{\min} \geq 0, \quad l_{\max} < \infty. \tag{4.10}$$

Note that the equation at some order k only involves $\hat{T}^{(n)}$ with $n \leq k$, which means that one can compute the $\hat{T}^{(n)}$ successively, starting with the lowest order. Given some $a(\varepsilon, \{x_j\})$, one first calculates h and $\hat{a}$, which in turn fixes the values of $l_{\min}$, $l_{\max}$ and $k_{\max}$. However, the value of $n_{\max}$ remains unknown until the solution for $\hat{T}$ is known. Therefore, one should test at each order k whether $k = n_{\max}$. In order to do so one has to check if $\hat{T}^{(n)} = 0$ for all $n > k$ solves the equations of the remaining $\max(k_{\max}, l_{\max} + 1)$ subsequent orders. The algorithm stops as soon as this test is successful and returns $T = \hat{T}/h$.

5. Solving for a rational transformation

The previous section showed that the problem of finding a transformation T to a canonical basis is equivalent to finding a rational solution of the finite number of differential equations, which appear by expanding (4.5). These differential equations do in general admit transcendental solutions for $\hat{T}^{(n)}$. However, one is only interested in rational solutions, therefore it suggests itself to solve the differential equations with a rational ansatz. This means that a subspace of the space of rational functions in the invariants has to be parameterized. There are many different ways of achieving this, but in order to end up with linear equations in the parameters one makes an ansatz of the following form for the Taylor coefficients of $\hat{T}$

$$\hat{T}^{(n)} = \sum_{k=1}^{|\mathscr{R}_T|} \tau_k^{(n)} r_k(\{x_j\}), \tag{5.1}$$

$$\mathscr{R}_T := \left\{ r_1(\{x_j\}), \ldots, r_{|\mathscr{R}_T|}(\{x_j\}) \right\}. \tag{5.2}$$

Here the $\tau_k^{(n)}$ are $m \times m$ matrices of unknown parameters and $\mathscr{R}_T$ a set of rational functions. It was shown in [42, 43] that every rational function can be written as a linear combination of *simple* rational functions. These simple rational functions have the property that their irreducible denominator factors are algebraically independent and share a common zero. The former implies that one does

not have to consider rational functions with more irreducible denominator factors than the number of variables, as those would necessarily be algebraically dependent. The set $\mathscr{R}_T$ can without loss of generality be restricted to contain only rational functions of this particular type. Moreover, the determinant of $\hat{T}$ is known by virtue of (3.12). Information on which irreducible factors have to be present in the elements of $\mathscr{R}_T$ can then be extracted from the determinant.

Note that $d\tilde{A}$ is also unknown in (4.5). However, $\tilde{A}$ is already very constrained, since we require it to be of the form

$$\tilde{A} = \sum_{l=1}^{N} \alpha_l \log(L_l(\{x_j\})). \tag{5.3}$$

Here, the α_l are considered as $m \times m$ matrices of unknown parameters. The alphabet

$$\mathscr{A} = \{L_1(\{x_j\}), \dots, L_N(\{x_j\})\} \tag{5.4}$$

has to be chosen such that it contains all letters which are necessary for a resulting ε-form. A natural choice is to take the set of all irreducible denominator factors occurring in $\hat{a}$. Also remember that (3.13) fixes the traces of all α_l and thereby reduces the number of free parameters. Consider the expansion of (4.5) with the above ansatz inserted. Upon requiring the resulting equations to hold for all allowed values of the invariants, one obtains a system of equations in the unknown parameters. It may happen that $\hat{T}^{(n)}$ is not fully determined by the equations of order n or lower. If $\tilde{A}$ is not fully determined as well, it can happen that terms, which are nonlinear in the parameters, arise at the equation of order $n+1$ due to the term $\varepsilon T d\tilde{A}$ in (4.1). So in general one has to solve a system of polynomial equations in the unknowns in order to determine the solution.

6. Conclusion

In this report we have shown that the transformation to a canonical basis of master integrals can be obtained by finding rational solutions of a finite number of differential equations. We argued that this can be achieved with an ansatz in terms of a special class of rational functions [42, 43]. The block-triangular form, which the differential equations inherit from the structure of the integration-by-parts identities, has not been exploited here. However, this can be incorporated in the above approach to give an algorithm, which then computes the transformation in a bottom-up approach.

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