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β-functions in higher dimensional field theories

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We review recent activity in the construction of the renormalization group functions for $O(N)$ scalar and gauge theories in six and higher dimensions. The theories lie in their respective universality classes at the Wilson-Fisher fixed point. The critical exponents at this fixed point in the various dimensions are all in agreement with the known exponents determined in the large N expansion.

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1. Introduction

Dimensional regularization is the canonical method of controlling the divergences when renormalizing a quantum field theory in perturbation theory. The theory is extended from the critical dimension D to d where $d = D - 2\varepsilon$ and ε is the regularizing parameter. The poles in ε are then absorbed in renormalization constants prior to the construction of the renormalization group functions. The process is complete when these functions are determined in $d = D$ dimensions. What is less apparent in the procedure is the connectivity different theories, with the same underlying symmetry, have with each other in different dimensions. This connection derives from the critical point renormalization group equation and the Wilson-Fisher (WF) fixed point which is defined as the non-trivial zero of the β-function in d-dimensions. This connectivity has seen a recent revival in analysing conformal field theories beyond two dimensions as well as related interest in the a-theorem, ultraviolet-infrared duality across dimensions and conformal windows. Here we review the renormalization group functions of scalar and gauge theories in dimensions higher than four in their respective universality classes.

2. Scalar theories

We begin by reprising the picture of a tower of theories in d-dimensions all lieing in the same universality class at the WF fixed point by considering scalar theories with an $O(N)$ symmetry. The base theory is in two dimensions which is the $O(N)$ nonlinear σ model with the Lagrangian

$$L^{\sigma} \;=\; \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 \;+\; \frac{1}{2}\sigma\left(\phi^i\phi^i - \frac{1}{\lambda}\right) \tag{2.1}$$

where λ is the coupling constant and σ is a Lagrange multiplier field. Throughout the set of theories in the same universality class the two fields have dimensions $[\phi^i] = d/2-1$ and $[\sigma] = 2$ in d-dimensions. While (2.1) is non-renormalizable perturbatively it is renormalizable in the $1/N$ expansion which is a dimensionless coupling parameter in d-dimensions. This point of view of the renormalizability is the key to understanding the dimensional connectivity. The common feature of theories in the same universality class is the core interaction such as that in (2.1). For instance, relative to four dimensions (2.1) is critically equivalent to $O(N)$ ϕ^4 theory which can be formulated in two ways since

$$L^{(4)} \;=\; \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 \;-\; \frac{g}{8}\left(\phi^i\phi^i\right)^2 \;=\; \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 \;+\; \frac{1}{2}\sigma\phi^i\phi^i - \frac{\sigma^2}{2g} \;. \tag{2.2}$$

In the latter Lagrangian σ is an auxiliary field and appears quadratically rather than linearly to ensure perturbative renormalizability in the critical dimension. These additional σ dependent parts of a Lagrangian over and above the core interaction are termed spectators since they are only present in specific dimensions. Once (2.1) and (2.2) are viewed in this way the method to construct the Lagrangians in the universal tower is evident and requires a core interaction and renormalizability. The six dimensional extension, $L^{(6)}$, was given in [1, 2, 3, 4, 5] and extended to eight in [6] and their respective Lagrangians are

$$L^{(6)} \;=\; \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 \;+\; \frac{1}{2}\left(\partial_\mu \sigma\right)^2 \;+\; \frac{g_1}{2}\sigma\phi^i\phi^i \;+\; \frac{g_2}{6}\sigma^3$$

$$L^{(8)} = \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 + \frac{1}{2}(\Box\sigma)^2 + \frac{1}{2}g_1\sigma\phi^i\phi^i + \frac{1}{6}g_2\sigma^2\Box\sigma + \frac{1}{24}g_3^2\sigma^4 \tag{2.3}$$

where the number of independent couplings increases with dimension. For all of these theories the critical exponents which are derived from the renormalization group functions have been computed in the large N expansion, [7, 8, 9, 10]. For instance, the first three terms of the exponents

$$\eta = \gamma_\phi(g_c) \quad , \quad \omega = \beta'(g_c) \tag{2.4}$$

are known, [7, 8, 9, 10], as functions of d where $\gamma_\phi(g)$ is the wave function anomalous dimension and g_c represents the vector of critical couplings. It is worth noting that the derivation of large N exponents uses analytic regularization of the Feynman integrals and not dimensional regularization which is effectively the reason why one can study field theories in d-dimensions. Moreover, the large N method is applicable to non-abelian gauge theories but N_f, the number of quark flavours, is used as the expansion parameter rather than the number of colours.

Recent activity for $L^{(6)}$ has involved the extension of the three loop results of [11, 12] to the $O(N)$ case, [3, 5], and then to four loops, [13]. For the latter computation one can derive all the basic renormalization group functions from evaluating a 2-point function. This is because in six dimensions a $1/(k^2)^2$ propagator is infrared safe unlike in four dimensions. Therefore to extract the coupling constant renormalization one can nullify one external leg momentum on the vertex function relegating it effectively to a 2-point function evaluation. Such a nullification can be accommodated within the usual renormalization of the field 2-point function, [13]. This approach substantially reduces the number of Feynman graphs to be evaluated. However, to achieve this we have used the Laporta algorithm, [14], and specifically the REDUZE encoding of it, [15, 16], to construct all the integration by parts relations between the required integrals. The final step requires the substitution of the basic master integrals. As the 2-point four loop master integrals are known in four dimensions, [17], we can determine the corresponding ones in six dimensions by applying Tarasov's method, [18, 19]. We use FORM, [20], throughout to handle the underlying algebra. The main results are provided in [13] but we note that the renormalization group functions for the $O(1)$ version of $L^{(6)}$ are, [11, 12, 13],

$$\begin{aligned}
\beta(g) &= \frac{3}{8}g^3 - \frac{125}{288}g^5 + 5[2592\zeta_3 + 6617]\frac{g^7}{41472} \\
&\quad + [-4225824\zeta_3 + 349920\zeta_4 + 1244160\zeta_5 - 3404365]\frac{g^9}{1492992} \\
\gamma_\phi(g) &= -\frac{1}{12}g^2 + \frac{13}{432}g^4 + [2592\zeta_3 - 5195]\frac{g^6}{62208} \\
&\quad + [10080\zeta_3 + 18144\zeta_4 - 69120\zeta_5 + 53449]\frac{g^8}{248832}
\end{aligned} \tag{2.5}$$

where ζ_n is the Riemann zeta-function. Finally, all the renormalization group functions evaluated for the $O(N)$ versions of (2.3), [2, 3, 6, 13], are in full agreement with the known large N exponents which supports the Wilson's vision of a tower of theories in d-dimensions within a universal underlying theory.

3. Gauge theories

We can also extend the process to non-supersymmetric gauge theories. First, we focus on six-dimensional QCD. The gauge fixed Lagrangian in a linear covariant gauge is, [6],

$$L^{(6)}_{\rm GI} = -\frac{1}{4}\left(D_\mu G^a_{\nu\sigma}\right)\left(D^\mu G^{a\,\nu\sigma}\right) + \frac{g_2}{6} f^{abc} G^a_{\mu\nu} G^{b\,\mu\sigma} G^{c\,\nu}{}_\sigma + i\bar{\psi}^{iI} \not{D} \psi^{iI} - \frac{1}{2\alpha}\left(\partial_\mu \partial^\nu A^a_\nu\right)\left(\partial^\mu \partial^\sigma A^a_\sigma\right) - \bar{c}^a \Box \left(\partial^\mu D_\mu c\right)^a \tag{3.1}$$

where α is the gauge parameter. In six dimensions there are two independent gluonic operators, [21], as a consequence of the Bianchi identities and we have chosen to use a 2-leg and a 3-leg operator. The latter is important in effective field theories in four dimensions. Though in six dimensions a 4-fermi operator has dimension 10 and hence is absent in (3.1). As a consequence the gluon and ghost propagators have double poles in the squared momentum unlike in four dimensions. Structurally (3.1) is similar to the eight dimensional $O(N)$ scalar theory, [6]. However, as an aside the propagators of (3.1) would lead to a confining inter-quark potential in *four* dimensions. Indeed there is a Schwinger-Dyson solution of four dimensional QCD which constructs the effective infrared Lagrangian, [22], and takes the precise form (3.1) with g_2 massive. Continuing this theme (3.1) can be extended to include lower dimension operators with

$$L^{(6)}_m = L^{(6)} + m_1 \bar{\psi}^{iI} \psi^{iI} - \frac{1}{4} m_2^2 G^a_{\mu\nu} G^{a\,\mu\nu} - \frac{1}{2\alpha} m_3^2 (\partial^\mu A^a_\mu)^2 - m_3^2 \bar{c}^a \left(\partial^\mu D_\mu c\right)^a - \frac{1}{2} m_4^4 A^a_\mu A^{a\,\mu} + m_4^4 \alpha \bar{c}^a c^a \,. \tag{3.2}$$

Here m_i are masses to ensure each term is dimension six. Interestingly the Landau gauge propagators derived from (3.2) are formally the same as those which are used to model the infrared structure of the propagators on the lattice in four dimensions, [23]. For instance,

$$\begin{aligned} \langle A^a_\mu(p) A^b_\nu(-p) \rangle \big|_{\alpha=0} &= -\frac{\delta^{ab}}{[(p^2)^2 + m_2^2 p^2 + m_4^4]} \left[\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] \\ \langle c^a(p) \bar{c}^b(-p) \rangle \big|_{\alpha=0} &= -\frac{\delta^{ab}}{p^2[p^2 + m_3^2]} \,. \end{aligned} \tag{3.3}$$

Returning to the massless Lagrangian we have calculated the wave function anomalous dimensions and the β-functions to two loops for non-zero α. For the latter we evaluated the three 3-point vertices at the fully symmetric point with non-zero momenta flowing through each external leg. Unlike in a renormalization of four dimensional QCD nullifying an external leg momentum is not infrared safe in six dimensions as such an operation would produce propagators of the form $1/(k^2)^4$ where k is a loop momentum. By renormalizing these three vertices and obtaining the same $\overline{\rm MS}$ results for the renormalization of g_1 is a non-trivial check on our computation. Like the scalar theory analysis we have again used the Laporta algorithm, [14], to obtain our renormalization group functions. For instance, the two $\overline{\rm MS}$ β-functions are, [6],

$$\begin{aligned} \beta_1(g_1,g_2) &= \left[-249 C_A - 16 N_f T_F\right] \frac{g_1^3}{120} \\ &\quad + \left[-50682 C_A^2 g_1^3 + 2439 C_A^2 g_1^2 g_2 + 3129 C_A^2 g_1 g_2^2 - 315 C_A^2 g_2^3 - 1328 C_A N_f T_F g_1^3 \right. \end{aligned}$$

$$- 624C_A N_f T_F g_1^2 g_2 + 96 C_A N_f T_F g_1 g_2^2 - 3040 C_F N_f T_F g_1^3 \big] \frac{g_1^2}{4320}$$
$$\beta_2(g_1,g_2) = \big[81 C_A g_1^3 - 552 C_A g_1^2 g_2 + 135 C_A g_1 g_2^2 - 15 C_A g_2^3 + 104 N_f T_F g_1^3 - 48 N_f T_F g_1^2 g_2 \big] \frac{1}{120} \tag{3.4}$$

where for space reasons we only note the one loop term of $\beta_2(g_1,g_2)$. The full expression is given in [6]. We note that the gauge coupling constant is asymptotically free for all (positive) values of N_f. A final check on the calculation of the renormalization group functions is if the critical exponents derived from them at the WF fixed point agree with the large N_f exponents computed using Vasiliev's method. In this instance the base theory in two dimensions which serves to seed this fixed point universal theory is the non-abelian Thirring model (NATM) which can be written in two ways as

$$L^{\mathrm{NATM}} = i \bar{\psi}^i \partial\!\!\!/ \psi^i + \frac{\tilde{g}}{2} \left(\bar{\psi}^i T^a \gamma^\mu \psi^i \right)^2 = i \bar{\psi}^i D\!\!\!/ \psi^i - \frac{1}{2} A^a_\mu A^{a\mu} \tag{3.5}$$

where in strictly two dimensions $A^a_\mu \propto \bar{\psi}^i T^a \gamma_\mu \psi^i$ is an auxiliary field. Various large N_f critical exponents are known, [24, 25, 26], and expanding these in an ε-expansion relative to six dimensions the two loop renormalization group functions are in total agreement. That this happens for (3.1) indicates that it is correct which is not as remarkable as the observation of [27] that (3.5) does not contain gluonic vertices. The large N_f d-dimensional critical exponents derived from (3.5) contain information on the contribution from the triple, quartic and *quintic* vertices of the higher dimensional gauge theories.

Evidence for the connection of the field theories across the dimensions is not just restricted to the basic renormalization group functions. We have also computed the two loop $\overline{\mathrm{MS}}$ corrections to the anomalous dimensions, $\gamma_{(n)}(g_1,g_2)$, of the flavour non-singlet twist-2 Wilson operators $\bar{\psi} \gamma^{\mu_1} D^{\mu_2} \ldots D^{\mu_n} \psi$ using (3.1). For the lowest moments we find

$$\begin{aligned}
\gamma_{(2)}(g_1,g_2) &= 2 C_F g_1^2 \\
&\quad + [26841 C_A g_1^2 - 1200 C_A g_1 g_2 - 600 C_A g_2^2 - 4200 C_F g_1^2 + 1264 g_1^2 N_f T_F] \frac{C_F g_1^2}{1800} \\
\gamma_{(3)}(g_1,g_2) &= \frac{49}{15} C_F g_1^2 \\
&\quad + [186321 C_A g_1^2 - 10950 C_A g_1 g_2 - 4200 C_A g_2^2 - 23564 C_F g_1^2 + 9104 g_1^2 N_f T_F] \frac{7 C_F g_1^2}{54000} \; .
\end{aligned} \tag{3.6}$$

Again evaluating these at the WF fixed point to determine the critical exponent they are in exact agreement with the leading order large N_f exponent of [28].

One application of (3.4) is that the location of the conformal window can be determined in an ε expansion with $d = 6 - 2\varepsilon$. In four dimensional $SU(3)$ QCD the window is at $9 \leq N_f \leq 16$, [29]. By numerically solving

$$\beta_1(g_1,g_2) = \beta_2(g_1,g_2) = 0 \; , \; \frac{\partial \beta_1}{\partial g_1} \frac{\partial \beta_2}{\partial g_2} - \frac{\partial \beta_1}{\partial g_2} \frac{\partial \beta_2}{\partial g_1} = 0 \tag{3.7}$$

we find one real solution labelled by A which is, [6],

$$N_{f\,(A))} \;=\; 2.797566\frac{C_A}{T_F} \;+\; [2.198165C_F - 3.432003C_A]\,\frac{\varepsilon}{T_F} \;+\; O(\varepsilon^2) \tag{3.8}$$

where we retain the four dimensional trace convention. For $SU(3)$ we have

$$N_{f\,(A)}\big|_{SU(3)} \;=\; 16.785398 \;-\; 14.730246\varepsilon \;+\; O(\varepsilon^2)\;. \tag{3.9}$$

Unlike the situation with $O(N)$ scalar theories the boundary in strictly six dimensions is the same as in four dimensions.

Further insight into the structure of higher dimensional gauge theories can be gained by focusing on the extension of Quantum Electrodynamics (QED) to six and *eight* dimensions. This is because one can compute to higher order than in the non-abelian case especially in eight dimensions where there are a substantial number of 4-gluon operators. The respective Lagrangians are, [30, 6],

$$\begin{aligned}
L^{(6)}\Big|_{U(1)} &= -\frac{1}{4}\left(\partial_\mu F_{\nu\sigma}\right)\left(\partial^\mu F^{\nu\sigma}\right) \;-\; \frac{1}{2\alpha}\left(\partial_\mu\partial^\nu A_\nu\right)\left(\partial^\mu\partial^\sigma A_\sigma\right) \;+\; i\bar{\psi}^i \not{D}\psi^i \\
L^{(8)}\Big|_{U(1)} &= -\frac{1}{4}\left(\partial_\mu\partial_\nu F_{\sigma\rho}\right)\left(\partial^\mu\partial^\nu F^{\sigma\rho}\right) \;-\; \frac{1}{2\alpha}\left(\partial_\mu\partial^\nu A_\nu\right)\left(\partial^\mu\partial^\sigma A_\sigma\right) \\
&\quad + i\bar{\psi}^i \not{D}\psi^i \;+\; \frac{g_2^2}{32}F_{\mu\nu}F^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} \;+\; \frac{g_3^2}{8}F_{\mu\nu}F^{\mu\sigma}F_{\nu\rho}F^{\sigma\rho}
\end{aligned} \tag{3.10}$$

where we have extended the gauge fixing sector to eight dimensions for a linear covariant gauge. The renormalization group functions of each theory are known to various loop orders, [30, 6], including the electron mass anomalous dimensions. In all cases the critical exponents derived from these are in full agreement with the corresponding critical exponents to whatever order they are known in large N_f, [31, 24, 26]. This again substantiates the picture of a tower of theories connected via the WF fixed point. Several general features emerge in the QED analysis. First, we have verified the result of [30] that six dimensional QED is asymptotically free. By contrast the eight dimensional theory is like its four dimensional partner. Under the assumption that there are no triple photon vertices in any QED formulation in higher dimensions then D-dimensional QED is asymptotically free in $D = 2 + 4r$ dimensions for integer $r \geq 1$. For both Lagrangians (3.10) we have explicitly checked that the Ward-Takahashi identity holds. In addition a novel feature emerges for the electron anomalous dimension in a linear covariant gauge. In four dimensions the gauge parameter appears only in the one loop term, [32, 33], and is absent thereafter in any explicit evaluation. The six and eight dimensional theories share the same property to three and two loops respectively. Therefore, if a rigorous proof of this observation emerges it should be applicable to all dimensions.

4. Discussion

We have reviewed recent work in six and higher dimensional scalar and gauge field theories and provided solid evidence for the picture of a tower of connected theories in d-dimensions accessed by the WF fixed point. This extends from strictly two dimensions where conformal field

theories give the foundation for the connection. Results in gauge theories show a similar vision but with the observation that structurally eight dimensional scalar theory is similar to six dimensional QCD. From the higher dimensional quantum field theory side the next stage is to extend loop computations to higher order to refine the fixed point structure as will as to gain more insight into the operators which drive any infrared fixed points in QCD in the context of the underlying universal theory. Form the computational point of view one question is whether there is a deeper connection of the Tarasov construction of relating d- and $(d+2)$-dimensional Feynman integrals with the underlying field theories. In other words is there a way of proceeding more fundamentally via a path integral construction without having to make the connection at the renormalization group function level?

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