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# Perturbative quantum field theory informs algebraic geometry



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We consider Feynman integrals, in perturbative quantum field theory, with evaluations that inform both the algebraic geometry revealed by their representation as integrals over Schwinger parameters and also the number theory of L-series, $L_n(s)$, defined by products over primes of data derived from $n$-th power moments of Kloosterman sums in finite fields. In co-ordinate space, Feynman diagrams with massive propagators yield integrals of products of Bessel functions. We consider vacuum diagrams and on-shell sunrise diagrams, with two vertices, evaluated in two spacetime dimensions, and label integrals, $S_{n,s}$, by the number of Bessel functions, $n$, and the number of loops, $s$. For $n<5$, is is proven that $S_{n,s}$ is an integer multiple of $L_n(s)$. At $n=5,6,8$, the L-series are obtained from the Fourier series of modular forms, with weight $n-2$, the sunrise integrals $S_{n,n-2}$ evaluate to multiples of $\zeta(2)L_n(n-4)$ and the vacuum integrals $S_{n,n-1}$ enter determinants that evaluate $L_n(n-1)$, outside the critical strip. At $n=7$, we find a functional equation for $L_7(s)$ and obtain $S_{7,4}=20\zeta(2)L_7(2)$. For each $n>2$, we give a conjectural evaluation of a determinant of Feynman integrals as a rational or algebraic multiple of a power of $\pi$. Empirical evaluations are indicated by question marks and have been checked at 1000-digit precision.

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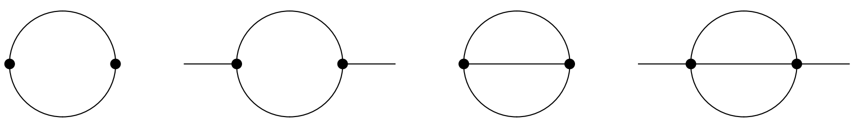
## 1. Feynman integrals

We consider Feynman diagrams, like those illustrated below, evaluated, in two spacetime dimensions, by integrals of the form

$$S_{n,s} := 2^s \int_0^\infty [I_0(t)]^{n-s-1}[K_0(t)]^{s+1} t\, dt \tag{1.1}$$

with $n$ Bessel functions and loop-number $s$ satisfying $s < n \le 2s+2$ and $s > 1$ for $n = 2s+2$, to ensure convergence. The internal scalar particles have unit mass and account for the Bessel function $K_0(t)$ in the integrand. Hence the two-loop vacuum integral $S_{3,2}$ has a propagator $K_0(t)$ associated with each of its three internal edges. $S_{4,2} := 2^2 \int_0^\infty I_0(t) K_0^3(t)\, t\, dt$ is a two-loop on-shell sunrise diagram, with the Bessel function $I_0(t)$ coming from external half-edges, whose momenta are on the unit mass shell. The one-loop diagram $S_{3,1}$ is obtained from $S_{3,2}$ by cutting an internal edge. Removing the external half-edges from $S_{3,1}$, we obtain the one-loop vacuum diagram $S_{2,1}$. If we join up the half-edges in $S_{4,2}$, we obtain a three-loop vacuum diagram, $S_{4,3}$.



$S_{2,1}$ $S_{3,1}$ $S_{3,2}$ $S_{4,2}$

## 2. L-series

For $n < 8$ and $s$ with a suitably large real part, we define the L-series

$$L_n(s) := \prod_{p \ge 2} \frac{1}{Z_n(p, p^{-s})} = \sum_{m>0} \frac{A_n(m)}{m^s} \tag{2.1}$$

with a product over primes of local factors defined by [4]

$$Z_n(p,T) := \exp\left(-\sum_{k>0} \frac{c_n(p^k)}{k} T^k\right) \tag{2.2}$$

where $c_n(q)$ are $n$-th power Kloosterman moments [7, 8, 9, 10, 14] in finite fields $\mathbf{F}_q$. Then (2.2) is a polynomial in $T$, with degree less than $n/2$, and $L_1(s) = L_2(s) = 1$,

$$L_3(s) = \sum_{k \ge 0} \left( \frac{1}{(3k+1)^s} - \frac{1}{(3k+2)^s} \right), \tag{2.3}$$

$$L_4(s) = (1 - 2^{-s})\zeta(s) = \sum_{k \ge 0} \frac{1}{(2k+1)^s}, \tag{2.4}$$

with a functional equation at $n = 3$ giving

$$\Lambda_3(s) := \left(\frac{3}{\pi}\right)^{s/2} \Gamma\left(\frac{s+1}{2}\right) L_3(s) = \Lambda_3(1-s). \tag{2.5}$$

For $n \leq 4$, the $s$-loop integral $S_{n,s}$ is an *integer* multiple [1] of $L_n(s)$:

$$S_{1,0} = L_1(0) = 1 \tag{2.6}$$
$$S_{2,1} = L_2(1) = 1 \tag{2.7}$$
$$S_{3,1} = 2L_3(1) = \frac{2\pi}{\sqrt{3^3}} \tag{2.8}$$
$$S_{3,2} = 3L_3(2) = 3\sum_{k\geq 0}\left(\frac{1}{(3k+1)^2} - \frac{1}{(3k+2)^2}\right) \tag{2.9}$$
$$S_{4,2} = 2L_4(2) = \frac{\pi^2}{4} \tag{2.10}$$
$$S_{4,3} = 8L_4(3) = 7\zeta(3). \tag{2.11}$$

## 3. Proofs for 5 Bessel functions

In a conference talk, *Reciprocal PSLQ and the tiny nome of Bologna*, given in June 2007 at the Zentrum für interdisziplinäre Forschung in Bielefeld, an empirical evaluation [3]

$$S_{5,3} = \frac{\pi^3}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\left(\sum_{n=-\infty}^{\infty} e^{-n^2\pi\sqrt{15}}\right)^4 \tag{3.1}$$

was given for the on-shell 3-loop sunrise diagram in two spacetime dimensions. This implies a neat evaluation as a product of values of the gamma function [11]

$$S_{5,3} = \frac{1}{30\sqrt{5}}\prod_{k=0}^{3}\Gamma\left(\frac{2^k}{15}\right) \tag{3.2}$$

by applying the Chowla-Selberg theorem to elliptic integrals at the 15th singular value [1]. Intense work in 2007 with Jon Borwein, in Halifax, Nova Scotia, showed that (3.2) is equivalent to

$$S_{5,3} = \frac{4\pi}{\sqrt{15}} S_{5,2}. \tag{3.3}$$

Discussions with Spencer Bloch and Francis Brown, at a summer school in Les Houches, organized by Dirk Kreimer in 2010, pointed to a connection with the weight-3 level-15 modular form

$$f_{3,15} := (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3 = \sum_{n>0} A_5(n)q^n \tag{3.4}$$

with $\eta_n := q^{n/24}\prod_{k>0}(1-q^{nk})$. This came from the representation of

$$S_{5,3} = \int_0^\infty\int_0^\infty\int_0^\infty \frac{da\,db\,dc}{(abc+ab+bc+ca)(a+b+c)+ab+bc+ca} \tag{3.5}$$

as an integral over Schwinger parameters. Then counts of the zeros of the denominator of the integrand, in finite fields $\mathbf{F}_p$, implicated the Fourier coefficients $A_5(p) = c_5(p)$ of $f_{3,15}$, at small prime $p$. The L-series obtained by setting $n = 5$ in (2.1) has a functional equation

$$\Lambda_5(s) := \left(\frac{15}{\pi^2}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L_5(s) = \Lambda_5(3-s) \tag{3.6}$$



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and analytic continuation yields the evaluations

$$S_{5,2} = 3L_5(2) \tag{3.7}$$

$$S_{5,3} = \frac{48}{5}\zeta(2)L_5(1). \tag{3.8}$$

All of the above results are proven, thanks to work in [1, 2, 12, 13].

## 4. Conjectures for 5 Bessel functions

The 4-loop vacuum integral with 5 Bessel functions has a representation

$$S_{5,4} = 2\int_0^\infty\int_0^\infty\int_0^\infty \frac{\log(a+b+c+1)\,da\,db\,dc}{(abc+ab+bc+ca)(a+b+c)+ab+bc+ca} \tag{4.1}$$

with the same denominator as for $S_{5,3}$ in (3.5), but a logarithmic numerator, resulting from integration over an extra Schwinger parameter. This resisted 8 years of effort to find a relation to the L-series $L_5(s)$ until the authors met at the Mainz Institute for Theoretical Physics in 2015 and experimented with determinants of matrices of Bessel moments. On the basis of numerical investigation, we arrived at the conjectures

$$\det \int_0^\infty I_0(t)K_0^3(t)\begin{bmatrix} K_0(t) & t^2K_0(t) \\ I_0(t) & t^2I_0(t)\end{bmatrix} t\,dt \stackrel{?}{=} \frac{2\pi^3}{\sqrt{3^3 5^3}} \tag{4.2}$$

$$\det \int_0^\infty K_0^3(t)\begin{bmatrix} K_0^2(t) & t^2K_0^2(t) \\ I_0^2(t) & t^2I_0^2(t)\end{bmatrix} t\,dt \stackrel{?}{=} \frac{45}{8\pi^2}L_5(4) \tag{4.3}$$

with question marks indicating that these evaluations are as yet unproven. Conjectures (4.2,4.3) may be combined with proven results to obtain the striking evaluation

$$\frac{L_5(4)}{L_5(2)\zeta(2)} \stackrel{?}{=} \frac{4}{5}\int_0^\infty (R-t^2)K_0^5(t)\,t\,dt \tag{4.4}$$

$$R := 13\left(\frac{2}{15}\right)^2 + 32\prod_{k=0}^{3}\frac{\Gamma(1-2^k/15)}{\Gamma(2^k/15)}. \tag{4.5}$$

## 5. Conjectures for 6 Bessel functions

At $n=6$, the first author found, with help from Francis Brown at Les Houches in 2010, a modular form of weight 4 and level 6

$$f_{4,6} := (\eta_1\eta_2\eta_3\eta_6)^2 = \sum_{n>0} A_6(n)q^n \tag{5.1}$$

with $A_6(p) = c_6(p)$ at the primes agreeing with counts in $\mathbf{F}_p$ of zeros of the denominator of the Feynman integrand for the 4-loop sunrise diagram $S_{6,4}$, with 6 Bessel functions. Then the functional equation

$$\Lambda_6(s) := \left(\frac{6}{\pi^2}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L_6(s) = \Lambda_6(4-s) \tag{5.2}$$



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yielded the conjectures [5]

$$S_{6,2} \stackrel{?}{=} 6L_6(2) \tag{5.3}$$
$$S_{6,3} \stackrel{?}{=} 12L_6(3) = 24\zeta(2)L_6(1) \tag{5.4}$$
$$S_{6,4} \stackrel{?}{=} 48\zeta(2)L_6(2) \tag{5.5}$$

which have been checked at 1000-digit precision. It is notable that $S_{6,4}$ is pulled down to $L_6(2)$, by a multiple of $\zeta(2)$. Hence conjectures (5.3,5.5) imply the sum rule [1]

$$\int_0^\infty I_0(t)K_0^3(t)\left(\pi^2 I_0^2(t) - 3K_0^2(t)\right) t\,dt \stackrel{?}{=} 0. \tag{5.6}$$

It was harder to relate Feynman integrals to $L_6(5)$, outside the critical strip. This problem was cracked by using determinants in conjectures

$$\det \int_0^\infty I_0(t)K_0^4(t) \begin{bmatrix} K_0(t) & t^2K_0(t) \\ I_0(t) & t^2I_0(t) \end{bmatrix} t\,dt \stackrel{?}{=} \frac{5}{32}\zeta(4) \tag{5.7}$$
$$\det \int_0^\infty K_0^4(t) \begin{bmatrix} K_0^2(t) & t^2K_0^2(t) \\ I_0^2(t) & t^2I_0^2(t) \end{bmatrix} t\,dt \stackrel{?}{=} \frac{27}{4\pi^2}L_6(5) \tag{5.8}$$

that neatly follow the pattern discovered at $n = 5$, in (4.2,4.3).

## 6. Conjectures for 7 Bessel functions

Until last year, no relation between L-series and Feynman integrals with 7 Bessel functions had been discovered. Working at Mainz, we used Kloosterman sums in $\mathbf{F}_q$ to determine the local factors (2.2) in the L-series

$$L_7(s) := \prod_{p\ge 2} \frac{1}{Z_7(p,p^{-s})} = \sum_{n>0} \frac{A_7(n)}{n^s} \tag{6.1}$$

and sought a functional relation that would allow analytic continuation to critical values with $0 < s < 5$. Crucial in this endeavour were the determinations of the local factors [4]

$$Z_7(2,2^{-s}) = \left(1-\frac{1}{2^{s-2}}\right)\left(1+\frac{5}{2^{s-2}}+\frac{1}{2^{2s-4}}\right) \tag{6.2}$$
$$Z_7(3,3^{-s}) = 1-\frac{10}{3^{s-2}}+\frac{1}{3^{2s-4}} \tag{6.3}$$
$$Z_7(5,5^{-s}) = 1-\frac{1}{5^{2s-4}} \tag{6.4}$$
$$Z_7(7,7^{-s}) = 1-\frac{10}{7^{s-3}}+\frac{1}{7^{2s-4}} \tag{6.5}$$

at $p \le 7$, where Ronald Evans had been silent [7]. Thereafter, it was sufficient to use `Sage`, with commands kindly provided by William Stein, to determine local factors from Hecke eigenvalues, $\lambda_p \in \mathbf{Q}(\sqrt{-1},\sqrt{6},\sqrt{14})$, at prime $p$, of a newform on $\Gamma_0(525)$, with weight 3 and quartic nebentypus, that gives $|\lambda_p|^2 = p^2 \pm A_7(p)$, for a Kronecker symbol $\left(\frac{p}{105}\right) = \pm 1$, and hence

$$Z_7(p,p^{-s}) = \left(1-\left(\frac{p}{105}\right)p^{2-s}\right)\left(1+\left(\frac{p}{105}\right)(2p^2-|\lambda_p|^2)p^{-s}+p^{4-2s}\right) \tag{6.6}$$

for $p > 7$. Having determined $A_7(n)$ for $n \leq 10^5$, we used Tim Dokchitser's code [6] `computel`, in `Pari-GP`, to determine the viability of a functional equation

$$\Lambda_7(s) := \left(\frac{105}{\pi^3}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) L_7(s) \stackrel{?}{=} \Lambda_7(5-s) \qquad (6.7)$$

with the crucial factor $\Gamma((s-1)/2)$ found empirically. Then `computel` professed itself ready to compute $L_7(s)$ and was used at low precision to determine a conjectural integer 20 in

$$S_{7,4} := 2^4 \int_0^\infty I_0^2(t) K_0^5(t)\, t\, dt \stackrel{?}{=} 20\zeta(2) L_7(2) \qquad (6.8)$$

which was then confirmed to 1000-digit precision, in less than 7 hours. The 4-loop Feynman integral $S_{7,4}$ is the sole integral with 7 Bessel functions that we have been able to relate to $L_7(s)$. It is not hard to see why there is only one. The factor $\Gamma((s-1)/2)$ in the functional equation (6.7) seems to render $s=1$ and hence $s=5-1=4$ inaccessible. Inside the critical strip, with $0<s<5$, that leaves only $s=2$, which is equivalent to $s=5-2=3$, by the functional equation.

### 6.1 Determinants of Feynman integrals with an odd number of Bessel functions

With $n=7$ Bessel functions, no determinant was found to permit an excursion to $s=6$, outside the critical strip. However a $3\times 3$ matrix of moments of 7 Bessel functions,

$$M_3 := \int_0^\infty I_0(t) K_0^4(t) \begin{bmatrix} K_0^2(t) & t^2 K_0^2(t) & t^4 K_0^2(t) \\ I_0(t)K_0(t) & t^2 I_0(t)K_0(t) & t^4 I_0(t)K_0(t) \\ I_0^2(t) & t^2 I_0^2(t) & t^4 I_0^2(t) \end{bmatrix} t\, dt \qquad (6.9)$$

gave the intriguing numerical result

$$\det M_3 \stackrel{?}{=} \frac{2^4\pi^6}{\sqrt{3^3 5^5 7^7}} \qquad (6.10)$$

with the square root of $3^3 5^5 7^7$ resonating with the square root of $3^3 5^5$ in (4.2). More generally, we define $M_k$ to be the $k\times k$ matrix with elements

$$(M_k)_{a,b} := \int_0^\infty [I_0(t)]^a [K_0(t)]^{2k+1-a} t^{2b-1} dt \qquad (6.11)$$

and conjecture that

$$\det M_k \stackrel{?}{=} \prod_{j=1}^{k} \frac{(2j)^{k-j}\pi^j}{\sqrt{(2j+1)^{2j+1}}}. \qquad (6.12)$$

## 7. Conjectures for 8 Bessel functions

At $n=8$, we need to modify the definition (2.1), which served well for $n<8$. The first author discovered that the modular form

$$f_{6,6} = \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3 = \sum_{n>0} A_8(n) q^n \qquad (7.1)$$

with weight 6 and level 6, gives $A_8(p) \equiv c_8(p) \bmod p$, at the primes. However, we do not have equality between $A_n(p)$ and $c_n(p)$ for $n = 8$ and prime $p > 2$. Instead we found that

$$c_8(p) = \begin{cases} A_8(p) & \text{if } p = 2 \\ p^4 + A_8(p) & \text{if } p > 2 \end{cases} \tag{7.2}$$

and hence that

$$L_8(s) := \prod_{p\geq 2} \frac{Z_4(p,p^{4-s})}{Z_8(p,p^{-s})} = \sum_{n>0} \frac{A_8(n)}{n^s} \tag{7.3}$$

is the L-series corresponding to the modular form (7.1), as was recently proven by Yun [14]. Then the functional equation

$$\Lambda_8(s) := \left(\frac{6}{\pi^2}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) L_8(s) = \Lambda_8(6-s) \tag{7.4}$$

enables analytic continuation inside the critical strip, $0 < s < 6$, where we find that

$$S_{8,3} \stackrel{?}{=} 8L_8(3) \tag{7.5}$$

$$S_{8,4} \stackrel{?}{=} 36L_8(4) \tag{7.6}$$

$$S_{8,5} \stackrel{?}{=} 216L_8(5) \tag{7.7}$$

$$S_{8,6} \stackrel{?}{=} 864\zeta(2)L_8(4). \tag{7.8}$$

### 7.1 Determinants of Feynman integrals with an even number of Bessel functions

Now consider the $3 \times 3$ matrix

$$N_3 := \int_0^\infty I_0(t)K_0^5(t) \begin{bmatrix} K_0^2(t) & t^2K_0^2(t) & t^4K_0^2(t) \\ I_0(t)K_0(t) & t^2I_0(t)K_0(t) & t^4I_0(t)K_0(t) \\ I_0^2(t) & t^2I_0^2(t) & t^4I_0^2(t) \end{bmatrix} t\,dt \tag{7.9}$$

obtained by adding an extra $K_0(t)$ to the integrand in (6.9). The elements of its first column are evaluated by $S_{8,6}$, in (7.8), by $S_{8,5}$, in (7.7), and by $S_{8,4}$, in (7.6). The elements in its first row are related to those in its third row, by the conjecture [4] that, for positive integer $n$,

$$A(n) := \left(\frac{2}{\pi}\right)^4 \int_0^\infty \left(\pi^2 I_0^2(t) - K_0^2(t)\right) I_0(t)K_0^5(t)\,(2t)^{2n-1}dt \tag{7.10}$$

yields integers, with $A(1) = 0$, $A(2) = 1$ and $A(3) = 2$. Thus we have 5 relations constraining the 9 elements. There is a 6th empirical constraint:

$$\det N_3 \stackrel{?}{=} \frac{5}{3}\frac{\pi^8}{2^{19}}. \tag{7.11}$$

More generally, let $N_k$ be the $k \times k$ matrix with elements

$$(N_k)_{a,b} := \int_0^\infty [I_0(t)]^a[K_0(t)]^{2k+2-a}t^{2b-1}dt, \tag{7.12}$$

which are moments of $2k+2$ Bessel functions. For integer $m > 0$, let

$$D_m := \frac{2\pi^{m^2/2}}{\Gamma(m/2)} \prod_{j=1}^{m} \frac{(2j-1)^{m-j}}{(2j)^j}. \tag{7.13}$$

Then for every integer $k > 0$, we conjecture that determinant of $N_k$ is $D_{k+1}$.

### 7.2 Evaluation of an L-series of weight 6 outside its critical strip

From the rationality of the moments (7.10), we were led to conjecture, and to test at 1000-digit precision, that the determinant of the $2 \times 2$ matrix

$$\mathscr{M}_2 := \int_0^\infty K_0^6(t) \begin{bmatrix} K_0^2(t) & t^2(1-2t^2)K_0^2(t) \\ I_0^2(t) & t^2(1-2t^2)I_0^2(t) \end{bmatrix} t\,dt \tag{7.14}$$

with 8-Bessel moments up to 7 loops, evaluates the L-series (7.3) for modular form (7.1), with weight 6, outside its critical strip, as follows:

$$L_8(7) \stackrel{?}{=} \frac{128\pi^2}{6075} \det \mathscr{M}_2. \tag{7.15}$$

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