# 30 years, some 700 integrals, and 1 dessert

or:

# Electroweak two-loop corrections to the $Z\bar{b}b$ vertex

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The one-loop corrections to the weak mixing angle $\sin^2\theta_{\text{eff}}^{\text{b}}$, derived from the $Z\bar{b}b$ vertex, are known since 1985. It took another 30 years to calculate the complete electroweak two-loop corrections to $\sin^2\theta_{\text{eff}}^{\text{b}}$. The main obstacle was the calculation of the $\mathscr{O}(700)$ bosonic two-loop vertex integrals with up to three mass scales, at $s = M_Z^2$. We did not perform the usual integral reduction and master evaluation, but chose a completely numerical approach, using two different calculational chains. One method relies on publicly available sector decomposition implementations. Further, we derived Mellin-Barnes (MB) representations, exploring the publicly available MB suite. We had to supplement the MB suite by two new packages: AMBRE 3, a Mathematica program, for the efficient treatment of non-planar integrals and MBnumerics for advanced numerics in the Minkowskian space-time. Our preliminary result for LL2016, the "dessert", for the electroweak bosonic two-loop contributions to $\sin^2\theta_{\text{eff}}^{\text{b}}$ is:

$\Delta\sin^2\theta_{\text{eff}}^{\text{b}(\alpha^2,\text{bos})} = \sin^2\theta_W\,\Delta\kappa_{\text{b}}^{(\alpha^2,\text{bos})}$, with $\Delta\kappa_{\text{b}}^{(\alpha^2,\text{bos})} = -1.0276 \times 10^{-4}$.

This contribution is about a quarter of the corresponding fermionic corrections and of about the same magnitude as several of the known higher-order QCD corrections. The $\sin^2\theta_{\text{eff}}^{\text{b}}$ is now predicited in the Standard Model with a relative error of $10^{-4}$ [1].

*Loops and Legs in Quantum Field Theory - LL 2016,*
*24 - 29 April 2016*
*Leipzig, Germany*

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*In Auerbach's Keller, where Faust met Mephisto: T. Riemann, J. Usovitsch, I. Dubovyk, J. Gluza*

## Preface

> *Der Wahnsinn*
> *ist nur eine schmale Brücke*
> *die Ufer sind Vernunft und Trieb*
>
> Rammstein, [http://www.magistrix.de/lyrics/Rammstein/Du-Riechst-So-Gut-26274.html](http://www.magistrix.de/lyrics/Rammstein/Du-Riechst-So-Gut-26274.html) [1]

LL2016 in Leipzig was the $13^{th}$ edition, and it was my last Loops and Legs conference as an organizer, and perhaps also as a participant. I founded it, together with Johannes Blümlein and Martina Mende, in April 1992 as a bi-annual event; it was a follow-up of the 1989 November changes in Germany. The aim was an overview of the recent developments in perturbative quantum field theory with focus on applications to precision collider experiments. We attempted a mix of all the relevant research directions, but also a mix of both younger and more experienced collegues. From time to time we had to invent some modernizations like parallel sessions or online proceedings. In the early years the focus was more at direct phenomenological applications, and now it is more on advanced technical developments. The steady high scientific level has been guaranteed by the participants, and the discussions were top-level, lively and sometimes even hot.

My own research reflects the above observations. Since 1977, I am engaged in complete electroweak radiative corrections for collider physics. A statement by a head of an institute: "For me you are one who calculates integrals." My most successful project is ZFITTER (with Dima Bardin et al., see [http://sanc.jinr.ru/users/zfitter](http://sanc.jinr.ru/users/zfitter)). ZFITTER became the standard software for the study of the Z boson resonance at LEP and elsewhere, and it was used to predict the masses of the top quark and the Higgs boson prior to their discoveries. Many PhD students used it. Recently, ZFITTER even got illegally copied by non-experts in order to promote their carriers. In 1985, we calculated the one-loop corrections to $Z \to \bar{b}b$. The corresponding FORTRAN code ZRATE, later ZWRATE and ROKANC, became the center of the Standard Model library of ZFITTER. 30 years later, at this year's Loops and Legs conference, I presented an electroweak Standard Model two-loop calculation for $Z \to \bar{b}b$, performed together with my coauthors. This project started in 2012 at a meeting at the Max-Planck Institute in Munich, where I met Ayres Freitas. At that meeting the second-oldest speaker was about 20 years younger than me . . . Our numerical fitting formula for the $Z\bar{b}b$ weak mixing angle will, presumably, get included into software packages like ZFITTER.

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[1] *Insanity – is just a narrow bridge – the shores are reason and urge*, Rammstein, [http://lyricstranslate.com/de/du-riechst-so-gut-you-smell-so-good.html](http://lyricstranslate.com/de/du-riechst-so-gut-you-smell-so-good.html) (22.8.2016)

## 1. Introduction

The study of the $Z$ boson resonance in $e^+e^-$ annihilation,

$$e^+e^- \to (\gamma, Z) \to f^+f^- \ (+ n\gamma), \tag{1.1}$$

has been performed with high precision at LEP and is planned with better precision at future $e^+e^-$ colliders. Correspondingly, the theoretical predictions in the Standard Model are needed with 2-loop accuracy in the weak sector, and even better for QED and QCD. Usually, the theoretical analysis is based not on cross sections as measured in reaction (1.1), including non-observed additional photons (and gluons), but on so-called pseudo-observables, corresponding to

$$e^+e^- \to (\gamma, Z) \to f^+f^-, \tag{1.2}$$

or even to the simpler reaction

$$Z \to f^+f^- \ (+ n\gamma). \tag{1.3}$$

This is quite similar to the analysis of LHC events, which one tries to focus on the underlying hard $2 \to 2$ process. As a result, the analysis of observables rests on two relatively independent steps:

- Unfolding the observed cross sections and representing them as $2 \to 2$ (pseudo-) observables; the theoretical frame has to be sufficiently general in order not to bias step 2.

- Confronting the pseudo observables with specific theory predictions.

A key element of the theoretical analysis is the $Z\bar{b}b$ vertex, whose prediction at two electroweak loops is the subject of our study. Two pseudo-observables are related to this vertex:

- The partial decay width $\Gamma(Z \to \bar{b}b)$.

- The decay asymmetry, related to the parameter $A_b$; we give its definition below in (1.9).

The partial decay width $\Gamma(Z \to \bar{b}b)$ is related to the peak cross-section $\sigma_T$ of $e^+ e^- \to (\gamma, Z) \to b^+ b^- (+n\gamma)$, also to the pseudo-cross section of $e^+ e^- \to (\gamma, Z) \to b^+ b^-$, and $A_b$ is related to the angular asymmetry of the cross-sections.

In Born approximation everything looks relatively easy. Neglecting here photon exchange, and using a Breit-Wigner resonance $Z$ propagator with a priori mass $M_Z$ and width $\Gamma_Z$ of the $Z$ boson, one derives a qualitatively good description of the $Z$ line shape close to the peak:

$$
\frac{d\sigma^B}{d\cos\theta} \sim G_F^2 \left| \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z} \right|^2 \tag{1.4}
$$
$$
\times \left[ (a_e^{B\,2} + v_e^{B\,2})(a_b^{B\,2} + v_b^{B\,2})(1 + \cos^2\theta) + (2a_e^B v_e^B)(2a_b^B v_b^B)(2\cos\theta) \right].
$$

Symmetric or anti-symmetric integration over $\cos\theta$ allows to determine the two independent contributions. One of them is the total cross section,

$$
\sigma_T^B \equiv \int_{-1}^{1} d\cos\theta \, \frac{d\sigma^B}{d\cos\theta} \sim \left| \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z} \right|^2 G_F^2 \left( a_e^{B\,2} + v_e^{B\,2} \right) \left( a_b^{B\,2} + v_b^{B\,2} \right) \sim \Gamma_e^B \Gamma_b^B, \tag{1.5}
$$

and the other one the forward-backward asymmetry,

$$
\sigma_{FB}^B \equiv \left[ \int_0^1 - \int_{-1}^0 \right] d\cos\theta \, \frac{d\sigma^B}{d\cos\theta} \sim \left| \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z} \right|^2 G_F^2 (2a_e^B v_e^B)(2a_b^B v_b^B), \tag{1.6}
$$

$$
A_{FB}^B \equiv \frac{\sigma_{FB}^B}{\sigma_T^B} = \frac{3}{4} \frac{2a_e^B v_e^B}{a_e^{B\,2} + v_e^{B\,2}} \frac{2a_b^B v_b^B}{a_b^{B\,2} + v_b^{B\,2}} \equiv \frac{3}{4} A_e^B A_b^B. \tag{1.7}
$$

We observe the factorization of $\sigma_T^B$ into the product of two partial widths,

$$
\Gamma_f^B = \frac{G_F M_Z^3}{\sqrt{2}\,6\pi} c_f \left( a_f^{B\,2} + v_f^{B\,2} \right), \tag{1.8}
$$

and of $A_{FB}^B$ into the product of two asymmetry functions,

$$
A_f^B = \frac{2a_f^B v_f^B}{a_f^{B\,2} + v_f^{B\,2}}. \tag{1.9}
$$

In Born approximation it is $a_f^B = \pm\frac{1}{2}$, $Q_e = -1$, $c_f$ the color factor, and

$$
\frac{v_f^B}{a_f^B} = 1 - 4|Q_f| \sin^2\theta_W. \tag{1.10}
$$

The vector and axial vector couplings will get loop corrections, which may be calculated from the vertex diagrams $\mathbf{V}_\mu^{Zf\bar{f}}(k^2)$; for the $Z\bar{b}b$-vertex:

$$
g_V^b(k^2)|_{k^2 = M_Z^2} = \frac{1}{2(2-D)k^2} \mathrm{Tr}[\gamma^\mu \not{p}_1 \mathbf{V}_\mu^{Zb\bar{b}} \not{p}_2]|_{k^2 = M_Z^2}, \tag{1.11}
$$

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$$g_A^b(k^2)_{k^2=M_Z^2} = \frac{1}{2(2-D)k^2} \text{Tr}[\gamma_5 \gamma^\mu \not{p}_1 \mathbf{V}_\mu^{Zb\bar{b}} \not{p}_2]_{k^2=M_Z^2}. \tag{1.12}$$

Here, we relate vertex corrections to effective couplings. In reality, realistic cross sections are measured, and one has to relate their couplings to $g_V^b(k^2)$ and $g_A^b(k^2)$.

Fitting programs like Gfitter are relating "experimental" values of e.g. $\Gamma_b$, $A_b$ with their theoretical predictions, e.g. in the standard model [2]. But does this fit the original pseudo-observables? To some approximation, it does, as can be seen in the Born formulae. But one has to control the quantum corrections safely. We know since long how to relate pseudo-observables in a strict way to the loop corrections [3, 4, 5, 6, 7]. The amplitude for $e^+e^-$ annihilation into two massless fermions, and we assume here the final state to be massless, may be described to all orders of perturbation theory by four complex-valued form factors, which depend on the masses and the invariants $s$ and $t$, and which are chosen here to be $\rho_{ef}, \kappa_e, \kappa_f, \kappa_{ef}$; we quote from [6], eq. (3.3.1):[2]

$$\begin{aligned}
\mathscr{A}_Z^{eff}(s,t) \sim\ & i e^2 4 I_e^{(3)} I_f^{(3)} \frac{\chi_Z(s)}{s} \rho_{ef}(s,t) \Big\{ \gamma_\mu (1+\gamma_5) \otimes \gamma_\mu (1+\gamma_5) \\
& -4|Q_e| s_w^2 \kappa_e(s,t) \gamma_\mu \otimes \gamma_\mu (1+\gamma_5) - 4|Q_f| s_w^2 \kappa_f(s,t) \gamma_\mu (1+\gamma_5) \otimes \gamma_\mu \\
& +16|Q_e Q_f| s_w^4 \kappa_{ef}(s,t) \gamma_\mu \otimes \gamma_\mu \Big\}.
\end{aligned} \tag{1.13}$$

We use the definitions

$$\chi_Z(s) = \frac{G_F M_Z^2}{\sqrt{2}\, 2\pi\alpha} \rho_Z(s), \tag{1.14}$$

$$\rho_Z(s) = \frac{s}{s - M_Z^2 + i M_Z \Gamma_Z(s)}. \tag{1.15}$$

For the complete amplitude one sums over all relevant diagrams so that the form factors are perturbative series:

$$\rho_{ef} = 1 + \delta\rho_{ef} = 1 + \frac{\alpha}{\pi} \delta\rho_{ef}^{(1)} + \cdots, \tag{1.16}$$

$$\kappa_a = 1 + \delta\kappa_a = 1 + \frac{\alpha}{\pi} \delta\kappa_a^{(1)} + \cdots, \quad a = e, f, ef. \tag{1.17}$$

Compared to a "naive" notation, we split from the rest of the amplitude the form factor $\rho_{ef}$ multiplicatively. If a diagram is represented by an original set $\{\rho_{ef}, \bar{\kappa}_e, \bar{\kappa}_f, \bar{\kappa}_{ef}\}$, this yields re-definitions for all the $\kappa_a$:

$$\kappa_a = \frac{\bar{\kappa}_a}{\rho_{ef}}, \quad a = e, f, ef. \tag{1.18}$$

The form factors, if introduced as it is done here, may be used for definitions of an effective Fermi constant and three effective weak mixing angles:

$$G_F^{\text{eff}} = \rho_{ef}(s,t)\, G_F, \tag{1.19}$$

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[2]The left-projector is in that notations $L = (1+\gamma_5)/2$, while it is usually $L = (1-\gamma_5)/2$. We further stress that the notation covers all kinds of contributions, including also box diagrams.

$$\sin^2\theta_{W,e}^{\text{eff}} = \kappa_e(s,t)\ \sin^2\theta_W, \tag{1.20}$$

$$\sin^2\theta_{W,f}^{\text{eff}} = \kappa_f(s,t)\ \sin^2\theta_W, \tag{1.21}$$

$$\sin^2\theta_{W,ef}^{\text{eff}} = \sqrt{\kappa_{ef}(s,t)}\ \sin^2\theta_W, \tag{1.22}$$

where

$$\sin^2\theta_W \equiv 1 - \frac{M_W^2}{M_Z^2}. \tag{1.23}$$

The unique definition of an effective weak mixing angle is lost.

The Breit-Wigner propagator $\rho_Z(s)$ contains a width function which is predicted by perturbation theory. Its calculation deserves special attention, and for the moment the notation $\Gamma_Z(s)$ emphasizes that it originates from summing over self-energies like $\Sigma_Z(s)$.

The amplitude may be further rewritten, in order to introduce the familiar couplings $v_f, a_f$, which now will cover the loop corrections:

$$\mathscr{A}_Z^{eff}(s,t) \sim ie^2\frac{\chi_Z(s)}{s}a_e a_f \rho_{ef}(s,t)\Big\{\gamma_\mu(1+\gamma_5)\otimes\gamma_\mu(1+\gamma_5) \tag{1.24}$$
$$+ \frac{v_f}{a_f}\ \gamma_\mu\otimes\gamma_\mu(1+\gamma_5) + \frac{v_e}{a_e}\ \gamma_\mu(1+\gamma_5)\otimes\gamma_\mu + \frac{v_{ef}}{a_e a_f}\ \gamma_\mu\otimes\gamma_\mu\Big\}.$$

Here, we made the choice that the axial couplings remain Born like,

$$a_e = -\frac{1}{2}, \tag{1.25}$$

$$a_f = \pm\frac{1}{2}. \tag{1.26}$$

This choice means that the axial couplings remain to be real constants here, and that the (axial $\times$ axial) radiative corrections coming from a product of two vertices like (1.12) will be collected in the definition of $\rho_{ef}$,

$$\delta\rho_{ef}^{Z\bar{e}eZ\bar{b}b} = \frac{g_A^e(k^2)_{k^2=M_Z^2}}{a_e}\frac{g_A^f(k^2)_{k^2=M_Z^2}}{a_f}, \tag{1.27}$$

while the vector couplings are understood to contain radiative corrections. From a final state $Z\bar{b}b$ vertex loop correction, one then gets e.g.:

$$v_e = v_e^B, \tag{1.28}$$

$$v_b = v_e^B + \delta v_f^{Z\bar{b}b} = v_e^B + g_V^f(k^2)_{k^2=M_Z^2}, \tag{1.29}$$

$$v_{eb} = v_e^B\left(v_b^B + \delta v_{ef}^{Z\bar{b}b}\right) = v_e^B\left(v_b^B + \delta v_f^{Z\bar{b}b}\right). \tag{1.30}$$

From the above definitions we get three relations between the vector couplings and the form factors $\kappa$:

$$\frac{v_e}{a_e} = 1 - 4|Q_e|s_w^2\kappa_e, \tag{1.31}$$

$$\frac{v_f}{a_f} = 1 - 4|Q_f|s_w^2 \kappa_f, \tag{1.32}$$

$$\frac{v_{ef}}{a_e a_f} = \frac{v_e v_f}{a_e a_f} + \Delta_{ef}, \tag{1.33}$$

with

$$\Delta_{ef} = 16|Q_e Q_f|s_W^4 (\kappa_{ef} - \kappa_e \kappa_f). \tag{1.34}$$

If $\kappa_{ef} - \kappa_e \kappa_f = 0$, there is factorization. Factorization is broken by photonic corrections and by box diagrams, while it is respected by weak vertex corrections and self-energies.

Having defined the amplitude, one may calculate, with standard text book methods, a $2 \to 2$ cross section. For unpolarized scattering one gets [4]:

$$\frac{d\sigma^{eff}}{d\cos\theta} = \frac{\pi\alpha^2}{2s}|\chi_Z(s)|^2 \left[(1 + \cos^2\theta)k_T + 2\cos\theta k_{FB}\right]. \tag{1.35}$$

The symmetric part depends on

$$\begin{aligned} k_T &= |\rho_{ef}|^2 \left[|a_e a_f|^2 + |v_e a_f|^2 + |a_e v_f|^2 + |v_{ef}|^2\right] \\ &= |\rho_{ef}|^2 |a_e|^2 |a_f|^2 \left[(1 + |\frac{v_e}{a_e}|^2)(1 + |\frac{v_f}{a_f}|^2) + \Delta_T\right], \end{aligned} \tag{1.36}$$

with

$$\Delta_T = |\Delta_{ef}|^2 + 2\Re e\left(\frac{v_e}{a_e}\frac{v_f}{a_f}\Delta_{ef}^*\right). \tag{1.37}$$

Assuming factorization, this becomes

$$k_T = |\rho_{ef}|^2 \left[(|a_e|^2 + |v_e|^2)(|a_f|^2 + |v_f|^2)\right], \tag{1.38}$$

and finally, neglecting additionally the imaginary parts of $v_e$ and $v_f$ (and of $\Delta_T$):

$$k_T = |\rho_{ef}|^2 \left[(a_e^2 + v_e^2)(a_f^2 + v_f^2)\right]. \tag{1.39}$$

This is the formula usually applied to analyses.

Similarly, for the anti-symmetric cross section part:

$$\begin{aligned} k_{FB} &= |\rho_{ef}|^2 a_e a_f \Re e(v_e v_f^* + v_{ef}) \\ &= a_e a_f \left(2\Re e\left[\frac{v_e}{a_e}\right] 2\Re e\left[\frac{v_f}{a_f}\right] + \Delta_{FB}\right), \end{aligned} \tag{1.40}$$

with

$$\Delta_{FB} = 2\Re e \Delta_{ef}, \tag{1.41}$$

and after again neglecting non-factorizing terms and imaginary parts:

$$k_{FB} = 2|\rho_{ef}|^2 (2a_e v_e)(2a_f v_f). \tag{1.42}$$

The cross section formula (1.35) is the exact result from the amplitude square and averaging over the initial final helicity states.

Here, a technical remark is at the place: Already in Born approximation, the photon exchange leads to non-factorization. It is numerically not small and has to be taken into account. One may, formally, assume that the photon exchange Born amplitude is contained in the above-introduced four form factors. This was exemplified in [8]. Conventionally, one works with two interfering amplitudes as it is described in detail in the publications describing the ZFITTER project [9, 4, 5, 6, 7].

Under the assumption that the form factors are independent of the scattering angle, we get for the total cross section and the forward-backward asymmetry:

$$\sigma_T = \frac{4\pi\alpha^2}{3s}|\chi_Z|^2 k_T, \tag{1.43}$$

$$\sigma_{FB} = \frac{\pi\alpha^2}{s}|\chi_Z|^2 k_{FB}, \tag{1.44}$$

and the forward-backward asymmetry becomes

$$A_{FB}^{eff} = \frac{3}{4}\frac{k_{FB}}{k_T}. \tag{1.45}$$

If the form factors depend on the scattering angles, as it is the case for corrections from box diagrams, one has to study the numerical effect of that.

Further observables may be introduced for polarized scattering, where the amplitude (1.14) is taken between helicity projected states. This may be easily investigated following [4].

The loop-corrected asymmetry parameter $A_b$ as defined in (1.9) will be set in relation to loop-corrected pseudo-observables at $s = M_Z^2$, in terms of the angular integrals $\sigma_{FB}, \sigma_T$ as defined in (1.36) and (1.40):

$$A_{FB}^{\bar{b}b} = \frac{\sigma_{FB}}{\sigma_T} \tag{1.46}$$
$$= \frac{3}{4}\frac{\Re e[2a_e v_e\, 2a_b v_b + 4\sin^2\theta_W |Q_e Q_b|^2(\kappa_{eb} - \kappa_e\kappa_b)]}{|a_e a_b|^2 + |v_e a_b|^2 + |a_e v_b|^2 + |v_{eb}|^2} + \text{ corrections}$$
$$= \frac{3}{4}A_e A_b + \text{ corrections}.$$

The first "corrections" are due to neglected angular dependences of the form factors, and the second "corrections" are due to neglected non-factorizations and imaginary parts.

As discussed in detail in [10], as well as in earlier work [11, 12], the weak mixing angle $\sin^2\theta_{\text{eff}}^b$ and $A_b$ are determined from the residue $R$ of the leading part of the resonance matrix element $\bar{\mathscr{M}}$ (1.48). This residue may be determined in a very good and controlled approximation from the renormalized vector and axial vector couplings of the vertices $V_\mu^{Ze\bar{e}}$ and $V_\mu^{Zb\bar{b}}$. To do so, we have to understand how the form factors are composed. Besides the terms from $s$ channel $Z$ boson exchange, they contain terms from $s$ channel photon exchange, from box diagrams (with weak bosons, but also with photon exchanges), vertices, self-energies.[3] Some of these terms are

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[3]We remark here that not only self-energies and vertices, but also arbitrary box diagrams may be inserted exactly into the form factors [3, 4, 6].

enhanced by the resonance form of the transition, others are not. One has to understand some summation of terms which otherwise would explode at $s = M_Z^2$.

We will now discuss shortly the consequences of the fact that we are studying a resonance. Here arises the question, what is the correct and model-independent formulation of the amplitude from the point of view of general quantum field theory? In order to respect general principles – unitarity, analyticity, gauge invariance – one may use the pole scheme [11]. In the pole scheme, one makes the following ansatz for the amplitude $\bar{\mathscr{M}}$ as a function of the scattering energy, or as a function of the corresponding relativistic invariant $s$. In a sufficiently small neighborhood around the pole position, it is a Laurent expansion with position of the pole $s_0$ defined by the mass $m_Z$ and its width $g_Z$,

$$s_0 = m_Z^2 - i m_Z g_Z, \tag{1.47}$$

and the residue $R$, plus a background term $B$. The latter is a Taylor expansion:

$$\bar{\mathscr{M}} \sim \frac{R}{s - m_Z^2 + i m_Z g_Z} + \sum_{n=0}^{\infty} \frac{b_n}{s_0} \left(1 - \frac{s}{s_0}\right)^n. \tag{1.48}$$

Here, $\bar{\mathscr{M}}$ stands for the functional form of the amplitude introduced in (1.11). Within our formalism, one may write in full generality:

$$\rho_{ef} = \frac{R_r}{s - s_0} + \sum_{n=0}^{\infty} \frac{b_{r,n}}{s_0} \left(1 - \frac{s}{s_0}\right)^n, \tag{1.49}$$

$$\kappa_e = \sum_{n=0}^{\infty} \frac{b_{e,n}}{s_0} \left(1 - \frac{s}{s_0}\right)^n, \tag{1.50}$$

$$\kappa_f = \sum_{n=0}^{\infty} \frac{b_{f,n}}{s_0} \left(1 - \frac{s}{s_0}\right)^n, \tag{1.51}$$

$$\kappa_{ef} = \sum_{n=0}^{\infty} \frac{b_{ef,n}}{s_0} \left(1 - \frac{s}{s_0}\right)^n. \tag{1.52}$$

Because $\rho$ is chosen to be an overall factor, it is appropriate to include the resonating part of the amplitude here.[4] From the generic formula (1.48), one gets all the expressions for $\sigma_T, A_{FB}, A_{LR}$ etc. as explained above [13, 14]. As a result, all these quantities $\sigma_A$ have the same form, but depend on different terms $R_{A,r}$ and $b_{A,f,n}$, which are bi-linear compositions of the coefficients $R_r$ and $b_{f,n}$.

The Breit-Wigner function used here deviates from the Breit-Wigner function as it was used by the LEP collaborations, where $\Gamma_Z(s) = s/M_Z^2 \, \Gamma_Z$ was used instead of $\Gamma_Z(s) = g_Z$. The difference is not negligible and amounts to [15]:

$$m_Z = \frac{M_Z}{\sqrt{1 + \Gamma_Z^2/M_Z^2}} \approx M_Z - \frac{1}{2}\Gamma_Z^2/M_Z \approx M_Z - 34 \text{ MeV}, \tag{1.53}$$

$$g_Z = \frac{\Gamma_Z}{\sqrt{1 + \Gamma_Z^2/M_Z^2}} \approx \Gamma_Z - \frac{1}{2}\Gamma_Z^3/M_Z^2 \approx \Gamma_Z - 1 \text{ MeV}. \tag{1.54}$$

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[4]One might, instead, hold a resonating overall factor of the amplitude $\chi_Z(s)$ outside the form factors. This is done in ZFITTER. Then $\rho$ has to be understood as a Taylor series, and if a specific contribution is non-resonating, e.g. because it is due to photon exchange, the first coefficient of $\rho$ would vanish, $b_{r,0} = 0$ (see [8] for more details).

We have expansions both around the pole position $s_0$ and in the coupling constants $\alpha$ and $\alpha_s$, and have to assume that $\alpha, \alpha_s$ and $g_Z/m_Z$ and also $1 - s/m_Z^2$ are of the same numerical order. As a consequence, in an electroweak calculation, $R$ is needed to order $\mathscr{O}(\alpha^2)$, the coefficients $b_0$ to order $\mathscr{O}(\alpha)$, and the $b_1$ etc. to leading order only. One has to observe that also the quantity $g_Z$ itself is a prediction of the theory, beginning at order $\mathscr{O}(\alpha^2)$.

A further complication comes from the fact that there are not so small higher order photonic corrections. There are two approaches to that. Either one assumes the photon exchange amplitude as a separate quantity, which interferes with the $Z$ boson amplitude, and takes this correctly into account. This was done in the ZFITTER approach [4, 16, 7]. To the perturbative orders covered by ZFITTER, this was a controlled approach. In general, it might be more consistent to work with only one amplitude and to understand photonic corrections a a part of background. Then, nevertheless it makes sense to calculate those parts of the photonic backgound with a precision needed by experiment, and to separate this from the unknown parts of the background, as was discussed in [17].

The above considerations help to understand the hierarchy of corrections. Weak vertex corrections as well as weak self-energies contribute to $R$, while all the photonic corrections and also the box diagrams go into the background $B$. This means that a two-loop calculation for the $Z$ resonance has to include only vertices and self-energies at two loops – these are the factorizing corrections. An immediate consequence is that for the calculation of an asymmetry like $A_{FB}$ close to the $Z$ peak, one needs only the $v_e/a_e$ and $v_f/a_f$, derived from (self-energy- and) vertex corrections, at two loops, and the other terms with less accuracy.[5] The photonic corrections, as well as the box terms are not resonating and thus suppressed compared to the resonance residue. As a consequence, all the complicated two-loop boxes are negligible here, while the photonic corrections and the one-loop box terms are well-known and may be considered as a correction. In ZFITTER, this is organized in the various interfaces [5, 16]. The numerical details have been carefully studied in [18], and never again since then.

We come now back to the definition of the effective weak mixing angle:

$$\sin^2\theta_{\rm eff}^{\rm f} \equiv \sin^2\theta_W \,\Re e\,\kappa_f = \frac{1}{4|Q_f|}\left(1 - \Re e\,\frac{v_f}{a_f}\right). \tag{1.55}$$

This means also

$$\sin^2\theta_{\rm eff}^{\rm f} = \left(1 - \frac{M_W^2}{M_Z^2}\right)(1 + \Delta\kappa_f). \tag{1.56}$$

According to its definition, the $A_f$ is a function of one variable only; for *b*-quarks:

$$A_{\rm b} \equiv \frac{2\Re e\,\frac{g_{\rm V}^b}{g_{\rm A}^b}}{1 + (\Re e\,\frac{g_{\rm V}^b}{g_{\rm A}^b})^2} \tag{1.57}$$

$$= \frac{1 - 4|Q_b|\sin^2\theta_{\rm eff}^{\rm b}}{1 - 4|Q_b|\sin^2\theta_{\rm eff}^{\rm b} + 8Q_b^2(\sin^2\theta_{\rm eff}^{\rm b})^2}$$

---

[5] Strictly speaking, the $A_f$ is dependent on the scattering channel for which it is measured or calculated. Using e.g. $A_e$ as it is measured from muon pair production for the determination of $A_b$ from $A_{FB}$ as it is measured from $\bar{b}b$ production, one has check that this is consistent to the accuracy aimed at.

The so far best measurement is due to LEP 1 measurements [19]:

$$A_{\mathrm{b}} = 0.899 \pm 0.013. \tag{1.58}$$

For the weak mixing angle, this means:

$$\sin^2 \theta_{\mathrm{eff}}^{\mathrm{b}} = 0.281 \pm 0.016. \tag{1.59}$$

This value corresponds to an experimental accuracy of about 5.7%. At the next lepton collider, one aims at electroweak per mille measurements, which motivates complete weak two-loop predictions. We will see later that this aim is achieved for $A_b$ with our new result.

To summarize this part of the discussion: The $Z\bar{b}b$ asymmetry parameter $A_b$ may be expressed by the effective weak mixing angle $\sin^2 \theta_{\mathrm{eff}}^{\mathrm{b}}$, or seen in a different way: One may determine the effective weak mixing angle $\sin^2 \theta_{\mathrm{eff}}^{\mathrm{b}}$ from the asymmetry parameter $A_b$, which by itself may be determined experimentally from combinations of pseudo-observables, and theoretically from the ratio $v_b/a_b$. Here, the vertex form factor $\rho_b$ drops out. The calculation of all the relevant radiative loop corrections at two loop order or more is involved. The relation of these radiative corrections to the $Zff$ width and asymmetry parameters is simple, while the relations of the various width and asymmetry parameters to realistic observables or to pseudo observables need a careful control.

What we did not discuss so far is the relation of "true", or "realistic", observables and pseudo-observables. It is constituted by the determination of the $2 \to 2$ hard scattering observable from the experimentally accessible cross sections with multi-particle final states, where the $2 \to 2$ amplitudes contribute together with more complicated final states which may not be distinguished. One has to cover additional soft photons, gluons, but also $e^+e^-$-pairs etc.

It is not the aim here to discuss this in detail. Up to additional corrections, at the $Z$ resonance the bulk of realistic observables may be described theoretically as a folding of the pseudo observables with some kernel functions. The experimentally accessible total cross section e.g. may be written as follows:

$$\sigma_T^{exp}(s) = \int d\frac{s'}{s} \, \rho_T\left(\frac{s'}{s}\right) \sigma_T(s') + \cdots. \tag{1.60}$$

Similarly, and with the same kernel function, one may describe polarization and helicity asymmetries. The notable exception is, due a different angular dependence, the forward-backward asymmetric cross section:

$$\sigma_{FB}^{exp}(s) = \int d\frac{s'}{s} \, \rho_{FB}\left(\frac{s'}{s}\right) \sigma_{FB}(s') + \cdots, \tag{1.61}$$

where $\rho_{FB}(s'/s) \neq \rho_T(s'/s)$. For the explicit expressions for $\rho_{FB}(s'/s)$ and $\rho_T(s'/s)$, as well for more involved contributions, see e.g. [20, 21, 22, 23, 24, 25]. Because at the $Z$ resonance the soft photon radiation dominates over hard photon emission, and the radiator kernels $\rho_{FB}(s'/s)$ and $\rho_T(s'/s)$ start to deviate from each other for hard emissions, one may use $\rho_T(s'/s)$ in some approximation for the prediction of all the realistic observables: At the resonance, hard radiative emissions are kinematically suppressed. But when non-resonating parts become important, one has to take notice of their difference.

The unfolding of realistic observables according to (1.60) and (1.61) can be performed with the analysis tools TOPAZ0 [26, 27, 28, 29] and ZFITTER. The latter one relies on the work quoted above for $\rho_T$ and $\rho_{FB}$. Evidently, the result of unfolding depends on the model chosen for the hard process $\sigma_T^0(s')$ or $\sigma_{FB}^0(s')$. This fact is reflected by the various model-dependent so-called interfaces of ZFITTER.

Finally, the following question has to be answered: How may one take into account the resonating Breit-Wigner form of the pseudo-observables when unfolding? The answer is given by the so-called S-matrix approach to the $Z$ resonance. This task was not covered in the original versions of ZFITTER. A relatively simple version was offered with the interface ZUSMAT of ZFITTER. This interface was finally replaced by a call to the independent Fortran package SMATASY [13, 14, 30, 31, 32, 33, 17]. It is not the intention here to describe details of the approach. We only mention that one may introduce in (1.60) and (1.61) the effective Born approximations as they are derived from amplitudes of the form (1.48), where the form factors are expressing the corresponding resonance parameters $R_A$, $s_{A,0}$ and $b_{A,n}$. Consequently, unfolding allows the determination of $s_0$ and of certain combinations of $R_A$ and, depending on the experimental accuracy, also of the background parameters $b_n$.

For the $b$-pair production, one has to derive from data e.g. the residues $R_T, R_{FB}$. Their ratio $R_{FB}/R_T$ gives then (with the approximations mentioned)

$$\frac{R_{FB}}{R_T} = \frac{3}{4} A_e A_b, \tag{1.62}$$

and for known $A_e$ from other measurements one may derive $A_b$.

If one intends to create a modernized per-mille version of ZFITTER for a study of the $Z$ resonance at a future lepton collider, one has to foresee interfaces which carefully take into account the notations and concepts described here.

Now we come back to the very determination of the weak bosonic two-loop corrections to $A_b$ and to $\sin^2 \theta_{\text{eff}}^{\text{b}}$. Explicit generic formulae for the residue $R$ of the $Z$ resonance amplitude, respecting general principles – unitarity, analyticity, gauge invariance –, are given with equations (12) and (13) of [10], with a reference to earlier work [12].

## 2. The $Z$-boson width

As explained above, the $Z\bar{e}e$ and $Z\bar{b}b$ vertices constitute the main contributions to the pseudo observables of the $Z$ resonance,

$$\Gamma(z \to \bar{b}b) \sim |\mathcal{M}_{1-loop} + \mathcal{M}_{2-loop} + \cdots|^2 + \cdots, \tag{2.1}$$

$$\mathcal{M}_{2-loop} \sim \cdots + \bar{u}\, \mathbf{V}_\mu^{Zb\bar{b}} u\, \varepsilon^\mu + \cdots \tag{2.2}$$

The two-loop electroweak *fermionic corrections* to $A_b$ and $\sin^2 \theta_{\text{eff}}^{\text{b}\bar{\text{b}}}$ were determined in [34]. We have calculated now the so far unknown *bosonic integrals* for the 2-loop diagrams for the vertex $\mathbf{V}_\mu^{Zb\bar{b}}$. They include the topologies shown in figure 1.
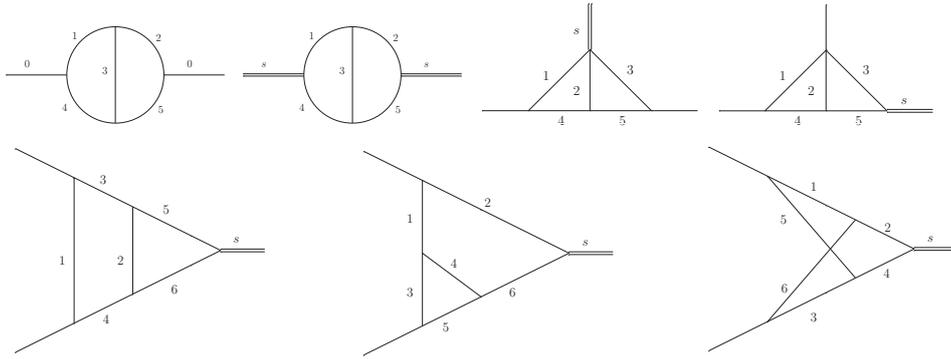
**Figure 1:** The bosonic electroweak two-loop topologies of the $Z\bar{b}b$ vertex.

## 2.1 The electroweak one-loop corrections to the $Z$-boson vertex

Around 1980 it became evident that the Glashow/Salam/Weinberg model might become the electroweak Standard Model. Consequently, some more elaborated loop calculations became meaningful.

Seen from today, the one-loop calculations of that time look quite simple. One had to understand complex logarithms, the Euler dilogarithm $\text{Li}_2$, and to read basically two seminal papers by 't Hooft and Veltman [35] and by Passarino and Veltman [36]. Among the first substantial, independent electroweak projects was a study in the unitary gauge by the Dubna group, founded by Dima Bardin. They studied complete electroweak radiative corrections for decays and scattering processes, including $e^+e^- \to \bar{f}f$ and $Z \to \bar{f}f$, assuming all fermions being massless. In [37], no expanded numerics was performed. Triggered partly by the detection of $Z \to l^+l^-$ by the UA1 and UA2 experiments at CERN, there were several calculations of $Z$-decay into leptons [38, 39], also under the assumption of massless fermions. The decay $Z \to \bar{b}b$ gets contributions from $m_t$-dependent vertex contributions, and it is not covered by these calculations, if the top quark is heavy. The top quark mass was unknown at that time, but the experimental mass limits were growing up. People at Dubna (Akhundov, Bardin, Riemann) observed that one can cover the amplitude with account of the additional $m_t$-dependent terms by adding up two known pieces:
$\mathcal{M}(Z \to \bar{q}q)$ with $m_q = 0$, and
$\mathcal{M}(Z \to \bar{q}_1 q_2)$ with $q_1, q_2$ having the same isospin, but $q_1 \neq q_2$.
The first piece was known from [37], and the second one, non-vanishing only if a loop fermion mass is non-vanishing, from a study of flavor-nondiagonal $Z$ decays [40]. So the two were combined and accomplished by an independent recalculation of the whole $Z\bar{b}b$ amplitude. The preprint JINR-E2-85-617 appeared in August 1985, and soon later the publication [41]. The language of $\rho_Z$ and $\kappa_Z$ for the radiative corrections was used there, and the Fortran program ZRATE became the first piece of the electroweak Standard Model library of the ZFITTER project [7, 16, 42]. At the official ZFITTER webpage http://sanc.jinr.ru/users/zfitter/ one may find the many additional publications on which the ZFITTER software is founded.

When the opening of LEP 1 at CERN approached, with a potential to observe the decays $Z \to \bar{b}b$, several further one-loop calculations were published: in October 1987 [43], in January

1988 [44], in January 1988 [45], in July 1988 [46], in 1990 [47]. Just to mention, the one-loop terms of our present code for the $Z\bar{b}b$ vertex remained unpublished (A. Freitas). [6]

## 2.2 Known higher order corrections to the $Z\bar{b}b$ vertex

There are several higher-order corrections to the $Z\bar{b}b$ vertex known: the $\mathcal{O}(\alpha\alpha_s)$ QCD corrections [48, 49, 50, 51, 52, 53, 54], partial corrections of order $\mathcal{O}(\alpha_t\alpha_s^2)$ [55, 56], partial corrections of order $\mathcal{O}(\alpha^2\alpha_t)$ and $\mathcal{O}(\alpha_t^3)$ [57, 58], the Standard Model two-loop prediction of $M_W$ from the Fermi constant $G_\mu$ [59], partial corrections of order $\mathcal{O}(\alpha_t\alpha_s^3)$ [60, 61, 62], the fermionic electroweak two-loop corrections [34]. Further references to be mentioned here are [15, 63, 64, 65, 10, 66, 67].

## 3. The bosonic $Z\bar{b}b$ topologies

The bosonic electroweak two-loop corrections to the $Z\bar{b}b$ vertex were the last missing piece for the prediction of $\sin^2\theta_{\mathrm{eff}}^b$. We could use the calculational scheme as it was worked out in [10, 34] and work quoted therein. We calculated the $\mathcal{O}(700)$ unknown bosonic Feynman diagrams with two methods, in order to have two independent numerical results. One method relied on sector decomposition, and we used the publicly available packages FIESTA 3 [68] and SecDec 3 [69]. Both of them can apply contour deformation and are applicable to Minkowskian kinematics, as it is met here. The second method uses Mellin-Barnes representations for the Feynman integrals. For this, one may use the MB suite, publicly available at the MBtools webpage [70], and software from the Katowice webpage [71]. We had to develop two new tools. For the treatment of non-planar Feynman integrals, we developed AMBRE 3 [72, 73, 74]. The package MBnumerics [75] delivers a stable 8-digit numerical treatment of Feynman integrals with presently up to four dimensionless scales in the Minkowskian region. It was also essential, that both methods can automatically treat ultraviolet and infrared singularities. While the sector decomposition method had problems with few infrared divergent one- or two-scale integrals, the MB-method tends to fail for integrals with a larger number of scales. Nevertheless, we derived two precise, independent results for all the integrals needed. Details of the complete calculation have been reported in [1, 76] and in the transparencies of this talk at LL2016 [77], so we may restrict ourselves here to some pedagogical remarks.

## 3.1 The non-planar one-scale integral $I_{15}$(**0H0W0txZ**)

As an example we compare several calculations of a non-planar two-loop vertex integral with one massive line and only one scale, $s = M_Z^2$, $I_{15}$(0H0W0txZ). It is shown in figure 2 and depends on one parameter $s/M_Z^2 = 1 + i\varepsilon$. A first calculation goes back to 1998 [78], so we could use the result as a cross-check of our own calculation with the Mellin-Barnes method [76, 1]. This was an important check, because an attempt to calculate the integral with the sector decomposition method in Minkowskian space-time failed.

---

[6]The one-loop corrections to $Z \to \bar{b}b$ in the Gfitter package of 2007 (version of 15 June 2008 by J. Haller, A. Hoecker, M. Goebel [2]) are not based on an independent calculation. That package makes use of the Standard Model implementation in ZFITTER [42].
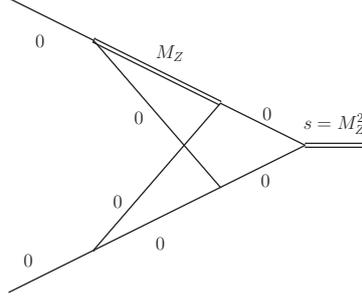
**Figure 2:** A non-planar vertex with one massive line.

The integral $I_{15}$(0H0W0txZ) is (up to some sign convention) the integral $N_3$ of [78]:

$$-N_3 = I_{15}(\text{0H0W0txZ}) \tag{3.1}$$
$$= \frac{e^{2\gamma_E \varepsilon}}{\pi^d} \int \frac{d^d k_1 d^d k_2}{D[k_1,0]D[k_1-k_2,0]D[k_2,0]D[k_2+p_2,0]D[k_1+k_2+p_2,0]D[k_1-k_2+p_1,M_Z]}.$$

The MB-representation is derived with calls to the packages PlanarityTest [72, 79] and AM-BRE 3 [73, 74]. The *U*- and *F*-polynomials are:

```
Upoly1 =   x[1]x[2] + x[1]x[3] + x[2]x[3] + x[1]x[4] + x[3]x[4] + x[1]x[5]
         + x[2]x[5] + x[4]x[5] + x[2]x[6] + x[3]x[6] + x[4]x[6] + x[5]x[6]

Fpoly2 = Upoly1 MZ^2 x[4] - s x[1]x[4]x[5] - s x[1]x[2]x[6] - s x[1]x[3]x[6]
                          - s x[2]x[3]x[6] - s x[1]x[4]x[6] - s x[1]x[5]x[6]
```

A naive MB-representation would become high-dimensional, and it would be plagued by the occurrence of terms containing the ill-defined expression $\Gamma[0]$. A dedicated introduction of Cheng-Wu variables leads to the following integrands for the *x*-integrations:

```
Upoly2 = v[1] + v[2] v[3]

Fpoly2 = + MZ^2 Upoly2 C[2]v[3] - s A[1]A[2]v[1]^2
           - s A[2]B[1]C[1]v[1]v[2]v[3] - s A[1]B[2]C[2]v[1]v[2]v[3]
```

The *x*-integrations over `v[i]` can be easily performed, and we remain, from the four additive terms in `Fpoly2`, with a three-dimensional MB-integral:

```
N3 ~  (-s)^(-2-2eps)  Gamma[-eps]
      (-(s/MZ^2))^-z2 Gamma[-eps-z1] Gamma[-z1] Gamma[-eps-z2] Gamma[-z2]
        Gamma[-1-2eps-z1-z3] Gamma[-1-2eps-z2-z3] Gamma[-1-2eps-z1-z2-z3]
        Gamma[-z3] Gamma[1+z3]^2 Gamma[1+z1+z3] Gamma[2+2eps+z1+z2+z3])
        / (Gamma[-2eps-z1] Gamma[-3eps-z2] Gamma[-2eps-z2] Gamma[-2eps-z1-z2])
```

When continuing in `eps` with the package MB.m [80], we derive for vanishing, but finite $\varepsilon$ a two-dimensional and a three-dimensional MB-representation:

```
N3 ~
{
MBint[ ((-s)^(-2-2eps) Gamma[-2eps] Gamma[-eps]
(-(s/MZ^2))^-z2 Gamma[-eps-z2] Gamma[-z2]^2 Gamma[1+z2] Gamma[-1-2eps-z2-z3]
Gamma[-z3] Gamma[1+z3] Gamma[1+eps+z3] Gamma[1+2eps+z3])
 / (Gamma[-3eps-z2] Gamma[-2eps-z2] Gamma[1-z2+z3]),
{{eps->0},{z2->-0.42644,z3->-0.826119}} ],

MBint[ ((-s)^(-2-2eps) Gamma[-eps]
 (-(s/MZ^2))^-z2 Gamma[-eps-z1] Gamma[-z1] Gamma[-eps-z2] Gamma[-z2]
 Gamma[-1-2eps-z1-z3] Gamma[-1-2eps-z2-z3] Gamma[-1-2eps-z1-z2-z3]
 Gamma[-z3]  Gamma[1+z3]^2 Gamma[1+z1+z3]  Gamma[2+2eps+z1+z2+z3])
 / (Gamma[-2eps-z1] Gamma[-3eps-z2] Gamma[-2eps-z2] Gamma[-2 eps-z1-z2]),
{{eps->0},{z1->-0.268281,z2->-1.00065,z3->-0.171895}} ]
}
```

Applying MB [80] and MBnumerics [75], we get:

$$N_3 = \frac{1}{\varepsilon^2}(1.23370055013617 - i\, 6.20475892887384 \times 10^{-13}) \tag{3.2}$$

$$+ \frac{1}{\varepsilon}(2.8902545096591976 + i\, 3.875784585038738)$$

$$+ (-0.7785996083247692 - i\, 4.123512600516016).$$

The integral N3, according to equation (D.11) of [78], is with $z = s/M_Z^2 = 1 + i\varepsilon$:

$$N_3 = \frac{1}{s^2} \sum_{n=1}^{\infty} (-z)^n \left\{ \frac{1}{\varepsilon^2}\left[ -\frac{1}{2}\zeta_2 + K_2(n-1) \right] \right. \tag{3.3}$$

$$+ \frac{1}{\varepsilon}\left[ -\frac{1}{2}\zeta_3 - 2\zeta_2 S_1(n-1) + 2S_3(n-1) - 2K_3(n-1) + 4S_1(n-1)K_2(n-1) \right.$$

$$\left. + (\zeta_2 - S_2(n-1))\ln(-z) \right]$$

$$+ \left[ -\zeta_4 - 2\zeta_3 S_1(n-1) - 7\zeta_2 S_2(n-1) - 4\zeta_2 S_1(n-1)^2 + 7\zeta_2 K_2(n-1) - \frac{7}{2}S_4(n-1) \right.$$

$$+ \frac{7}{2}S_2(n-1)^2 + 6S_1(n-1)S_3(n-1) + 2S_{13}(n-1) + 8K_4(n-1)$$

$$- 8S_1(n-1)K_3(n-1) + 8S_1(n-1)^2 K_2(n-1)$$

$$+ \left( \zeta_3 + 4\zeta_2 S_1(n-1) - S_{12}(n-1) - 3S_1(n-1)S_2(n-1) - 4K_3(n-1) \right)\ln(-z)$$

$$\left. \left. + \left( -\zeta_2 + \frac{1}{2}S_2(n-1) + K_2(n-1) \right)\ln^2(-z) \right] \right\}.$$

The expression contains harmonic sums [81, 82, 83]:

$$S_a(n) = \sum_{j=1}^{n} 1/j^a, \tag{3.4}$$

$$K_a(n) = -\sum_{j=1}^{n} (-1)^j/j^a, \tag{3.5}$$

$$S_{ab}(n) = \sum_{j=1}^{n} S_b(j-1)/j^a, \tag{3.6}$$

The sum $N_3$ converges both in the Euklidean and the Minkowskian kinematics, but very slowly, so that it would need many terms in order to get our accuracy goal of eight digits. The N3 evaluated with 200 terms gives e.g.:

```
time = 4.060519 sec        for 200 terms of the sum

N3 = (0.4 + 4 x I)

   + 1/eps x (2.8 + 3.87 x I) + 1 /eps^2 x (1.23 + 0 x I)
```

The agreement, with several thousand terms (few hours running time), is suffiently good in order to see that the results from the numerical MB-approach are reasonable. One may improve the comparison. In fact, in appendix E of [78], the necessary harmonic sums are explicitly performed. We derive:[7]

$$\phi(z) = \frac{1}{2}S_{1,2}(z^2) - S_{1,2}(z) - S_{1,2}(-z) + \ln(1-z)\mathrm{Li}_2(-z) \tag{3.7}$$

and

$$-\frac{s^2 z}{1+z} N_3 = \frac{1}{\varepsilon^2}\left[-\frac{\zeta_2}{2} - \mathrm{Li}_2(z)\right] \tag{3.8}$$

$$+ \frac{1}{\varepsilon}\left[-\frac{1}{2}\zeta_3 - 2\zeta_2\mathrm{Li}_2(-z) + 2\mathrm{Li}_3(-z) + 2\mathrm{Li}_3(z) + 4\left[\phi(-z) - S_{1,2}(z) - \mathrm{Li}_3(-z)\right]\right.$$

$$\left. + (\zeta_2 - \mathrm{Li}_2(-z))\ln(-z)\right]$$

$$- \frac{s^2 z}{1+z} N_3^{\mathrm{const}}$$

Further,

$$\frac{s^2(-z)}{1+z} N_3^{\mathrm{const}} = -\zeta_4 + 2\zeta_3\ln(1+z) - 7\zeta_2\mathrm{Li}_2(-z) - 4\,\zeta_2\left[-\ln(1-r)^2 + \mathrm{Li}_2(-z)\right] \tag{3.9}$$

$$+ 7\,\zeta_2\mathrm{Li}_2(z) - \frac{7}{2}\mathrm{Li}_4(-z) + \frac{7}{2}\left[\mathrm{Li}_2(-z)^2 + \mathrm{Li}_4(-z) - S_{2,2}(-z)\right]$$

$$+ 6\left[(\mathrm{Li}_4(-z) - \ln(1+z)\mathrm{Li}_3(-z) - \frac{1}{2}\mathrm{Li}_2(-z)^2 + S_{2,2}(-z)\right]$$

$$+ 2\left[-\ln(1+z)\mathrm{Li}_3(-z) - \frac{1}{2}\mathrm{Li}_2(-z)^2\right] + 8\mathrm{Li}_4(z)$$

$$- 8\left[-\mathrm{Li}_4(z) - S_{2,2}(z) - \int_0^{-z}\frac{dt}{1-t}\mathrm{Li}_3(-t)\right]$$

---

[7]In [78], the overall sign of (E.7) is wrong, and in the r.h.s. of (E.36) one has to replace $S_{1,3}$ under the integral by $S_{1,2}$ and to change the sign of $2\ln(1-z)$. We thank A. Kotikov for clarifying this.

$$+8\left[-\mathrm{Li}_4(z)-\frac{1}{2}S_{2,2}(z)-2S_{1,3}(z)+\phi(-z)[\ln(-z)-2\ln(1+z)]\right.$$

$$\left.-\int_0^{-z}\frac{dt}{1-t}[(2\ln(1-t)-\ln(t))\mathrm{Li}_2(-t)+2\mathrm{Li}_3(-t)+2S_{1,2}(-t)]\right]$$

$$+\left[\zeta_3-4\zeta_2\ln(1+z)+\left[\ln(1+z)\mathrm{Li}_2(-z)+S_{1,2}(-z)\right]\right.$$

$$\left.-3\left(\mathrm{Li}_3(-z)+(-\ln(1+z)\mathrm{Li}_2(-z)-S_{1,2}(-z))-4\mathrm{Li}_3(z)\right]\ln(-z)$$

$$+\left[-\zeta_2+\frac{1}{2}\mathrm{Li}_2(-z)+\mathrm{Li}_2(z)\right]\ln^2(-z)$$

The result is expressed in terms of polylogarithms, plus the few harmonic polylogarithms which are needed to close the basis of weight four. For a systematic numerical calculation of the expressions here in terms of harmonic polylogarithms $H[x,a,b,c,d],a,b,c,d=\pm1,0$, see e.g. appendix B of [84], which is implemented in the Mathematica package HPL4num.m [85] and checked with the Mathematica package HPL [86]. The most compact representation of the integral at the $Z$ boson mass shell was obtained with the aid of Jacob Ablinger, Johannes Blümlein, Carsten Schneider and Arnd Behring (priv. commun.):

$$N_{3,-2}(z) = (z\zeta_2)/(2(1+z))+(z\mathrm{Li}_2(z))/(1+z), \tag{3.10}$$

$$N_{3,-2}(1+i\,\varepsilon) = \frac{3\zeta_2}{4}, \tag{3.11}$$

$$N_{3,-1}(z) = \frac{z}{2(z+1)}\left[(-4H(-1,z)(2H(0,1,z)+\zeta_2)+8H(0,-1,1,z)\right. \tag{3.12}$$

$$-4H(0,0,-1,-z)+4H(0,0,-1,z)+8H(0,1,-1,z)+8H(0,1,1,z)$$

$$\left.-2\ln(-z)H(0,-1,z)-2\,\zeta_2\ln(-z)+\zeta_3)\right],$$

$$N_{3,-1}(1+i\,\varepsilon) = -3\ln(2)\zeta_2+\frac{21}{4}\zeta_3+i\,\frac{3}{4}\pi\zeta_2. \tag{3.13}$$

We confirm with $N_3(1+i\,\varepsilon)$ the 9 digits accuracy obtained with AMBRE/MB/MBnumerics given in (3.2):

$$N_{3,0}(1+i\,\varepsilon) = 24\mathrm{Li}_4(1/2)+\ln^4(2)-\frac{351}{80}\zeta_2^2-I\,3\ln(2)2\pi\zeta_2-i\,\frac{63}{16}\pi\zeta_3 \tag{3.14}$$

$$= 0.778599608987621685634528056900$$

$$-i\,4.123512593336422726481033653830.$$

## 4. Results

The electroweak bosonic two-loop contribution to the weak mixing angle is:

$$\Delta\kappa_b^{(\alpha^2,\mathrm{bos})} = -0.9855\times10^{-4}. \tag{4.1}$$

The value $-1.0276\times10^{-4}$, presented as preliminary result at LL2016, was based on the input parameter list of [77] which differs slightly from the input list of table 1 of [1], which is applied here. This value amounts to about $\frac{1}{4}$ of the leptonic corrections to $\kappa_b$ and $\sin^2\theta_{\mathrm{eff}}^b$. The corrections

**Table 1:** Reference values used in the numerical analysis, from Ref. [87].

| Parameter | Value | Range |
|---|---|---|
| $M_Z$ | 91.1876 GeV | $\pm 0.0042$ GeV |
| $\Gamma_Z$ | 2.4952 GeV | |
| $M_W$ | 80.385 GeV | $\pm 0.030$ GeV |
| $\Gamma_W$ | 2.085 GeV | |
| $M_H$ | 125.1 GeV | $\pm 5.0$ GeV |
| $m_t$ | 173.2 GeV | $\pm 4.0$ GeV |
| $\alpha_s$ | 0.1184 | $\pm 0.0050$ |
| $\Delta\alpha$ | 0.0590 | $\pm 0.0005$ |

**Table 2:** Comparison of different orders of radiative corrections to $\Delta\kappa_b$, using the input parameters in 1. Numerical values taken from [1].

| Order | Value $[10^{-4}]$ |
|---|---|
| $\alpha$ | 468.945 |
| $\alpha\alpha_s$ | $-42.655$ |
| $\alpha_t\alpha_s^2$ | $-7.074$ |
| $\alpha_t\alpha_s^3$ | $-1.196$ |
| $\alpha_t^2\alpha_s$ | 1.362 |
| $\alpha_t^3$ | 0.123 |
| $\alpha_{\text{ferm}}^2$ | 3.866 |
| $\alpha_{\text{bos}}^2$ | $-0.986$ |

to the weak mixing angle are shown in table 2. The biggest corrections come from the one-loop electroweak contributions, followed by mixed electroweak-QCD corrections of order $\alpha\alpha_s$. All the other corrections, including the new bosonic electroweak two-loop corections, are of the same order, at the $10^{-4}$ level. For a per mille measurement, it is good to know them, but they will not influence the data analysis numerically.

For the corresponding fitting formula for $\sin^2\theta_{\text{eff}}^b$, we refer to [1]. An analysis tool for the consistent 1 per mille treatment of realistic observables, pseudo-observables, and two-loop predictions to them, is not available for the $Z$ boson resonance, although ZFITTER is a very good approximation and suffices for the presently available accuracy of data.

## Acknowledgements

We would like to thank Peter Marquard for discussions and Peter Uwer and his group "Phenomenology of Elementary Particle Physics beyond the Standard Model" at Humboldt-Universität zu Berlin for providing computer resources.

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