PROCEEDINGS OF SCIENCE

Vacuum currents in braneworlds

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Vacuum expectation value of the current density is investigated for a massive charged scalar field with arbitrary curvature coupling in the geometry of a brane on the background of AdS spacetime with partial toral compact dimensions. The presence of a gauge field flux, enclosed by compact dimensions, is assumed. On the brane the field obeys Robin boundary condition and along compact dimensions periodicity conditions with general phases are imposed. The vacuum charge density and the components of the current along non-compact dimensions vanish. The expectation value of the current density along compact dimensions is a periodic function of the gauge field flux with the period equal to the flux quantum. It is decomposed into the boundary-free and brane-induced contributions. Both these contributions vanish on the AdS boundary. The brane-induced contribution vanishes on the horizon and for points near the horizon the current is dominated by the boundary-free part. Depending on the value of the Robin coefficient, the presence of the brane can either increase or decrease the vacuum currents. Applications are given for a higher-dimensional version of the Randall-Sundrum braneworld model.

The Modern Physics of Compact Stars 2015
30 September 2015 - 3 October 2015
Yerevan, Armenia

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1. Introduction

Recent proposals of large extra dimensions use the concept of brane as a submanifold embedded in a higher dimensional spacetime, on which the standard model particles are confined (for reviews see [1]). Braneworlds naturally appear in string/M theory context and provide a novel setting for discussing phenomenological and cosmological issues related to extra dimensions. The model introduced by Randall and Sundrum [2] is particularly attractive. The corresponding background solution consists of two parallel flat 3-branes in a 5-dimensional anti-de Sitter (AdS) bulk. The fifth coordinate is compactified on S^1/Z_2 and the branes are on the two fixed points. It is assumed that all matter fields are confined on the branes and only the gravity propagates freely in the 5-dimensional bulk. More recently, scenarios with additional bulk fields have been considered.

From the point of view of embedding the Randall-Sundrum model into a more fundamental theory, such as string/M theory, one may expect that a more complete version of this scenario must admit the presence of additional extra dimensions compactified on an internal manifold. From a phenomenological point of view, higher dimensional theories have a richer geometrical and topological structure. The consideration of more general spacetimes may provide interesting extensions of the Randall-Sundrum mechanism for the geometric origin of the hierarchy. More extra dimensions also relax the fine-tunings of the fundamental parameters.

Motivated by the problems of the radion stabilization and the generation of cosmological constant, the role of quantum effects in braneworlds has attracted a great deal of attention. In models with compact dimensions, the periodicity conditions imposed on the operator of a quantum field lead to a number of interesting physical effects that include topological mass generation, instabilities in interacting field theories and symmetry breaking. The periodicity conditions modify the spectrum of the zero-point fluctuations, as a result the vacuum energy density and the stresses are changed. This is the well-known topological Casimir effect. It has been investigated for large number of geometries and has important implications on all scales, from mesoscopic physics to cosmology (for reviews see [3]). The vacuum energy depends on the size of extra dimensions and this provides a stabilization mechanism for moduli fields in Kaluza-Klein-type models and in braneworld scenario. In particular, motivated by the problem of the radion stabilization in Randall-Sundrum-type braneworlds, the investigations of the Casimir energy on AdS bulk have attracted a great deal of attention. The Casimir effect in AdS spacetime with compact internal spaces has been considered in [4]. The vacuum energy generated by the compactification of extra dimensions can also serve as a model of dark energy needed for the explanation of the present accelerated expansion of the universe.

For charged fields, another important local characteristic of the vacuum state is the expectation value of the current density. In the present paper we investigate the current density for a charged scalar field in AdS spacetime, covered by Poincaré coordinates, assuming that a part of spatial dimensions are compactified to a torus. In addition, we assume the presence of a brane parallel to the AdS boundary and a constant gauge field. The VEV of the current density for a fermionic field in flat spaces with toral dimensions has been investigated in [5]. The finite temperature effects on the current densities for scalar and fermionic fields in topologically nontrivial spaces have been studied in [6]. The VEV of the current density for charged scalar and Dirac spinor fields in de Sitter spacetime with toroidally compact spatial dimensions are considered in [7]. The case of AdS background has been considered in [8]. The influence of boundaries on the vacuum currents in topologically nontrivial flat spaces are studied in [9, 10] for scalar and fermionic fields. The effects induced in AdS bulk with toral dimensions by branes were studied in [11, 12].

2. Geometry and the field content

In Poincaré coordinates, the line element for $(D+1)$-dimensional AdS spacetime is given by

$$ds^2 = e^{-2y/a}\eta_{ik}dx^i dx^k - dy^2,\ i,k = 0,1,\dots,D-1, \tag{2.1}$$

where a is the AdS curvature radius, $\eta_{ik} = \mathrm{diag}(1,-1,\dots,-1)$ is the metric tensor for D-dimensional Minkowski spacetime and $-\infty < y < +\infty$. The coordinates (x^i, y) cover a part of the AdS manifold and there is a horizon corresponding to the hypersurface $y = +\infty$. In what follows we assume that the coordinates x^l, with $l = p+1,\dots,D-1$, are compactified to circles with the lengths L_l, so $0 \leqslant x^l \leqslant L_l$. For the remaining coordinates x^l, with $l = 1,2,\dots,p$, one has $-\infty < x^l < +\infty$. Hence, the subspace perpendicular to the y-axis has a topology $R^p \times T^q$, where $q+p = D-1$ and T^q stands for a q-dimensional torus. Introducing a new coordinate z, $0 \leqslant z < \infty$, in accordance with the relation $z = ae^{y/a}$, the line element is presented in a conformally-flat form: $ds^2 = (a/z)^2\eta_{\mu\nu}dx^\mu dx^\nu$. In terms of the new coordinate, the AdS boundary and horizon are presented by the hypersurfaces $z = 0$ and $z = \infty$, respectively. Note that, for an observer with a fixed value of z, the proper length of the lth compact dimension is given by $L_{(p)l} = aL_l/z$ and it decreases with increasing z.

The physical quantity we are interested in is the vacuum expectation value (VEV) of the current density

$$j_\mu(x) = ie[\varphi^+(x)D_\mu\varphi(x) - (D_\mu\varphi^+(x))\varphi(x)], \tag{2.2}$$

for a charged scalar field, $\varphi(x)$, in the presence of an external classical gauge field A_μ. In (2.2), $D_\mu = \nabla_\mu + ieA_\mu$, where ∇_μ is the standard covariant derivative operator associated with the metric tensor $g_{\mu\nu}$ and e is the charge of the field quanta. The equation for the field operator reads

$$\left(g^{\mu\nu}D_\mu D_\nu + m^2 + \xi R\right)\varphi(x) = 0, \tag{2.3}$$

with ξ being the curvature coupling parameter and for the Ricci scalar one has $R = -D(D+1)/a^2$. In the most important special cases of minimally and conformally coupled fields $\xi = 0$ and $\xi = (D-1)/(4D)$, respectively. In models with nontrivial topology, in addition to the field equation, we need also to specify the periodicity conditions along compact dimensions. Here we impose quasiperiodicity conditions

$$\varphi(\dots,x^l + L_l,\dots) = e^{i\alpha_l}\varphi(\dots,x^l,\dots),\ l = p+1,\dots,D-1, \tag{2.4}$$

with constant phases α_l. The special cases of untwisted and twisted scalars correspond to $\alpha_l = 0$ and $\alpha_l = \pi$, respectively. In what follows we assume that the gauge field is constant, $A_\mu = \mathrm{const}$. Though the corresponding field strength vanishes, the nontrivial topology of the background space gives rise to Aharaonov-Bohm-like effects on the VEVs of physical observables.

3. Vacuum current density

First we consider the vacuum currents in the absence of branes. The corresponding VEVs are evaluated by making use of a complete set of modes for the field under consideration obeying the periodicity conditions (2.4). The charge density and the components along uncompactified dimensions vanish: $\langle j^l \rangle_0 = 0$, $l = 0, \ldots, p, D$. For the component of the current density along the lth compact dimension one gets

$$\langle j^l \rangle_0 = \frac{4ea^{-1-D}L_l}{(2\pi)^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \cos(\sum_{i\neq l} \tilde{\alpha}_i n_i) q_{\nu-1/2}^{(D+1)/2}(1+\sum_i n_i^2 L_i^2/(2z^2)), \tag{3.1}$$

where $\mathbf{n}_{q-1} = (n_{p+1}, \ldots, n_{l-1}, n_{l+1}, \ldots, n_{D-1})$,

$$\begin{aligned} \tilde{\alpha}_l &= \alpha_l + eA_l L_l, \\ \nu &= \sqrt{D^2/4 - D(D+1)\xi + m^2 a^2}, \end{aligned} \tag{3.2}$$

and the summation goes over $-\infty < n_i < +\infty$, $i \neq l$. In (3.1), $q_\alpha^\mu(x) = e^{-i\pi\mu}(x^2-1)^{-\mu/2} Q_\alpha^\mu(x)$, with $Q_\alpha^\mu(x)$ being the associated Legendre function of the second kind. The phases α_l and the components A_l of the vector potential along compact dimensions enter in the expressions for the VEVs in the form of $\tilde{\alpha}_l$. Note that $eA_l L_l = -2\pi\Phi_l/\Phi_0$, where Φ_l is the magnetic flux enclosed by the lth compact dimension and $\Phi_0 = 2\pi/e$ is the flux quantum. As is seen from (3.1), the current density along the lth compact dimension is an odd periodic function of the phase $\tilde{\alpha}_l$ and an even periodic function of the phases $\tilde{\alpha}_i$, $i \neq l$. In both cases the period is equal to 2π. In particular, the current density is a periodic function of the magnetic fluxes with the period equal to the flux quantum.

For the charge flux through the $(D-1)$-dimensional spatial hypersurface $x^l = \text{const}$ one has $n_l \langle j^l \rangle_0$, where $n_l = a/z$ is the corresponding normal. The VEV $n_l \langle j^l \rangle_0$ depends on the coordinate lengths of the compact dimensions L_i and on the coordinate z in the form of the ratio L_i/z. The latter is the proper length of the compact dimension measured in units of the curvature radius a.

For a conformally coupled massless field one has $\nu = 1/2$ and for the current density we get $\langle j^l \rangle = (z/a)^{D+1} \langle j^l \rangle_M^{(b)}$, where

$$\begin{aligned} \langle j^l \rangle_{\mathrm{M}}^{(b)} &= 2eL_l \frac{\Gamma((D+1)/2)}{\pi^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \cos(\sum_{i\neq l} \tilde{\alpha}_i n_i) \\ &\times \left[(\sum_i n_i^2 L_i^2)^{-(D+1)/2} - (\sum_i n_i^2 L_i^2 + 4z^2)^{-(D+1)/2} \right], \end{aligned} \tag{3.3}$$

is the current density for a massless scalar field in Minkowski spacetime with toroidally compactified dimensions, in the presence of Dirichlet boundary at $z = 0$. The part with the first term in square brackets corresponds to the VEV in the boundary-free Minkowski spacetime. It is obtained from the general result of [6] in the zero mass limit. The appearance of the boundary-induced term in (3.3) is related to the boundary conditions imposed on the field operator on AdS boundary.

The Minkowskian limit corresponds to the limiting transition $a \to \infty$ for a fixed value of the coordinate y. In this limit one has $\nu \approx ma \gg 1$ and $z \approx a + y$. To the leading order we get $\langle j^l \rangle_0 \approx$

$\langle j^l \rangle_{\rm M}$, where

$$\langle j^l \rangle_{\rm M} = \frac{4eL_l m^{D+1}}{(2\pi)^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \cos(\sum_{i\neq l} \tilde{\alpha}_i n_i) f_{(D+1)/2}(m(\sum_i n_i^2 L_i^2)^{1/2}), \tag{3.4}$$

is the VEV of the current density in Minkowski spacetime with toroidally compactified dimensions [6]. In (3.4) we have used the notation $f_\nu(x) = K_\nu(x)/x^\nu$ with $K_\nu(x)$ being the MacDonald function.

If the proper length of the one of the compact dimensions, say x^i, $i \neq l$, is large compared with the AdS curvature radius, $L_i/z \gg 1$, the dominant contribution into (3.1) comes from the $n_i = 0$ term and the contribution of the remaining terms is suppressed by the factor $(z/L_i)^{D+2\nu+2}$. To the leading order, we obtain the current density for the topology $R^{p+1} \times T^{q-1}$ with the uncompactified direction x^i. In the opposite limit corresponding to small proper length of the dimension x^i, $L_i/z \ll 1$, the behavior of the current density depends crucially on whether the phase $\tilde{\alpha}_i$ is zero or not. For $\tilde{\alpha}_i = 0$, to the leading order, for the combination $(aL_i/z)\langle j^l \rangle_0$ we obtain the expression which coincides with the formula for $\langle j^l \rangle_0$ in D-dimensional AdS spacetime, obtained from the geometry under consideration by excluding the dimension x^i. For $\tilde{\alpha}_i \neq 0$, to the leading order one gets

$$\langle j^l \rangle_0 \approx \frac{2eL_l \sigma_i^{(D-1)/2} \sin(\tilde{\alpha}_l)}{(2\pi)^{(D-1)/2} a^{D+1}} \frac{z^{D+1} e^{-L_l \sigma_i / L_i}}{(L_i L_l)^{(D+1)/2}}, \tag{3.5}$$

where $\sigma_i = \min(\tilde{\alpha}_i, 2\pi - \tilde{\alpha}_i)$, $0 \leqslant \tilde{\alpha}_i < 2\pi$. In this case the current density is exponentially small.

For large values of the proper length compared with the AdS curvature radius, $L_l/z \gg 1$ and for $\tilde{\alpha}_i = 0$, $i = p+1, \ldots, D-1$, $i \neq l$, to the leading order we have

$$\langle j^l \rangle_0 \approx \frac{4e\Gamma(p/2+\nu+2)}{\pi^{p/2+1}\Gamma(\nu+1)a^{D+1}V_q} \frac{z^{D+2\nu+2}}{L_l^{p+2\nu+2}} \sum_{n_l=1}^{\infty} \frac{\sin(\tilde{\alpha}_l n_l)}{n_l^{p+2\nu+3}}, \tag{3.6}$$

with the power-law decay as a function of L_l for both massless and massive fields. In this sense, the situation for the AdS bulk is essentially different from that in the corresponding problem for Minkowski background. For the latter, in the massless case and for large values of L_l the current density decays as $1/L_l^p$, whereas for a massive field the current is suppressed exponentially, by the factor e^{-mL_l}. This shows that the influence of the background gravitational field on the VEV is crucial. If $L_l/z \gg 1$ and at least one of the phases $\tilde{\alpha}_i$, $i \neq l$, is not equal to zero, one gets

$$\langle j^l \rangle_0 \approx \frac{2ea^{-1-D}}{\pi^{(p+1)/2}} \frac{\sin(\tilde{\alpha}_l) z^{D+2\nu+2}}{\Gamma(\nu+1)V_q e^{\beta_{q-1}L_l}} \frac{\beta_{q-1}^{(p+3)/2+\nu}}{(2L_l)^{(p+1)/2+\nu}}, \tag{3.7}$$

where $\beta_{q-1} = (\sum_{i=p+1,\neq l}^{D-1} \tilde{\alpha}_i^2 / L_i^2)^{1/2}$. In this case the current density, as a function of L_l, decays exponentially.

For small values of L_l, $L_l/z \ll 1$, the dominant contribution to the current density comes from the term with $\mathbf{n}_{q-1} = 0$ and to the leading order we obtain

$$\langle j^l \rangle_0 \approx \frac{2e\Gamma((D+1)/2)}{\pi^{(D+1)/2}(a/z)^{D+1}L_l^D} \sum_{n_l=1}^{\infty} \frac{\sin(\tilde{\alpha}_l n_l)}{n_l^D}. \tag{3.8}$$

The right-hand side, multiplied by $(a/z)^{D+1}$, coincides with the VEV of the current density for a massless scalar field in $(D+1)$-dimensional Minkowski spacetime compactified along the direction x^l to the circle with the length L_l.

Near the AdS boundary, $z \to 0$, from the general expression (3.1), to the leading order we get

$$\langle j^l \rangle_0 \approx \frac{4eL_l\Gamma(\nu+D/2+1)}{\pi^{D/2}\Gamma(\nu+1)a^{D+1}} z^{D+2\nu+2} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \frac{\cos(\sum_{i\neq l}\tilde{\alpha}_i n_i)}{(\sum_i n_i^2 L_i^2)^{D/2+\nu+1}}, \tag{3.9}$$

which shows that the current density vanishes on the AdS boundary. Near the horizon one has $z \to \infty$, and one finds $\langle j^l \rangle_0 \approx (z/a)^{D+1} \langle j^l \rangle_M$, where $\langle j^l \rangle_M$ is the corresponding current density in Minkowski spacetime for a massless scalar field. The latter is directly obtained from (3.4) taking the limit $m \to 0$.

In what follows, the numerical results are presented for the $D = 4$ model with a single compact dimension with the length L. The corresponding value of the phase we will denote by $\tilde{\alpha}$. In figure 1 we have displayed the quantity $a^D n_l \langle j^l \rangle_0 / e$ as a function of the phase in the periodicity condition for $ma = 0.5$. The numbers near the curves correspond to the values of the ratio z/L and the full/dashed curves are for minimally/conformally coupled fields.

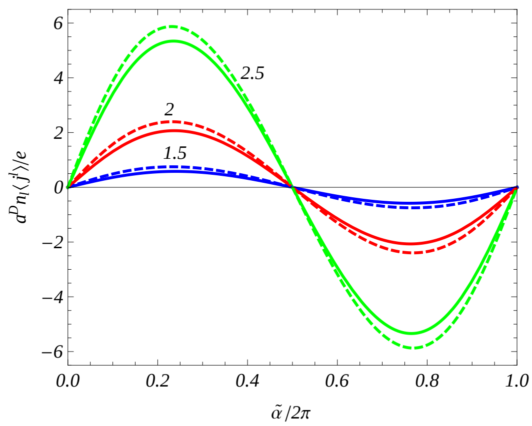

Figure 1: The quantity $a^D n_l \langle j^l \rangle / e$ as a function of $\tilde{\alpha}$ for $D = 4$ AdS space with a single compact dimension. The numbers near the curves correspond to the values of the ratio z/L and the full/dashed curves are for minimally/conformally coupled fields.

For the same model with $D = 4$, in figure 2 we have plotted the ratio of the current densities in AdS and Minkowski bulks for the same proper lengths of the compact dimension, $L_{(p)}$, as a function of the proper length measured in units of the AdS curvature radius. The current density in Minkowski bulk is given by the right-hand side of (3.4), specified to the special case under consideration. The graphs are plotted for $\tilde{\alpha} = \pi/2$ and the numbers near the curves are the corresponding values of the parameter ma (mass measured in units of the AdS energy scale). As before the full and dashed curves correspond to minimally and conformally coupled fields. We see the feature already described before: for a massive field and for large values of the proper length the decay of the current density in the Minkowski bulk is stronger than that for AdS background.

4. Vacuum currents induced by a brane

In this section we consider the effects induced by a brane parallel to the AdS boundary and located at $y = y_0$. The corresponding value for the conformal coordinate z will be denoted by

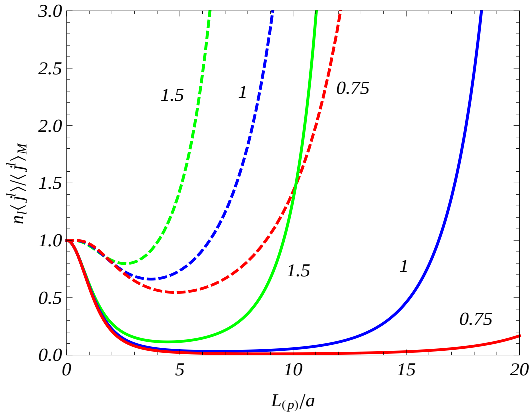

Figure 2: The ratio of the current densities in AdS and Minkowski backgrounds as a function of the proper length of the compact dimension. The numbers near the curves correspond to the values of *ma* and we have taken $\tilde{\alpha} = \pi/2$. The full and dashed curves are for minimally and conformally coupled fields.

$z_0 = ae^{y_0/a}$. On the brane we assume a gauge invariant boundary condition of the Robin type:

$$(1+\beta n^\mu D_\mu)\varphi(x) = 0, \quad y = y_0, \tag{4.1}$$

where β is a constant with the dimension of length and n^μ is the inward pointing normal to the brane. For the latter one has $n^\mu = \delta^\mu_D$ in the region $y > y_0$ and $n^\mu = -\delta^\mu_D$ in the region $y < y_0$. The Robin boundary condition is a generalization of Dirichlet and Neumann conditions and naturally appears in a number of physical problems, including those in branworld scenario. In the presence of the brane, the component of the current density along the lth compact dimension can be decomposed as

$$\langle j^l\rangle = \langle j^l\rangle_0 + \langle j^l\rangle_b,\ l = p+1,\dots,D-1, \tag{4.2}$$

where $\langle j^l\rangle_0$ is the corresponding VEV in the absence of the brane and the part $\langle j^l\rangle_b$ is induced by the brane. We consider the brane-induced contribution in the VEVs of the current density for the regions $y > y_0$ (R-region) and $y < y_0$ (L-region) separately.

4.1 R-region

In the R-region the brane-induced contribution in (4.2) is given by the expression

$$\langle j^l\rangle_b = -\frac{eC_p z^{D+2}}{2^{p-1}a^{D+1}V_q}\sum_{\mathbf{n}_q} k_l \int_{k_{(q)}}^{\infty} dx\, x(x^2-k_{(q)}^2)^{\frac{p-1}{2}}\frac{\bar{I}_\nu(z_0x)}{\bar{K}_\nu(z_0x)}K_\nu^2(zx), \tag{4.3}$$

where $\mathbf{n}_q = (n_{p+1},\dots,n_{D-1})$, $-\infty < n_i < +\infty$, $k_{(q)}^2 = \sum_{i=p+1}^{D-1}(2\pi n_i+\tilde{\alpha}_i)^2/L_i^2$, $I_\nu(x)$ is the modified Bessel function and

$$C_p = \frac{\pi^{-(p+1)/2}}{\Gamma((p+1)/2)}. \tag{4.4}$$

Here and in what follows, for a given function $F(x)$, the notation with the bar is defined in accordance with

$$\bar{F}(x) = \left(1+\delta_y\frac{D\beta}{2a}\right)F(x) + \delta_y\frac{\beta}{a}xF'(x), \tag{4.5}$$

where $\delta_y = 1$ for the R-region and $\delta_y = -1$ for the L-region. Similar to the case of the brane-free part, the brane-induced contribution to the current density along the lth compact dimension is an odd periodic function of the phase $\tilde{\alpha}_l$ with the period 2π and an even periodic function of the remaining phases $\tilde{\alpha}_i$, $i \neq l$, with the same period. In particular, the VEV of the current density is a periodic function of the magnetic flux with the period equal to the flux quantum. The brane-induced contribution to the charge flux through the $(D-1)$-dimensional spatial hypersurface $x^l = \text{const}$, $n_l \langle j^l \rangle_b$, depends on the lengths of compact dimensions and on the coordinate z in the form of the ratios L_i/z_0 and z/z_0. The latter is expressed in terms of the proper distance from the brane, $y-y_0$, as $z/z_0 = e^{(y-y_0)/a}$.

First of all, it can be seen that in the flat spacetime limit, corresponding to $a \to \infty$ for fixed values of y and y_0, from (4.3) we obtain the boundary-induced part of the current density for the geometry of a single Robin plate in $(D+1)$-dimensional Minkowski spacetime with spatial topology $R^{p+1} \times T^q$ (see [10]). For a conformally coupled massless field the modified Bessel functions in (4.3) are expressed in terms of the elementary functions. In this case the expression for the total current density takes the form

$$\langle j^l \rangle = (z/a)^{D+1} \left[\langle j^l \rangle_{\text{M}} + \frac{eC_p}{2^p V_q} \sum_{\mathbf{n}_q} k_l \int_{k_{(q)}}^{\infty} dx \, (x^2 - k_{(q)}^2)^{\frac{p-1}{2}} e^{-2x(z-z_0)} \frac{\beta_{\text{M}}^{+} x + 1}{\beta_{\text{M}}^{+} x - 1} \right], \tag{4.6}$$

where

$$\beta_{\text{M}}^{\pm} = \frac{\beta z_0/a}{1 \pm (D-1)\beta/(2a)}. \tag{4.7}$$

The right-hand side of (4.6), divided by the conformal factor $(z/a)^{D+1}$, coincides with the current density in the corresponding problem on Minkowski bulk with the plate at $z = z_0$ on which the field obeys the Robin boundary condition (4.1) with the replacement $\beta \to \beta_{\text{M}}^{+}$.

At large distances from the brane compared with the AdS curvature radius, $y - y_0 \gg a$, one has $z \gg z_0$. In addition, assuming that $z \gg L_i$, we can see that the dominant contribution to the integral in (4.3) comes from the region near the lower limit and the contribution of the mode with a given $\mathbf{n}_q$ is suppressed by the factor $e^{-2zk_{(q)}}$. Under the condition $|\tilde{\alpha}_i| < \pi$, assuming that all the lengths L_i are of the same order, the main contribution comes from the term with $n_i = 0$, $i = p+1, \dots, D-1$, and to the leading order we find

$$\langle j^l \rangle_b \approx -\frac{e z^{D-(p-1)/2} \tilde{\alpha}_l k_{(q)}^{(0)(p-1)/2}}{2^{p+1} \pi^{(p-1)/2} a^{D+1} V_q L_l} \frac{\bar{I}_\nu(z_0 k_{(q)}^{(0)})}{\bar{K}_\nu(z_0 k_{(q)}^{(0)})} e^{-2zk_{(q)}^{(0)}}, \tag{4.8}$$

where $k_{(q)}^{(0)2} = \sum_{i=p+1}^{D-1} \tilde{\alpha}_i^2 / L_i^2$. This asymptotic corresponds to points near the AdS horizon. In this limit, for the boundary-free part one has $\langle j^l \rangle_0 \approx (z/a)^{D+1} \langle j^l \rangle_{\text{M}}$, where $\langle j^l \rangle_{\text{M}}$ is the current density for a massless scalar field in $(D+1)$-dimensional Minkowski spacetime with spatial topology $R^{p+1} \times T^q$ and with the lengths of the compact dimensions L_i, $i = p+1, \dots, D-1$. From here we conclude that near the horizon the boundary-free part dominates in the total VEV.

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For fixed values of z and L_i, when the location of the brane tends to the AdS boundary, $z_0 \to 0$, to the leading order, from (4.3) one finds

$$\langle j^l \rangle_b \approx -\frac{4eC_p z^{D+2} z_0^{2\nu}}{2^{2\nu+p}\nu\Gamma^2(\nu)a^{D+1}V_q}\frac{a+(D/2+\nu)\beta}{a+(D/2-\nu)\beta}\sum_{\mathbf{n}_q} k_l k_{(q)}^{2\nu+p+1} \times \int_1^\infty dx x^{2\nu+1}(x^2-1)^{(p-1)/2}K_\nu^2(zk_{(q)}x), \tag{4.9}$$

and the VEV vanishes as $z_0^{2\nu}$.

Now, let us consider the limit when the length of the lth dimension is much smaller than the lengths of the other compact dimensions, $L_l \ll L_i$. In this case, in (4.3) the dominant contribution to the sum over n_i, $i = p+1,\ldots,D-1$, $i \neq l$, comes from large values of $|n_i|$ and we can replace the summation by the integration. As a result, to the leading order we get

$$\langle j^l \rangle_b \approx -\frac{eC_{D-2}z^{D+2}}{2^{D-3}a^{D+1}L_l}\sum_{n_l=-\infty}^{+\infty} k_l \int_{|k_l|}^\infty dx x(x^2-k_l^2)^{\frac{D-3}{2}}\frac{\bar{I}_\nu(z_0x)}{\bar{K}_\nu(z_0x)}K_\nu^2(zx), \tag{4.10}$$

with $k_l = (2\pi n_l + \tilde{\alpha}_l)/L_l$. The expression in the right-hand side coincides with the brane-induced contribution in the model with a single compact dimension of the length L_l ($q = 1$, $p = D-2$). If in addition to $L_l \ll L_i$ one has $L_l \ll z_0$, the arguments z_0x of the modified Bessel functions in (4.10) are large. By using the corresponding asymptotic expressions, after the integration over x we find

$$\langle j^l \rangle_b \approx \frac{(1-2\delta_{0\beta})\,e(z/a)^{D+1}}{2^{D-2}\pi^{D/2}L_l\,(z-z_0)^{D/2-1}}\sum_{n_l=-\infty}^{+\infty} k_l|k_l|^{D/2-1}K_{D/2-1}(2\,(z-z_0)\,|k_l|). \tag{4.11}$$

Here, for non-Dirichlet boundary conditions we have assumed that $|\beta|/a \gg L_l/z_0$. From (4.11) it follows that the brane-induced contribution is located near the brane within the region $z-z_0 \lesssim L_l$ and has opposite signs for Dirichlet and non-Dirichlet boundary conditions. At distances $z-z_0 \gg L_l$ it is suppressed by the factor $e^{-2(z-z_0)\tilde{\alpha}_l/L_l}$.

The VEV of the current density is finite on the brane. For Dirichlet boundary condition both the current density and its normal derivative vanish on the brane. The finiteness of the current density is in clear contrast to the behavior of the VEVs for the field squared and the energy-momentum tensor which suffer surface divergences. In quantum field theory the ultraviolet divergences in the VEVs of physical observables bilinear in the field are determined by the local geometrical characteristics of the bulk and boundary. On the background of standard AdS geometry with non-compact dimensions the VEV of the current density in the problem under consideration vanishes by the symmetry. The compactification of the part of spatial dimensions to q-dimensional torus does not change the local bulk and boundary geometries and, consequently, does not add new divergences to the expectation values compared with the case of trivial topology.

In figure 3, for the $D = 4$ model with a single compact dimension of the length L, we have depicted the current density for a minimally coupled field as a function of the phase $\tilde{\alpha}$ for fixed values of the parameters $z_0/L = 1$, $z/z_0 = 1.2$. The graphs are plotted for Dirichlet (D), Neumann (N) and for Robin (with $\beta/a = -1$, the number near the curve) boundary conditions. The dashed curve presents the current density in the same model when the brane is absent. As is seen, depending on the boundary condition, the presence of the brane leads to the increase or decrease of the current density.

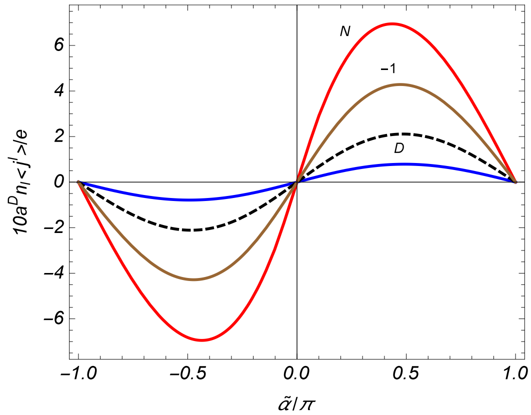

Figure 3: The VEV of the current density as a function of the phase in the periodicity condition for $D = 4$ AdS space with a single compact dimension and for Dirichlet, Neumann and Robin ($\beta/a = -1$) boundary conditions. The graphs are plotted for $z_0/L = 1$, $z/z_0 = 1.2$.

Figure 4 presents the ratio $n_l \langle j^l \rangle / \langle j^l \rangle_{\rm M}$ as a function of z/z_0 in the case of Robin boundary condition for several values of β/a (numbers near the curves). Here,

$$\langle j^l \rangle_{\rm M} = \frac{2e\Gamma((D+1)/2)}{\pi^{(D+1)/2}(aL/z)^D} \sum_{n=1}^{\infty} \frac{\sin(\tilde{\alpha} n)}{n^D}, \tag{4.12}$$

is the current density for a massless scalar field in $(D+1)$-dimensional Minkowski spacetime with topology $R^{D-1} \times S^1$ and with the length of the compact dimension aL/z. Note that the latter is the proper length of the compact dimension in AdS spacetime measured by an observer with a given z. The graphs are plotted for $\tilde{\alpha} = \pi/2$ and $z_0/L = 1$.

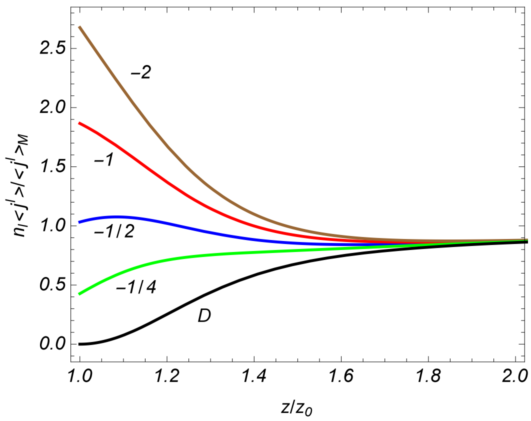

Figure 4: The dependence of the quantity $n_l \langle j^l \rangle / \langle j^l \rangle_{\rm M}$ on z/z_0 for Robin boundary conditions. The graphs are plotted for $\tilde{\alpha} = \pi/2$, $z_0/L = 1$ and the numbers near the curves correspond to the values of β/a.

From the results in this section we can obtain the current density in Z_2-symmetric braneworld models of the Randall–Sundrum type with a single brane. In the original Randall–Sundrum 1-brane model [2] the universe is realized as a Z_2-symmetric positive tension brane in 5-dimensional

AdS spacetime. However, most scenarios motivated from string theories predict the presence of small compact dimensions originating from 10D string backgrounds. In a generalized $(D+1)$-dimensional version of the Randall-Sundrum 1-brane model the line element is given by (2.1) with the warp factor $e^{-|y-y_0|/a}$ where y_0 is the location of the brane. The background geometry contains two patches $y > y_0$ of the AdS glued by the brane and related by the Z_2-symmetry identification $y - y_0 \longleftrightarrow y_0 - y$. The expressions for the VEV of the current density in the generalized Randall-Sundrum 1-brane model with compact dimensions are obtained from those given above with an additional factor $1/2$ and with the Robin coefficient $\beta = -1/(c + 2D\xi/a)$ for untwisted fields and with $\beta = 0$ for twisted fields. Here c is the so-called brane mass term for a scalar field.

4.2 L-region

Now we turn to the current density in the L-region. It is decomposed as (4.2) with the brane-induced part

$$\langle j^l \rangle_b = -\frac{eC_p z^{D+2}}{2^{p-1} a^{D+1} V_q} \sum_{\mathbf{n}_q} k_l \int_{k_{(q)}}^{\infty} dx\, x (x^2 - k_{(q)}^2)^{\frac{p-1}{2}} \frac{\bar{K}_\nu(z_0 x)}{I_\nu(z_0 x)} I_\nu^2(zx). \tag{4.13}$$

Note that the expressions in the R- and L-regions are related by the replacements $I_\nu \rightleftarrows K_\nu$. For large values of the AdS curvature radius a, we can see the limiting transition to the corresponding formula for a plate in Minkowski bulk.

For a conformally coupled massless field, the expression of the total current density takes the form

$$\begin{aligned} \langle j^l \rangle &= (z/a)^{D+1} \left\{ \langle j^l \rangle_{\mathrm{M}} - \frac{eC_p}{2^p V_q} \sum_{\mathbf{n}_q} k_l \int_{k_{(q)}}^{\infty} dx \right. \\ &\quad \left. \times (x^2 - k_{(q)}^2)^{\frac{p-1}{2}} \left[e^{-2zx} + \frac{4\sinh^2(zx)}{\frac{1-\beta_{\mathrm{M}}^- x}{1+\beta_{\mathrm{M}}^- x} e^{2z_0 x} - 1} \right] \right\}, \end{aligned} \tag{4.14}$$

with β_{M}^- defined by (4.7). Here, the first term in the figure braces and the part with the first term in the square brackets come from $\langle j^l \rangle_0$. The expression on the right of (4.14), divided by the conformal factor $(z/a)^{D+1}$, coincides with the current density in the region between two plates on Minkowski bulk with Dirichlet boundary condition on the left plate and Robin condition (4.1), with $\beta \to \beta_{\mathrm{M}}^-$, on the right one (see [10] for the problem with Robin boundary conditions on both the plates). The fact that the problem with a single brane in AdS bulk in the L-region is conformaly related to the problem with two plates in Minkowski bulk is a consequence of the boundary condition we have imposed on the AdS boundary.

Near the AdS boundary, $z \to 0$, to the leading order, we get

$$\begin{aligned} \langle j^l \rangle_b &\approx -\frac{2^{1-2\nu-p} eC_p z^{D+2\nu+2}}{a^{D+1} V_q \Gamma^2(\nu+1)} \sum_{\mathbf{n}_q} k_l k_{(q)}^{2\nu+p+1} \\ &\quad \times \int_1^{\infty} dx\, x^{2\nu+1} (x^2-1)^{(p-1)/2} \frac{\bar{K}_\nu(z_0 k_{(q)} x)}{I_\nu(z_0 k_{(q)} x)}, \end{aligned} \tag{4.15}$$

and the brane-induced contribution vanishes as $z^{D+2\nu+2}$. Recall that near the AdS boundary the part $\langle j^l \rangle_0$ in the VEV of the current density behaves in a similar way and, hence, on the AdS boundary the ratio of the brane-induced and boundary-free contributions tend to a finite limiting value.

In the limit when the brane tends to the AdS horizon, $z_0 \to \infty$, the dominant contribution to the integral in (4.13) comes from the region near the lower limit. To the leading order one has

$$\langle j^l \rangle_b \approx \frac{(1-2\delta_{0\beta}) e\tilde{\alpha}_l z^{D+2} e^{-2z_0 k_{(q)}^{(0)}}}{2^p \pi^{(p-1)/2} a^{D+1} V_q L_l z_0^{(p+1)/2}} k_{(q)}^{(0)(p+1)/2} I_\nu^2 (z k_{(q)}^{(0)}). \tag{4.16}$$

This shows that, for a fixed value of z, when the brane location tends to the AdS horizon, the brane-induced contribution is exponentially suppressed.

If the length of the lth dimension is much smaller than the lengths of the remaining compact dimensions, $L_l \ll L_i$, to the leading order, the brane-induced contribution coincides with the corresponding quantity in the model with a single compact dimension of the length L_l. The expression for the latter is obtained from the right-hand side of (4.10) by the replacements $I_\nu \rightleftarrows K_\nu$. If in addition $L_l \ll z$, the corresponding asymptotic expression is given by the right-hand side of (4.11) with $z_0 - z$ instead of $z - z_0$, and the brane-induced contribution is concentrated near the brane in the region $z_0 - z \lesssim L_l$. Similar to the VEV in the R-region, the current density is finite on the brane and vanishes on the brane for Dirichlet boundary condition.

Figure 5 displays the vacuum current density in the L-region for Dirichlet, Neumann and Robin (with $\beta/a = -1/2$) boundary conditions as a function of the phase in the quasiperiodicity condition along the compact dimensions. The graphs are plotted for $z_0/L = 1$, $z/z_0 = 0.8$. The dashed curve corresponds to the current density in the absence of the brane.

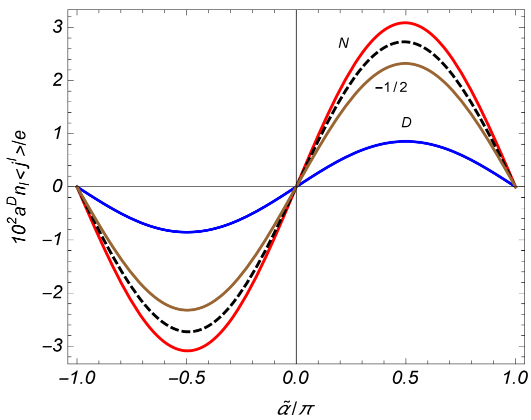

Figure 5: The current density in the L-region as a function of $\tilde{\alpha}$ for Dirichlet, Neumann and Robin (with $\beta/a = -1/2$) boundary conditions. The graphs are plotted for $z_0/L = 1$, $z/z_0 = 0.8$. The dashed curve corresponds to the current density in the geometry without the brane.

In figure 6, for $\tilde{\alpha} = \pi/2$, we show the dependence of the ratio $n_l \langle j^l \rangle / \langle j^l \rangle_{\rm M}$ on z/z_0 in the case of Robin boundary condition for several values of β/a (numbers near the curves). As it follows from the asymptotic (4.15), near the AdS boundary the charge flux density $n_l \langle j^l \rangle$ behaves as $z^{D+2\nu+1}$. For the Minkowskian VEV with the length of the compact dimension aL/z, equal to the proper length on the AdS bulk, one has $\langle j^l \rangle_{\rm M} \propto z^{D+1}$. Hence, the ratio plotted in figure 6 vanishes on the AdS boundary as $z^{2\nu}$.

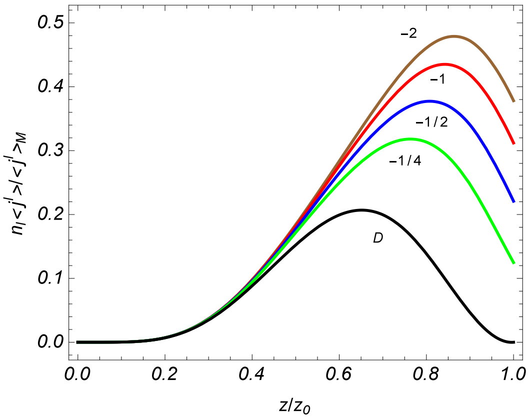

Figure 6: The ratio $n_l\langle j^l\rangle/\langle j^l\rangle_{\rm M}$ versus z/z_0 in the case of Robin boundary condition. The numbers near the curves correspond to the values of β/a. For the phase we have taken $\tilde{\alpha} = \pi/2$ and $z_0/L = 1$.

5. Conclusion

We have discussed combined effects of the gravitational field and topology on the VEV of the current density for a charged scalar field. In order to have an exactly solvable problem we have taken highly symmetric background geometry corresponding to a slice of AdS spacetime with a toroidally compactified subspace. Currently, the AdS spacetime is among the most popular backgrounds in gravitational physics and appears in a number of contexts. The vacuum charge density and the components of the current along uncompactified dimensions vanish and the current density along lth compact dimension is an odd periodic function of the phase $\tilde{\alpha}_l$ and an even periodic function of the phases $\tilde{\alpha}_i$, $i \neq l$. In particular, the current density is a periodic function of the magnetic fluxes with the period equal to the flux quantum. The appearance of the vacuum currents is an Aharonov-Bohm type effect and is closely related to the nontrivial topology of the background spacetime. In the presence of the brane, in both the R- and L-regions the VEV of the current density is decomposed into the boundary-free and brane-induced contributions. Both these contributions vanish on the AdS boundary. Near the horizon, the leading term in the asymptotic expansion of the current density is conformally related to the corresponding quantity on the Minkowski bulk with compact dimensions. From the results in the R-region we can obtain the vacuum currents in the generalized Randall-Sundrum 1-brane model with additional compact dimensions. We have shown that, depending on the value of the Robin coefficient, the presence of the brane can either increase or decrease the current density. In particular, in the examples considered, the modulus of the current density takes its minimal value for Dirichlet boundary condition. Note that the current density discussed above is a source of magnetic fields in the uncompactified subspace. In particular, they induce magnetic fields on the visible brane in braneworld models of the Randall-Sundrum type.

Acknowledgments

A. A. S. and N. A. S. were supported by the State Committee of Science Ministry of Education and Science RA, within the frame of Grant No. SCS 15T-1C110, and by the Armenian National

Science and Education Fund (ANSEF) Grant No. hepth-4172.

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