Leading and next to leading large $n_f$ terms in the cusp anomalous dimension and the quark–antiquark potential

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I discuss 3 related quantities: the cusp anomalous dimension, the HQET heavy-quark field anomalous dimension, and the quark–antiquark potential. Leading large $n_f$ terms can be calculated to all orders in $\alpha_s$. Next to leading terms with the abelian color structure $C_F^2$ also can be found to all orders (but not non-abelian $C_F C_A$ terms). This talk is based on Appendices C and D in [1].
1. Introduction

The one-loop cusp anomalous dimension
\[
\Gamma(\alpha_s, \phi) = C_F \frac{\alpha_s}{\pi} (\phi \coth \phi - 1)
\] (1.1)
follows from the soft radiation function in classical electrodynamics: when a charge suddenly changes its velocity, it emits electromagnetic waves; integrating the intensity over directions, one obtains \(2 \phi \coth \phi - 1\). This result is probably known for more than 100 years, and should be included in The Guinness Book of Records as the anomalous dimension known for a longest time. The two-loop term has been calculated 30 years ago [3] (and rewritten via \(\text{Li}_2, \text{Li}_3\) in [4]). The three-loop term has been calculated recently [5, 6, 1].

The HQET heavy-quark field anomalous dimension (or the anomalous dimension of a straight Wilson line) is known up to 3 loops. At 2 loops, after a wrong calculation [7], the correct result has been obtained in [8], and later in [9, 10, 11, 12]. The three-loop result has been obtained in [13, 14] (in the first paper [13] it has been found as a by-product of the calculation of the QCD on-shell heavy-quark field renormalization constant, from the requirement that the QCD/HQET matching coefficient for the heavy-quark field [15] is finite; at 2 loops this has been done in [11]).

The quark–antiquark potential is known at two [16, 17] and three [18, 19, 20] loops.

Some terms in perturbative series for these quantities can be obtained to all orders in \(\alpha_s\).

2. Large \(n_f\) terms

The terms with the highest power of \(n_f\) at each order of perturbation theory for the cusp anomalous dimension \(\Gamma\) have the structures \(C_F(T_F n_f)^{L-1} \alpha_s^L (L \geq 1)\). They are known to all orders in \(\alpha_s\). The terms with next to highest power of \(n_f\) have the structures \(C_F^2(T_F n_f)^{L-2} \alpha_s^L\) and \(C_F C_A(T_F n_f)^{L-2} \alpha_s^L (L \geq 3)\). The abelian ones (without \(C_A\)) can be also found to all orders in \(\alpha_s\).

For this purpose it is sufficient to consider QED with \(n_f\) massless lepton flavors: \(C_F = T_F = 1, C_A = 0, \beta_0 = -\frac{4}{3} n_f\). Let’s introduce
\[
b = \beta_0 \frac{\alpha_s}{4\pi}.
\] (2.1)
We assume \(b \sim 1\) and take into account all powers of \(b\); \(1/\beta_0 \ll 1\) is our small parameter, and we consider only a few terms in expansions in \(1/\beta_0\).

At the leading and next-to-leading large-\(b_0\) orders (\(L\beta_0\) and \(NL\beta_0\)), the coordinate-space Wilson line of any shape is equal to
\[
\log W = \text{thick photon line with NL\beta_0 accuracy}.
\] (2.2)
where the thick photon line is the full photon propagator with the NL\(b_0\) accuracy. This simple exponentiation formula is first broken at \(\text{NNL}\beta_0\) order by the light-by-light diagram (figure 1).
Figure 1: The light-by-light diagram is $n_f \alpha^4$, and hence NNL$\beta_0$.

With the NNL$\beta_0$ accuracy the renormalization constant $Z$ of the heavy-to-heavy current (the cusp) is given by

$$\log W(t,t';\varphi) - \log W(t,t';0) = \log Z + \text{finite} \quad (2.3)$$

(diagrams where both photon-interaction vertices are before the cusp, or after the cusp, cancel in this difference). Going to momentum space, we can express it via the vertex function $V(\omega, \omega'; \varphi)$ (it is convenient to set $\omega' = \omega$, in order to have a single-scale problem):

$$V(\omega, \omega; \varphi) - V(\omega, \omega; 0) = \log Z + \text{finite}. \quad (2.4)$$

The HQET field renormalization can be obtained from $V(\omega, \omega; 0)$.

The static quark–antiquark potential can be considered similarly. The terms with the highest power of $n_f$ in each order of perturbation theory have the structures $C_F (T_F n_f)^L \alpha_s^{L+1}$ ($L \geq 0$). The terms with next to highest power of $n_f$ have the structures $C_F^2 (T_F n_f)^{L-1} \alpha_s^{L+1}$ and $C_F C_A (T_F n_f)^{L-1} \alpha_s^{L+1}$ ($L \geq 2$); we’ll consider only the abelian ones. In the Coulomb gauge, up to NNL$\beta_0$ the potential is given by the full Coulomb photon propagator

$$V(\vec{q}) = \frac{e^2}{\vec{q}^2} \frac{1}{1 - \Pi(-\vec{q}^2)} \quad (2.5)$$

($\Pi(q^2)$ is gauge invariant in QED, and can be taken from covariant-gauge calculations). This simple equality is first broken at NNL$\beta_0$ order by the light-by-light diagram (figure 2).

Figure 2: The light-by-light diagram is $n_f \alpha^4$, and hence NNL$\beta_0$.

As discussed in [1], conformal symmetry leads to the relation between $\Gamma(\pi - \delta)$ at $\delta \to 0$ and $V(\vec{q})$:

$$\Delta \equiv [\delta \Gamma(\pi - \delta; \alpha_s)]_{\delta \to 0} - \frac{\vec{q}^2 V(\vec{q}; \alpha_s)}{4\pi} = 0 \quad (2.6)$$
(this relation has been observed in [21] at 2 loops). In QCD (and QED) conformal symmetry is anomalous (thus leading to non-zero $\beta$ function), and [1]

$$\Delta = \frac{\pi}{108} \beta_0 C_F \left( \frac{\alpha_s}{\pi} \right)^3 (47C_A - 28T_F n_f) + O(\alpha_s^4). \quad (2.7)$$

3. Leading $\beta_0$ order

The photon self energy at the L$\beta_0$ order is $\sim 1$:

$$\Pi_0(k^2) = \frac{\varepsilon_0^2}{(4\pi)^{d/2}} e^{-\varepsilon \varepsilon} \frac{D(\varepsilon)}{\varepsilon} (-k^2)^{-\varepsilon},$$

$$D(\varepsilon) = e^{\varepsilon \varepsilon} \frac{(1 - \varepsilon)\Gamma(1 + \varepsilon)\Gamma^2(1 - \varepsilon)}{(1 - 2\varepsilon)(1 - \frac{\varepsilon}{2})\Gamma(1 - 2\varepsilon)} = 1 + \frac{5}{3} \varepsilon + \cdots \quad (3.1)$$

The charge renormalization in the $\overline{MS}$ scheme is

$$\frac{\varepsilon_0^2}{(4\pi)^{d/2}} e^{-\varepsilon \varepsilon} = bZ_\alpha(b) \mu^{2\varepsilon}. \quad (3.2)$$

At the L$\beta_0$ order we can solve the RG equation

$$\frac{d \log Z_\alpha}{d \log b} = -\frac{b}{\varepsilon + b}$$

and obtain

$$Z_\alpha = \frac{1}{1 + b/\varepsilon}. \quad (3.3)$$

The vertex $V(\omega, \omega; \varphi)$ is given by the one-loop diagram with the factor $1/(1 - \Pi(k^2))$ inserted in the integrand. At the L$\beta_0$ order (figure 3) the result can be written in the form

$$V(\omega, \omega; \varphi) = \frac{1}{\beta_0^2} \sum_{L=1}^{\infty} \frac{f(\varepsilon, L\varepsilon; \varphi)}{L} \Pi_0^L + O\left( \frac{1}{\beta_0^2} \right), \quad (3.4)$$

where $L$ is the number of loops and $\Pi_0 (3.1)$ is taken at $-k^2 = (-2\omega)^2$. Reduction of such integrals to master ones, as well as evaluation of these master integrals, has been considered in [22].
Landau gauge we obtain

\[
f(\varepsilon, u; \varphi) = \frac{(1 - \varepsilon^2)\Gamma(2 - 2\varepsilon)\Gamma(1 - u)\Gamma(1 + 2u)}{(1 - \varepsilon)\Gamma^2(1 - \varepsilon)\Gamma(1 + \varepsilon)\Gamma(2 + u - \varepsilon)} \times \left[ (2 + u - 2\varepsilon) \cos \varphi - u \right] _2F_1 \left( \frac{1, 1 - u}{3/2}; \frac{1 - \cos \varphi}{2} \right) + 1 \]

(3.5)

(in an arbitrary covariant gauge, a one-loop gauge-dependent contribution should be added). The function \( f(\varepsilon, u; \varphi) \) is regular at the origin:

\[
f(\varepsilon, u; \varphi) = \sum_{n,m=0}^{\infty} f_{nm}(\varphi) \varepsilon^n u^m. \tag{3.6}
\]

The renormalization constant \( Z \) can be written as

\[
\log Z = \frac{Z_1}{\varepsilon} + \frac{Z_2}{\varepsilon^2} + \cdots, \quad Z_n = O(b^n).
\]

Only \( Z_1 \) is needed in order to obtain

\[
\Gamma(b; \varphi) = -2\frac{dZ_1(b; \varphi)}{d\log b};
\]

higher \( Z_n \) contain no new information, and are uniquely reconstructed from \( Z_1 \) using self-consistency conditions. Choosing

\[
\mu^2 = D(\varepsilon)^{-1/2}( -2\omega)^2 \rightarrow e^{-\frac{\varepsilon^2}{2}}( -2\omega)^2
\]

we have

\[
V(\omega, \omega; \varphi) - V(\omega, \omega; 0) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \tilde{f}(\varepsilon, L\varepsilon; \varphi) \left( \frac{b}{\varepsilon + b} \right)^L + O\left( \frac{1}{\beta_0^2} \right), \tag{3.7}
\]

where \( \tilde{f}(\varepsilon, u; \varphi) = f(\varepsilon, u; \varphi) - f(\varepsilon, u; 0) \). We expand in \( b \), expand \( \tilde{f}(\varepsilon, u; \varphi) \) in \( \varepsilon \) and \( u \) and select only \( \varepsilon^{-1} \) terms in order to obtain \( Z_1 \). All coefficients but \( f_{n0} \) cancel:

\[
Z_1(b; \varphi) = 2\frac{\varphi \cot \varphi - 1}{\beta_0} \sum_{n=0}^{\infty} \frac{\hat{f}_n(-b)}{n+1} (-b)^{n+1},
\]

where

\[
\hat{f}(\varepsilon, 0; \varphi) = -2\hat{f}(\varepsilon)(\varphi \cot \varphi - 1), \quad \hat{f}(\varepsilon) = \sum_{n=0}^{\infty} \hat{f}_n \varepsilon^n.
\]

Therefore at the \( L\beta_0 \) we obtain [23]

\[
\Gamma(b; \varphi) = 4b \frac{\beta_0}{b} \gamma_0(b)(\varphi \cot \varphi - 1) + O\left( \frac{1}{\beta_0^2} \right),
\]

\[
\gamma_0(b) = \hat{f}(-b) = \frac{1}{(1 + b)\Gamma^3(1 + b)^2}\Gamma(1 - b)
\]

\[
= 1 + \frac{5}{3} b - \frac{1}{3} b^2 - \left( 2\zeta_3 - \frac{1}{3} \right) b^3 + \left( \frac{\pi^4}{30} - \frac{10}{3} \zeta_3 - \frac{1}{3} \right) b^4 + \cdots \tag{3.8}
\]
As a free bonus, we can obtain the HQET field anomalous dimension. The vertex function $V$ at $\varphi = 0$ is related to the HQET propagator $S$ by the Ward identity
\[
V(\omega, \omega'; 0) = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega}, \quad V(\omega, \omega; 0) = \frac{dS^{-1}(\omega)}{d\omega}.
\] (3.9)
Therefore the renormalization constant of the HQET quark field $Z_h$ is given by
\[
\log V(\omega, \omega'; 0) = -\log Z_h + \text{finite}.
\]
Using
\[
f(e, u; 0) = -\frac{(1 - \frac{2}{3}e)^2\Gamma(2 - 2\epsilon)\Gamma(1 - u)}{(1 - e)^2(1 - e)\Gamma(1 + 2u)},
\]
we obtain in the Landau gauge [24]
\[
\gamma_h(b) = \frac{b}{b_0} \gamma_0(b) + \mathcal{O}\left(\frac{1}{b_0}\right),
\]
\[
\gamma_0(b) = f(-b, 0; 0) = \frac{(1 + \frac{2}{3}b)^2\Gamma(2 + 2b)}{(1 + b)^2\Gamma(1 + b)}
\]
\[
= 1 + \frac{4}{3}b - \frac{5}{9}b^2 - \left(2\zeta_3 - \frac{2}{3}\right)b^3 + \left(\frac{\pi^4}{30} - \frac{8}{3}\zeta_3 - \frac{7}{9}\right)b^4 + \cdots
\] (3.10)
(in an arbitrary covariant gauge, a one-loop gauge-dependent contribution should be added).

Now we consider the potential $V(\bar{q})$ at the $L\beta_0$ order. Choosing $\mu^2 = \bar{q}^2$ we have
\[
V(\bar{q}) = -\frac{(4\pi)^{D/2}e^{\gamma}}{b_0D(e)(\bar{q}^2)^{1-\epsilon}} \sum_{L=1}^{\infty} \left(D(e) \frac{b}{\epsilon + b}\right)^L + \mathcal{O}\left(\frac{1}{b_0}\right).
\]
The sum here can be written as
\[
\sum_{L=1}^{\infty} g(e, L\epsilon) \left(\frac{b}{\epsilon + b}\right)^L, \quad g(e, u) = D(e)^u = \sum_{n,m=0}^{\infty} g_{nm} \epsilon^n u^m.
\]
This sum is equal to
\[
\frac{b}{\epsilon} \sum_{n=0}^{\infty} n! g_{0n} b^n + \mathcal{O}(\epsilon^0)
\]
(1/\epsilon^n terms with $n > 1$ vanish, so that $V(\bar{q})$ is automatically finite), where
\[
g(0, u) = e^{\gamma u}, \quad g_{0n} = \frac{1}{n!} \left(\frac{5}{3}\right)^n.
\] (3.11)
Therefore
\[
V(\bar{q}) = -\frac{(4\pi)^2}{\bar{q}^2} \frac{b}{b_0} V_0(b) + \mathcal{O}\left(\frac{1}{b_0}\right), \quad V_0(b) = \frac{1}{1 - \frac{5}{3}b}.
\] (3.12)
The conformal anomaly (2.6) at the $L\beta_0$ order is
\[
\Delta = 4\pi \frac{b^3}{b_0} \delta_0(b) + \mathcal{O}\left(\frac{1}{b_0}\right),
\]
\[
\delta_0(b) = \frac{V_0(b) - \gamma_0(b)}{b^2} = \frac{28}{9} + 2 \left(\zeta_3 + \frac{58}{27}\right) b - \frac{1}{3} \left(\frac{\pi^4}{10} - 10\zeta_3 - \frac{652}{27}\right) b^2 + \cdots
\] (3.13)
The first term here reproduces the $T_Fn_f$ term in (2.7).
4. Next to leading $\beta_0$ order

To obtain the photon propagator with the NL$\beta_0$ accuracy, we need the photon self-energy up to $1/\beta_0$:

$$\Pi(k^2) = \Pi_0(k^2) + \Pi_1(k^2) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \quad (4.1)$$

where the photon propagators in $\Pi_1$ are taken at the L$\beta_0$ order. The NL$\beta_0$ contribution can be written in the form [25, 26]

$$\Pi_1(k^2) = 3\varepsilon \sum_{L=2}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \Pi_0(k^2)^L. \quad (4.2)$$

Using integration by parts, one can reduce it to

$$F(\varepsilon, u) = \frac{2(1-2\varepsilon)^2(3-2\varepsilon)\Gamma^2(1-2\varepsilon)}{9(1-\varepsilon)(1-u)(2-u)\Gamma^2(1-\varepsilon)}I(1+u-2\varepsilon)$$

$$\times \left[ -u \frac{4-3\varepsilon-\varepsilon^2+\varepsilon(2+\varepsilon)u-\varepsilon u^2}{\Gamma^2(1-\varepsilon)} \right]$$

$$+ \frac{2(1+\varepsilon)(3-2\varepsilon)-(4+11\varepsilon-7\varepsilon^2)\varepsilon u+8(3-3\varepsilon)u^2-\varepsilon u^3}{(1-u)(2-u)(1-u-\varepsilon)(2-u-\varepsilon)} \frac{\Gamma(1+u)\Gamma(1+u-\varepsilon)}{\Gamma(1-u-\varepsilon)\Gamma(1+u-2\varepsilon)} \right]$$

$$= \sum_{n,m=0}^{\infty} F_{nm} \varepsilon^n u^m, \quad (4.3)$$

where the integral

$$I(n) = \int_0^1 \frac{d^d k_1 d^d k_2}{k_1^2 k_2^2 (k_1 + p)^2 (k_2 + p)^2 ((k_1 - k_2)^2)^n}$$

(euclidean, $p^2 = 1$) can be expressed via a $3F_2$ function of unit argument [27, 28] (see the review [29] for more references). The $3F_2$ function can be expanded up to any desired order using known algorithms, the coefficients are expressed via multiple $\zeta$ values; therefore, the coefficients $F_{nm}$ can be calculated to any desired order.

The function $F(\varepsilon, u)$ simplifies in some cases. In particular [25],

$$F(\varepsilon, 0) = \frac{(1+\varepsilon)(1-2\varepsilon)^2(1-\frac{3}{2}\varepsilon)\Gamma(1-2\varepsilon)}{(1-\varepsilon)^2(1-\frac{1}{4}\varepsilon)\Gamma(1+\varepsilon)\Gamma^3(1-\varepsilon)}, \quad (4.4)$$

so that $F_{n0}$ contain no multiple $\zeta$ values, only $\zeta_n$. Also [26]

$$F(0, u) = \frac{2\psi'\left(2-\frac{u}{2}\right) - \psi'\left(1+\frac{u}{2}\right) - \psi'\left(\frac{3-u}{2}\right) + \psi'\left(\frac{1+u}{2}\right)}{(1-u)(2-u)}, \quad (4.5)$$

so that $F_{0m}$ contains only $\zeta_{2n+1}$ [26]:

$$F_{0m} = -\frac{32}{3} \sum_{n=1}^{(m+1)/2} s (1-2^{-2s})(1-2^{2s-2m-2}) \zeta_{2n+1} + \frac{4}{3} (m+1) (m+(m+6)2^{-m-3}). \quad (4.6)$$
The two-loop case is, of course, trivial:

\[
F(\varepsilon, 2\varepsilon) = \frac{2}{9\varepsilon^2} \frac{3 - 2\varepsilon}{1 - \varepsilon} \left[ \frac{2 (1 - 2\varepsilon)^2 (2 - 2\varepsilon + \varepsilon^2)}{(1 - 3\varepsilon)(2 - 3\varepsilon)} \right] \frac{\Gamma(1 + 2\varepsilon) \Gamma^2(1 - 2\varepsilon)}{\Gamma^2(1 + \varepsilon) \Gamma(1 - \varepsilon) \Gamma(1 - 3\varepsilon)} - 2 + \varepsilon - 2\varepsilon^2.
\]

Let’s write the charge renormalization constant \( Z_a \) with the NL\( \beta_0 \) accuracy as

\[
Z_a(b) = \frac{1}{1 + b/\varepsilon} \left[ 1 + \frac{Z_{a1}(b)}{\beta_0} \right],
\]

\[
Z_{a1}(b) = \frac{Z_{a11}(b)}{\varepsilon} + \frac{Z_{a12}(b)}{\varepsilon^2} + \cdots, \quad Z_{a1n} = O(b^{n+1}). \tag{4.7}
\]

In the abelian theory, \( \log(1 - \Pi) \) expressed (3.2) via renormalized \( b \) should be equal to \( \log Z_a + \) finite. Equating the coefficients of \( \varepsilon^{-1} \) in the \( 1/\beta_0 \) terms in this relation, we see that \( Z_{a11} \) (4.7) is given by the coefficient of \( \varepsilon^{-1} \) in

\[
- \left( 1 + \frac{b}{\varepsilon} \right) \Pi_1.
\]

It is convenient to choose

\[
\mu^2 = D(\varepsilon)^{-1/\varepsilon}(-k^2) \rightarrow e^{-\frac{1}{2}\varepsilon}(-k^2),
\]

then

\[
\Pi_1 = 3\varepsilon \sum_{L=2}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \left( \frac{b}{\varepsilon + b} \right)^L.
\]

We expand in \( b \) and expand \( F(\varepsilon, u) \) in \( \varepsilon \) and \( u \); selecting \( \varepsilon^{-1} \) terms, we find that all coefficients but \( F_{n0} \) cancel:

\[
Z_{a11} = -3 \sum_{n=0}^{\infty} \frac{F_{n0}(-b)^{n+2}}{(n+1)(n+2)}. \tag{4.8}
\]

The \( \beta \) function with NL\( \beta_0 \) accuracy is

\[
\beta(b) = b + \frac{\beta_1(b)}{\beta_0} + O\left( \frac{1}{\beta_0^2} \right), \tag{4.9}
\]

where [25, 26]

\[
\beta_1(b) = -\frac{dZ_{a11}(b)}{d\log b} = 3 \sum_{n=0}^{\infty} \frac{F_{n0}(-b)^{n+2}}{n + 1}
\]

\[
= 3b^2 + \frac{11}{4} b^3 - \frac{77}{36} b^4 - \frac{1}{2} \left( 3\zeta_3 + \frac{107}{48} \right) b^5 + \frac{1}{5} \left( \frac{\pi^4}{10} - 11\zeta_3 + \frac{251}{48} \right) b^6 + \cdots. \tag{4.10}
\]

(the coefficients \( F_{n0} \) follow from \( F(\varepsilon, 0) \) (4.4)). The corresponding terms in the 5-loop QED \( \beta \) function [30] are reproduced. We shall need the full \( Z_{a1} \), not just \( Z_{a11} \); integrating the RG equation with the \( 1/\beta_0 \) accuracy we obtain

\[
Z_{a1}(b) = -\varepsilon \int_0^b \frac{\beta_1(b)}{b(\varepsilon + b)^2} db = -\frac{3}{2} \frac{b^2}{\varepsilon} + \frac{1}{2} \left( 4 + F_{10} \varepsilon \right) \frac{b^3}{\varepsilon^2} - \frac{1}{4} \left( 9 + 3F_{10} \varepsilon + F_{20} \varepsilon^2 \right) \frac{b^4}{\varepsilon^3} + \cdots
\]

\[7\]
At the $\text{NL} \beta_0$ order we should expand the photon propagator $(1 - \Pi_0 - \Pi_1/\beta_0)^{-1}$ up to $1/\beta_0$ (Fig. 4). The vertex function (3.7) becomes

\[
V(\omega, \omega; \varphi) - V(\omega, \omega; 0) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \tilde{f}(\varepsilon, L\varepsilon; \varphi) \left( \frac{b}{\varepsilon + b} \right)^L \times \left[ 1 + L \frac{Z_{\alpha_1}}{\beta_0} + \frac{3 \varepsilon}{\beta_0} \sum_{L'=2}^{L-1} \frac{L-L'}{L'} F(\varepsilon, L'\varepsilon) \right] + \mathcal{O}\left( \frac{1}{\beta_0^3} \right),
\]

where $L'$ is the number of loops in the $\Pi_1$ insertion, and the $1/\beta_0$ correction $Z_{\alpha_1}$ to the charge renormalization (4.7) is taken into account. We expand in $b$ and substitute the expansions (4.3) and (3.6); in $Z_1$, the coefficient of $\varepsilon^{-1}$, all $\tilde{f}_{nm}$ except $\tilde{f}_{n0}$ cancel. At the $\text{NL} \beta_0$ order the cusp anomalous dimension is determined by the same $\tilde{f}_n$ coefficients as at the $L \beta_0$ order:

\[
\Gamma(b; \varphi) = 4 \left[ \frac{b}{\beta_0} \gamma_0(b) - \frac{b^3}{\beta_0^2} \gamma_1(b) + \varepsilon \cot \varphi - 1 \right] + \mathcal{O}\left( \frac{1}{\beta_0^3} \right),
\]

where

\[
\gamma_0(b) = \frac{3}{2} \left( F_{10} + 2 F_{01} - 2 \hat{f}_1 \right) + \left[ 2 F_{20} + 3 \left( F_{11} + F_{02} \right) + 3 F_{01} \hat{f}_1 - 6 \hat{f}_2 \right] b
\]

\[
+ \left[ \frac{3}{4} \left( 3 F_{30} + 4 \left( F_{21} + F_{12} + F_{03} \right) \right) + \left( F_{20} + 3 \left( F_{11} + F_{02} \right) \right) \hat{f}_1 - \frac{3}{2} \left( F_{10} - 2 F_{01} \right) \hat{f}_2 - 9 \hat{f}_3 \right] b^2 + \cdots
\]

**Figure 4:** NL$\beta_0$ order diagrams contain one $\Pi_1$ insertion (with any number of $\Pi_0$ insertions inside) and any number of $\Pi_0$ insertions to the left and to the right of it.
Substituting $F_{nm}$ we obtain

$$\gamma_1(b) = 12 \zeta_3 - \frac{55}{4} + \left( -\frac{\pi^4}{5} + 40 \zeta_3 - \frac{299}{18} \right) b$$

$$+ \left( 24 \zeta_3 - \frac{2}{3} \pi^4 + \frac{233}{6} \zeta_3 + \frac{15211}{864} \right) b^2$$

$$+ \left( -48 \zeta_3^2 - \frac{2}{63} \pi^6 + 80 \zeta_3 - \frac{167}{225} \pi^4 + \frac{1168}{15} \zeta_3 - \frac{971}{240} \right) b^3$$

$$+ \left( 36 \zeta_3^7 + \frac{8}{5} \pi^4 \zeta_3 - 160 \zeta_3^2 - \frac{20}{189} \pi^6 + \frac{377}{15} \zeta_3 - \frac{23}{15} \pi^4 + \frac{929}{12} \zeta_3 - \frac{8017}{1728} \right) b^4$$

$$+ \left( -240 \zeta_3^5 - \frac{4}{225} \pi^8 + 120 \zeta_3 + \frac{36}{5} \pi^4 \zeta_3 - \frac{2776}{21} \zeta_3^2 - \frac{914}{3969} \pi^6 \right.$$

$$\left. + \frac{6826}{21} \zeta_3 - \frac{1793}{1350} \pi^4 - \frac{31693}{315} \zeta_3 + \frac{79433}{4320} \right) b^5 + \cdots$$

This expansion can be extended to any number of loops. The first term in (4.13) agrees with the $C_\gamma^2 T_F n_f$ term in the three-loop result [5, 6, 1]. The next term coincides with the $C_\gamma^2 (T_F n_f)^2 \alpha_s^4$ term in $\Gamma$ recently calculated in [31]. Note that the last (8-loop) term here contains $F_{nm}$ with $n + m = 6$, $n > 0$, $m > 0$, which contain $\zeta_{5, 3}$; but they enter as the combination $F_{51} + F_{42} + F_{33} + F_{24} + F_{15}$ in which this $\zeta_{5, 3}$ cancels.

Similarly, the field anomalous dimension in Landau gauge at the NL$\beta_0$ order is

$$\gamma_0(b) = -6 \left[ \frac{b}{\beta_0} \gamma_{00}(b) - \frac{b^3}{\beta_0^2} \gamma_{11}(b) \right] + \mathcal{O} \left( \frac{1}{\beta_0^3} \right),$$

$$\gamma_1(b) = 3 \left( 4 \zeta_3 - \frac{17}{4} + \frac{36 \zeta_3 - 103}{9} \right) b$$

$$+ \left( 24 \zeta_3 - \frac{3}{5} \pi^4 + \frac{59}{2} \zeta_3 + \frac{14579}{864} \right) b^2$$

$$+ \left( -48 \zeta_3^2 - \frac{2}{63} \pi^6 + 72 \zeta_3 - \frac{44}{75} \pi^4 + \frac{3229}{45} \zeta_3 - \frac{5191}{540} \right) b^3$$

$$+ \left( 36 \zeta_3^7 + \frac{8}{5} \pi^4 \zeta_3 - 144 \zeta_3^2 - \frac{2}{21} \pi^6 + 107 \zeta_3 - \frac{946}{675} \pi^4 + \frac{9601}{180} \zeta_3 + \frac{22859}{8640} \right) b^4$$

$$+ \left( -240 \zeta_3 \zeta_3^5 - \frac{4}{225} \pi^8 + 108 \zeta_3 + \frac{24}{5} \pi^4 \zeta_3 - \frac{664}{7} \zeta_3^2 - \frac{272}{1323} \pi^6 \right.$$

$$\left. + \frac{18574}{63} \zeta_3 - \frac{119}{135} \pi^4 - \frac{6263}{63} \zeta_3 + \frac{16103}{1296} \right) b^5 + \cdots$$

The first term here coincides with the $C_\gamma^2 T_F n_f$ term in the three-loop result obtained by a direct calculation [13, 14]. The last term contains the same combination of $F_{nm}$ with $n + m = 6$, so that $\zeta_{5, 3}$ cancels.

The static potential at the NL$\beta_0$ level is

$$V(\vec{q}) = -\frac{(4\pi^2)}{\bar{q}^2} \sum_{L=1}^{\infty} g(\epsilon, L\epsilon) \left( \frac{b}{\epsilon + \bar{b}} \right)^L \left[ 1 + L \frac{Z_{a_1}}{\beta_0} + \frac{3 \epsilon}{\beta_0} \sum_{L'=2}^{L-1} \frac{L - L'}{L'} F(\epsilon, L'\epsilon) \right] + \mathcal{O} \left( \frac{1}{\beta_0^3} \right)$$

$$= -\frac{(4\pi^2)}{\bar{q}^2} \left[ \frac{b}{\beta_0} V_0(b) - \frac{b^3}{\beta_0^3} V_1(b) \right] + \mathcal{O} \left( \frac{1}{\beta_0^3} \right)$$

(4.15)
where

\[
V_1(b) = -\frac{3}{2} [F_{10} + 2F_{01} + 2g_{01}] + \frac{1}{2} [F_{20} - 6F_{02} - 6(F_{10} + 3F_{01})g_{01} - 30g_{02}] b \\
- \frac{1}{4} [F_{30} + 24F_{03} - 4(F_{20} + 12F_{02})g_{01} + 36(F_{10} + 4F_{01})g_{02} + 312g_{03}] b^2 + \cdots
\]

contains only the same coefficients \(g_{0n}\) (3.11) as the \(L_2\) result, and only \(F_{n0}\) and \(F_{0m}\) are involved (see (4.4–4.6)). We obtain

\[
V_1(b) = 12\zeta_3 - \frac{55}{4} + \left(78\zeta_5 - \frac{7001}{72}\right) b + \left(60\zeta_5 + \frac{723}{2} \zeta_3 - \frac{147851}{288}\right) b^2
\]

\[
+ \left(770\zeta_5 + \frac{\pi^4}{200} + \frac{276901}{180} \zeta_3 - \frac{70418923}{25920}\right) b^3
\]

\[
+ \left(1134\zeta_7 + \frac{32297}{5} \zeta_5 + \frac{41}{1800} \pi^4 + \frac{402479}{60} \zeta_3 - \frac{1249510621}{77760}\right) b^4
\]

\[
+ \left(21735\zeta_7 + \frac{5\pi^6}{1323} + \frac{5911849}{126} \zeta_5 + \frac{41}{720} \pi^4 + \frac{48558187}{1512} \zeta_3 - \frac{46318004889}{93312}\right) b^5 + \cdots
\]

(4.16)

Thus we have reproduced the \(C_F(T_F n_f)^2\alpha_s^3\) and \(C_F^2(T_F n_f)^3\alpha_s^3\) terms in the two-loop potential [17], as well as the \(C_F(T_F n_f)^3\alpha_s^4\) and \(C_F^2(T_F n_f)^3\alpha_s^4\) terms in the three-loop one [18]. This expansion can be extended to any order; it contains only \(\zeta_n\) because only \(F_{n0}\) and \(F_{0m}\) are present. Note the pattern of the highest weights in (4.16): 3, 3, 5, 5, 7, 7, whereas one would expect 3, 4, 5, 6, 7, 8, as in (4.13), (4.14). The conformal anomaly (2.6) at the NL\(\beta_0\) order is

\[
\Delta = 4\pi \left[ \frac{b^3}{\beta_0} \delta_0(b) - \frac{b^4}{\beta_0^2} \delta_1(b) \right] + \mathcal{O} \left( \frac{1}{\beta_0^3} \right),
\]

\[
\delta_0(b) = \frac{\pi^4}{5} + 38\zeta_3 - \frac{645}{8} + \left(36\zeta_5 + \frac{2}{3} \pi^4 + \frac{968}{3} \zeta_3 - \frac{114691}{216}\right) b
\]

\[
+ \left(48\zeta_3 + \frac{2}{63} \pi^6 + 690\zeta_5 + \frac{269}{360} \pi^4 + \frac{52577}{36} \zeta_3 - \frac{14062811}{5184}\right) b^2
\]

\[
+ \left(1098\zeta_7 - \frac{8}{5} \pi^4 \zeta_3 + 160\pi^6 + \frac{20}{189} \zeta_5 + \frac{95006}{15} \zeta_3 + \frac{2801}{1800} \pi^4 + \frac{198917}{30} \zeta_3 - \frac{39035933}{2430}\right) b^3
\]

\[
+ \left(240\zeta_3 + \frac{1}{225} \pi^6 + 21615\zeta_7 - \frac{16}{3} \pi^4 \zeta_3 + \frac{397}{3} \zeta_5 + \frac{131}{567} \pi^6
\]

\[
+ \frac{838699}{18} \zeta_5 + \frac{14959}{10800} \pi^4 + \frac{34793081}{1080} \zeta_3 - \frac{51287121209}{466560}\right) b^4 + \cdots
\]

(4.17)

The \(b^3/\beta_0^2\) term has canceled, so that the coefficient of \(C_F\) in the bracket in (2.7) is 0.

5. Conclusion

The terms with the highest powers of \(n_f\) at each order of perturbation theory \((C_F(T_F n_f)^{l-1}\alpha_s^l)\) in \(\Gamma, g_0, g_F = (T_F n_f)^l\alpha_s^{l+1}\) in \(V(q)\) are known, and given by explicit formulas (3.8), (3.10), (3.12). The terms with the next to highest power of \(n_f\) can have abelian \((C_F^2)\) or non-abelian \((C_F C_A)\) color
structure. The abelian terms $(C_2^L(T_F n_f)^{L-2} \alpha_s^L (L \geq 3) \text{ in } \Gamma)$, $(C_2^L(T_F n_f)^{L-1} \alpha_s^{L+1} (L \geq 2) \text{ in } V(\vec{q}))$ are also known to all orders in $\alpha_s$, but only as algorithms which allow one to obtain (in principle) any number of terms, see (4.13), (4.14), (4.16). The simple method used here is not applicable to non-abelian terms.

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References


