PROCEEDINGS OF SCIENCE

Quantum Statistics Parton Distribution Functions

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The isospin asymmetry, $\bar{d} - \bar{u} > 0$, has been since a long time interpreted as a consequence of Pauli principle. This implies a role of statistics for parton distributions, which is confirmed by the correlation between the shapes and the first moments of the valence partons. The successful description of both unpolarized and polarized distributions of the light partons with the statistical distributions is confirmed by the spin asymmetries measured at RHIC for W production with polarized beams and the statistical distributions describe the different shapes of the valence partons better than the standard empirical polynomial expressions.

Proceedings of the Corfu Summer Institute 2015 "School and Workshops on Elementary Particle Physics and Gravity"
1-27 September 2015
Corfu, Greece

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PoS(CORFU2015)055

1. DEDICATION

It is a great honor for us to dedicate this contribution to the memory of the great scientist Guido Altarelli, who left us September 30th 2015. In particular for one of us (F. B.), who happened to meet him almost 70 years ago, in the fall of 1946 the first day in the elementary school. The two young boys lived in the same street at adiacent buildings about hundred meters distant. after they were in the same class at the beginning of the secondary schools and attended the same classical liceum. Their thesis in physics discussed November 26th 1963, with Prof. Gatto as a tutor, gave rise to the analytical form in the ultra-relativistic emission for the single photon emission in electron-positron scattering [1] quoted in the book by Landau and his collaborators. They both had the privilege in Florence to work with their tutor. In the States Guido worked on light-cone physics with Richard Brandt and Giuliano Preparata and, back in Rome, in the group lead by Cabibbo. In Paris, in collaboration with Giorgio Parisi, was able to formulate the DGLAP equations [2] in a way, which allowed the experiments to confirm the scaling violations predicted by QCD. During the long period spent at CERN, he played a leading role in the study of the consequences of the standard model and of neutrino oscillations to look for physics beyond it. Together with Ferruccio Feruglio he wrote brilliant papers on the role of discrete groups in understanding the form of PMNS matrix [3] and with Meloni, a young physicist of the third University of Rome, an intriguing paper to account for strong CP and dark matter in the successful framework of unified gauge theories [4]. Also the research here described, carried out in collaboration with Claude Bourrely and Jacques Soffer and, at the beginning, with young physicist from Naples, should complete his fundamental work for parton distribution evolution by a proper parametrization of them at a definite Q^2.

2. PHENOMENOLOGICAL MOTIVATIONS TO INTRODUCE THE QUANTUM STATISTICAL PARTON MODEL

The isospin asymmetry in the sea of the proton ;

$$\bar{d}(x) > \bar{u}(x)$$

has been advocated many years ago by Niegawa and Sisiki and by Feynman and Field [5] as a consequence of Pauli principle and confirmed by the defect in the Gottfried sum rule [6, 7], and by the larger Drell-Yan [8] production of muon pairs in pn scattering than in pp scattering [9, 10]. The correlation between the first moments of the valence partons and the shapes of their x distributions is the one expected for a quantum gas: broader shapes for higher first moments, as it is shown by the approximate relationship [11]

$$\Delta u(x) = u(x) - d(x),$$

which follows from the assumption :

$$2u^{\downarrow}(x) = d(x)$$

and relates the dramatic decrease at high x of the ratio $\frac{F_2^n(x)}{F_2^p(x)}$ [12], which is a consequence of a similar behaviour of the ratio $\frac{d(x)}{u(x)}$, to the shape of $\Delta u(x)$, which gives the main contribution

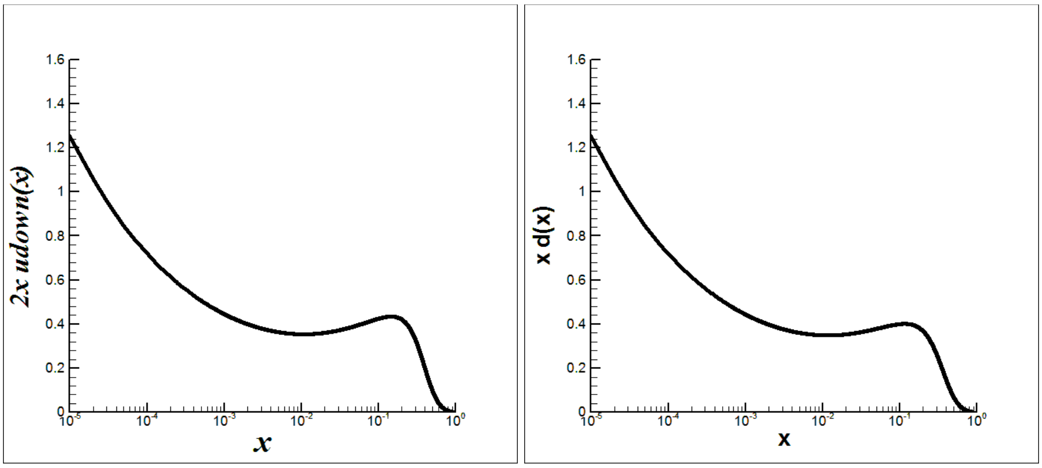

Figure 1: Our results at $Q^2 = 4(GeV)^2$ for $2xu^{\downarrow}(x)$ and $xd(x)$.

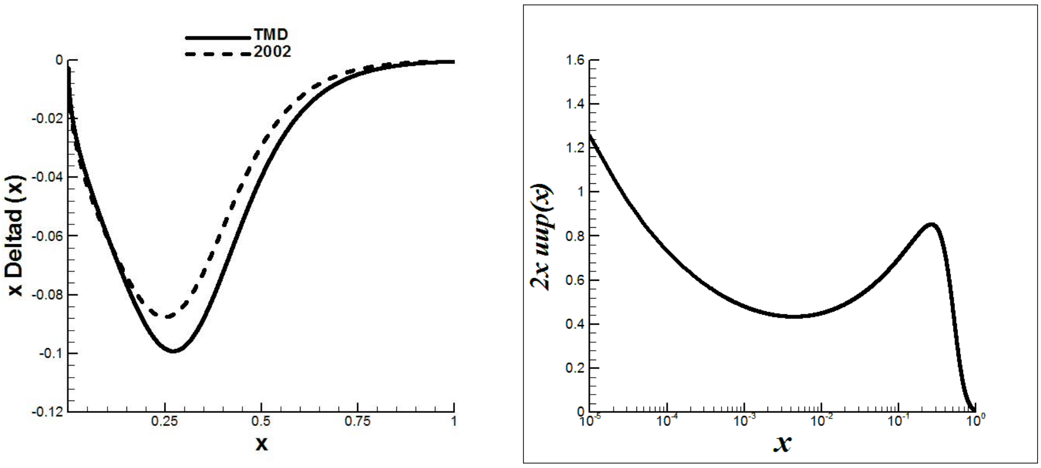

Figure 2: Our results at $Q^2 = 4(GeV)^2$ for $x\triangle d(x)$ and $2xu^{\uparrow}(x)$.

to $xg_1^p(x)$ The decreasing with x of the negative ratio $\frac{\Delta d(x)}{d(x)}$ is also expected within the statistical approach. The x dependance of $xg_1^n(x)$, negative at small x and positive at high x, follows from the different shapes and opposite signs of $\Delta u(x)$ and $\Delta d(x)$ [13, 14]. The role of Pauli principle suggests to write Fermi-Dirac functions for the quarks in the variable x, which is the one appearing in the parton model sum rules. Remember that the usual choice of the energy as the variable appearing in statistical mechanics follows from its appearing in the constraint, which fixes the total energy available.

$$\Sigma n_i \varepsilon_i = E$$

3. THE PARTON DISTRIBUTIONS PROPOSED IN 2002

After a first attempt [15] to describe parton distributions in the framework of quantum statistical mechanics, a progress has been achieved first by adding a diffractive term [16, 17], by imposing

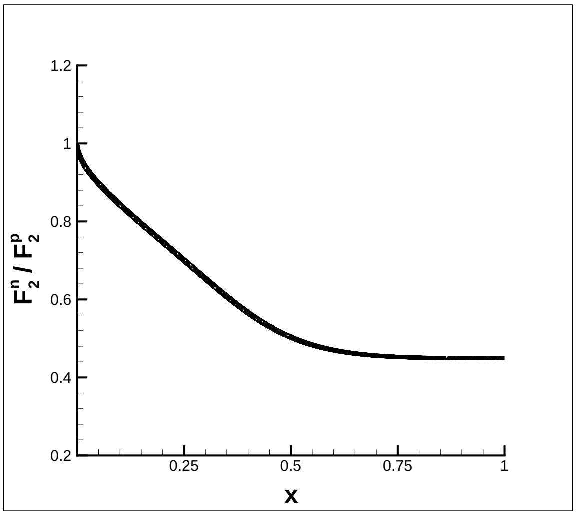

Figure 3: Our results at $Q^2 = 4(GeV)^2$ for $\frac{F_n^2(x)}{F_p^2(x)}$.

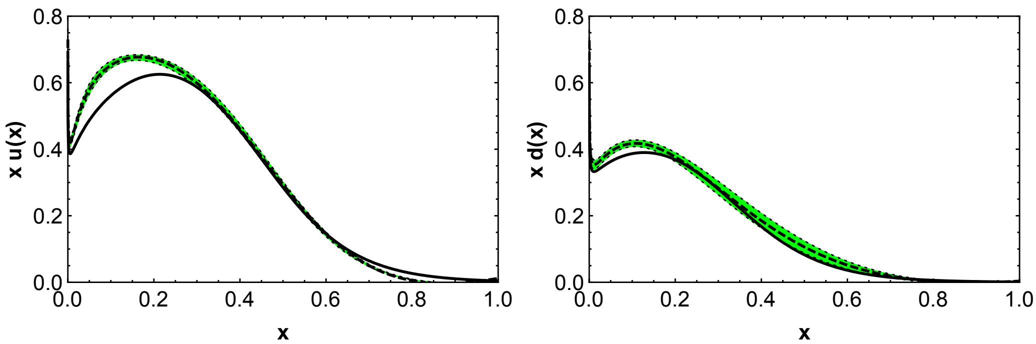

Figure 4: Distribution of light quarks (Solid line) in comparison to Hera (Dashed line) at $Q^2 = 4(GeV)^2$.

the QCD equilibrium conditions [18, 19], and finally by extending the role of quantum statistical mechanics to the transverse degrees of freedom [20] .

Let us begin to describe [19]. To costruct parton distributions $xq(x)$ able to reproduce data one had to modify the Fermi-Dirac function [19]:

$$\frac{1}{(\exp \frac{x-\tilde{X}_q}{\bar{x}} + 1)}$$

where $\bar{x}$ plays the role of the"temperature" and $\tilde{X}_q$ is the potential of the parton depending on its flavor and its helicity with the factor :

$$A\tilde{X}_q x^b$$

and add the diffractive contribution:

$$\frac{\tilde{A}x^{\tilde{b}}}{(\exp \frac{x}{\bar{x}} + 1)}$$

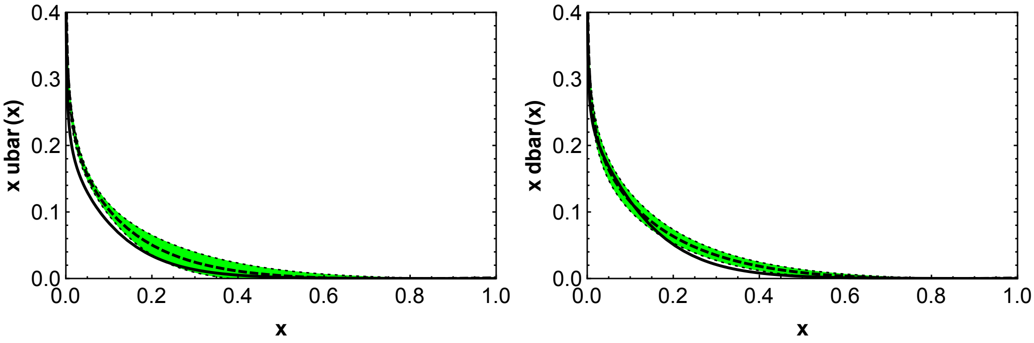

Figure 5: Distribution of light anti- quarks (Solid line) in comparison to Hera (Dashed line) at $Q^2 = 4(GeV)^2$.

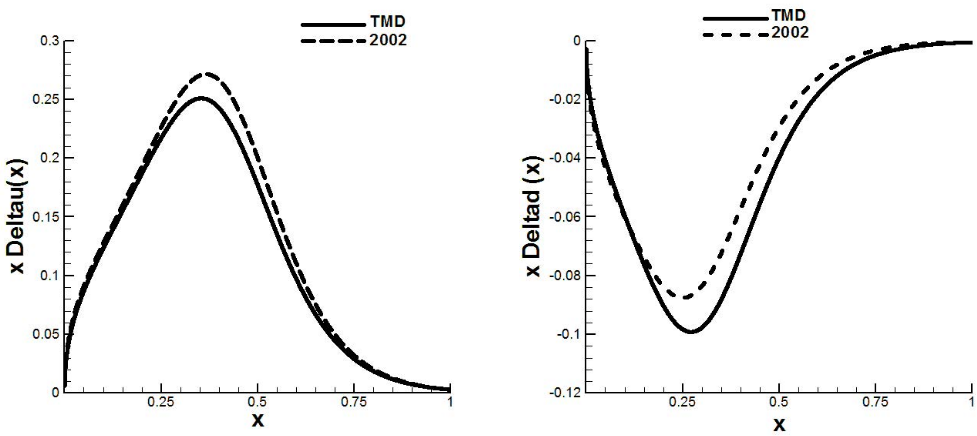

Figure 6: Distribution of $x\Delta u$ and $x\Delta d$ in comparison to 2002 result at $Q^2 = 4(GeV)^2$.

isoscalar and unpolarized to avoid an infinite contribution to the parton model sum rules, since $\tilde{b} = -0.25 < 0$.

For the light antiquarks we have the same diffractive contribution:

$$\frac{\tilde{A}x^{\tilde{b}}}{(\exp\frac{x}{\bar{x}} + 1)}$$

to be added to:

$$\frac{\bar{A}x^{2b}}{\tilde{X}_q} \times \frac{1}{(\exp\frac{x+\tilde{X}_q}{\bar{x}} + 1)}$$

where q and $\bar{q}$ have opposite helicities

Finally for the gluon we have the Planck form, namely a Bose-Einstein formula with vanishing potential:

$$xG(x) = \frac{A_G x^{b_G}}{(\exp\frac{x}{\bar{x}} - 1)}$$

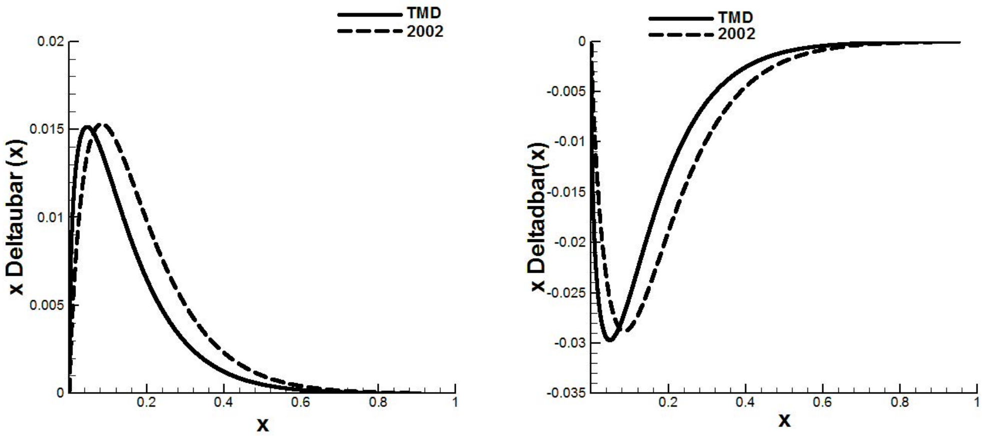

Figure 7: Distribution of $x\Delta\bar{u}$ and $x\Delta\bar{d}$ in comparison to 2002 result at $Q^2 = 4(GeV)^2$.

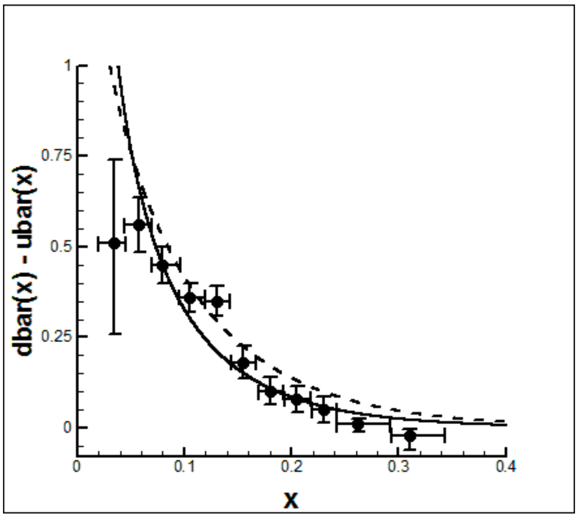

Figure 8: Distribution of $\bar{d}(x) - \bar{u}(x)$ in comparison to 2002 result and E866.

By requiring equilibrium for the two elementary QCD processes [18], the emission of a gluon by a fermion parton and the conversion of the gluon into a $q\bar{q}$ pair with opposite helicity, one has the important consequence to have a vanishing potential for the gluons of both helicities and opposite values for the potentials for a quark and its antiparticle with opposite helicity .

So the Bose-Einstein expression for the gluons $xG(x)$ turns into a Planck form :

$$\frac{1}{\exp(x/\bar{x}) - 1}$$

and $\Delta G(x) = 0$
while the relation:

$$\tilde{X}_q^h + \tilde{X}_{\bar{q}}^{-h} = 0$$

allows to disentangle the quark and antiquark contributions in the e. m. DIS.

While for the unpolarized distributions the disentangling is obtained from the obvious conditions :

$$u-\bar{u}=2$$
$$d-\bar{d}=1$$

for the polarized distributions the equilibrium conditions allow to determine the polarization of the light antiquarks from the knowledge of the shapes of the valence quark distributions.

4. THE EXTENSION TO THE TRANSVERSE DEGREES OF FREEDHOM AND THE TRANSVERSE ENERGY SUM RULE

In [19] to comply with data one introduced the "ad hoc" factors $\tilde{X}_q$ for the valence partons and guessed opposite factors for their antiparticles with opposite helicity.

The statistical model has been extended to the transverse degrees of freedhom [20] .

A crucial role to fix the p_T dependance is played by the sum rule on the transverse energy, the difference between the energy and the momentum.

The r. h. s. is P_0-P_z, which for $M \ll P_z$, is $\frac{M^2}{2P_z}$

The contribution of a light parton to the sum rule multiplied by $2P_z$ is

$$\frac{2P_zP_T^2}{p_0+p_z}=\frac{2P_T^2}{x+\sqrt{x^2+\frac{p_T^2}{P_z^2}}}$$

This implies the following dependance on p_T^2:

$$\frac{1}{\exp\left([\frac{2P_T^2}{\mu^2(x+\sqrt{x^2+\frac{p_T^2}{P_z^2}})]-\tilde{Y}_q}\right)+1}$$

which with the transformation:

$$P_T^2=\frac{\mu^2\eta(x+\sqrt{x^2+\frac{p_T^2}{P_z^2}})}{2}$$

gives rise to the integral in η of:

$$\frac{1+\frac{2\mu^2(1-x)}{Q^2}}{e^{(\eta-\tilde{Y}_q)}+1}$$

leading to:

$$\ln(1+\exp\tilde{Y}_q)+\frac{(1-x)2\mu^2}{Q^2}Polylog(-2,-\exp\tilde{Y}_q)$$

So we have instead of the factor $A\tilde{X}_q$, where the second term is absent in the polarized distribution as a consequence of the Wigner-Melosh rotation [21, 22].

$$xq(x) = \frac{A'x^b}{e^{(x-\tilde{X}_q^+)/\bar{x}}+1}\left[ln(1+e^{\tilde{Y}_q^+}) + \frac{2\mu^2(1-x)}{Q^2}(-Li_2(-e^{\tilde{Y}_q^+}))\right] + \frac{A'x^b}{e^{(x-\tilde{X}_q^-)/\bar{x}}+1}\left[ln(1+e^{\tilde{Y}_q^-}) + \frac{2\mu^2(1-x)}{Q^2}(-Li_2(-e^{\tilde{Y}_q^-}))\right]$$

$$x\Delta q(x) = \frac{A'x^b}{e^{(x-\tilde{X}_q^+)/\bar{x}}+1}\left[ln(1+e^{\tilde{Y}_q^+})\right] - \frac{A'x^b}{e^{(x-\tilde{X}_q^-)/\bar{x}}+1}\left[ln(1+e^{\tilde{Y}_q^-})\right]$$

By the substitution $u \to d$ one gets the distributions for d, while to get the ones for the anti-quarks one has to write $\bar{A}'$ instead of A and relate their potentials to the ones for the valence partons by the equilibrium conditions:

$$\tilde{X}_q^h + \tilde{X}_{\bar{q}}^{-h} = 0$$
$$\tilde{Y}_q^h + \tilde{Y}_{\bar{q}}^{-h} = 0$$

For the diffractive contributions one has the same form of [19] paper:

$$\frac{\tilde{A}x^{\tilde{b}}}{(\exp\frac{x}{\bar{x}}+1)}$$

5. THE COMPARISON WITH THE HERA FIT FOR THE LIGHT FERMION AND WITH NNPDF FOR THE GLUON DISTRIBUTION

Some years ago a joint analysis of the DIS data measured in the H_1 and $ZEUS$ [23] experiments has been performed to give the unpolarized parton distributions and Jacques Soffer immediately realized the similarity with the statistical distributions. To perform a check for the quantum statistical parton distributions [24] the parameters introduced in the statistical approach are fixed in order to reproduce the Hera result for the unpolarized distributions of the light parton fermions, while for the polarized ones the goal is to reproduce the expressions found in [19], which have been successful to describe the polarized structure functions $g_1^{p,d,He_3}(x)$ [19, 14, 25], and the production of the $W^{\pm}$ weak bosons [26, 28, 29]. In the Table we write in the first two columns the values found in [19] and [24], respectively, in the third one the coefficients obtained with the extension to the transverse momenta are compared with the "ad hoc" factors, X_q^h introduced in [19]; finally in the fourth one a recent evaluation [29] of the parameters of [19].

The good agreement between the numbers in the same row is a good point in favor of the statistical approach, since the numbers in the first column have been obtained before the measurements performed at HERA. The difference for b_G and $\tilde{A}$ is a consequence of the change of data in the small x region. In particular we have fixed $b_G = 1$, slightly larger than the value one should obtain from data, since it corresponds to the assumption that the hadrons in the deep inelastic regime are black body cavities for the chromomagnetic radiation. The slightly smaller value of $\bar{x}$ in the fourth

	2002	2014	$\frac{A'}{A} ln(1+\exp\tilde{Y}_q)$	Ref. [29]
$\tilde{X}_{u_+}$	0.46188	0.446	0.4650	0.475
$\tilde{X}_{d_-}$	0.30174	0.320	0.3115	0.309
$\tilde{X}_{u_-}$	0.29766	0.297	0.2975	0.307
$\tilde{X}_{d_+}$	0.22775	0.222	0.2345	0.245
b	0.40962	0.43		0.471
$\tilde{A}$	0.08318	0.070		
$\tilde{b}$	-0.25347	-0.240		
$\bar{x}$	0.09907	0.102		0.09

column may be the consequence of the different scale considered. Indeed, since we expect that the statistical description holds better at the scale Q_0^2 at the boundary between the non perturbative and the perturbative regime for the evolution, the choice of that value might be fixed by the requirement of $b_G = 1$ and just by chance the value of Q_0^2 chosen in [19] corresponds to a b_G, a value decreasing with the scale, near to 1. Indeed the evolution dictated by DGLAP equations [30] [2] spoils slowly the statistical form [31] and therefore the dependance on the chosen scale is not so relevant. As we expected, the biggest potential is $\tilde{X_u^\uparrow}$ and the smallest is $\tilde{X_d^\uparrow}$

The equilibrium conditions imply.

$$\Delta\bar{u}(x) > 0$$
$$\Delta\bar{d}(x) < 0$$

which is confirmed by the asymmetries in the production of $W^\pm$ [27] [28] [29] and implies a positive contribution to the Bjorken sum rule [32]. The nice property to relate the shapes and the first moments of d and u partons and automatically obtain the isospin asymmetry

$$\bar{d} - \bar{u} > 0$$

as expected by Pauli principle comes again twice for the polarized distributions .

The property of $\Delta u(x)$ to be positive and have its support mainly in the range:

$$\tilde{X_u^\downarrow}, \tilde{X_u^\uparrow}$$

while $\Delta d(x)$ is negative and has its support mainly in the range :

$$\tilde{X_d^\uparrow}, \tilde{X_d^\downarrow}$$

accounts for the shape of $xg_1^n(x)$, negative at small x and positive at large x.

Also it implies a positive value of $\Delta u(\bar{x})$ and a negative value for $\Delta d(\bar{x})$, which has been confirmed also quantitatively [28] [29] in $W^\pm$ production at RHIC with polarized beams [27].

6. COMPARISON WITH THE STANDARD PARAMETRIZATION

Despite the fact that $x = 0$ ($Q^2 = 0$) and the neighboring of $x = 1$ (elastic and resonance production) are not in the domain of DIS, the standard parametrization for parton distribution is

$$Ax^B(1-x)^C P(x)$$

with A, B and C and P(x) fixed by the comparison with experiment for each parton distribution and a separate analysis for unpolarized and polarized distributions. Indeed the diffractive component has a power behaviour near x = 0, which requires an infinite number of partons, while the valence partons, which dominate the intermediate and the high x regions have a different (more soft) power behaviour at small x, while the positive value of C gives rise to a decrease with x and also to a different weight for the valence partons, 2 (u and d) for the unpolarized distributions and 4 if one considers also the polarized ones. For the statistical distributions the decrease at high x is naturally explained by the Boltzmann behaviour of the parton distributions for x larger than the "potential" of each parton

$$\exp(-x/\bar{x})$$

The variation of the ratios between the different valence parton distributions:

$$\frac{d(x)}{u(x)}, \frac{\Delta u(x)}{u(x)} \text{ and } \frac{\Delta d(x)}{d(x)}$$

is concentrated in the range between the lowest and highest potential:

$$(X_{d^\uparrow}, X_{u^\uparrow})$$

while in the Boltzmann regime their ratios vary more slowly. This behaviour is the opposite for the standard parametrization, for which the effect of the different exponents for the power $(1-x)^C$ becomes more important as x approaches 1. The ratio at high x of $\frac{F_2^n(x)}{F_2^p(x)}$ is difficult to obtain, since the Fermi motion of the two nucleons in the deuteron gives rise to large uncertainties in the determination of the neutron unpolarized structure function at high x from the ones measured for the proton and for the deuton. So to get the ratio $\frac{d(x)}{u(x)}$ in that region is not a trivial task. The small statistics and the choice of the standard parametrization give rise to a big uncertainty on that ratio, which depends on the C values for u and d partons. In the statistical approach the free parameters, from which that ratio depends, the "temperature" and the "longitudinal and transverse potentials":

$$\bar{x}, X_q \text{ and } Y_q$$

are fixed in the intermediate x region $(0.222, 0.446)$, where the statistics is large and the systematic errors are small. The perfect agreement of the prediction for
$\frac{d(1)}{u(1)}$ = 0.22 with the result of the careful analysis by Owens, Accardi and Melnitchuk [33] is a good confirm for the statistical parton distributions. More in general the ratios of the valence partons distributions show a variation in the region expected in the statistical approach: $(X_{d^\uparrow}, X_{u^\uparrow})$ = $(0.222, 0.446)$ rather than in the x region expected with the standard parametrization.

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THE PLANCK FORM

The equilibrium conditions fix the "potentials" for the gluon to vanish for both helicities, which implies:

$$\Delta G(x) = 0$$

and a Planck form:

$$xG(x) = \frac{A_g x}{[\exp{(x/\bar{x})} - 1]}$$

where the exponent 1 for the power follows by the idea that the hadron in the deep inelastic processes behaves as a black body cavity for the chromomagnetic radiation and A_g is fixed by the sum rule for the longitudinal momentum. Indeed the fact that HERA data show that $xG(x)$ is growing at small x for $Q^2 = 1.9(GeV)^2$ and decreasing at $Q^2 = 10(GeV^2)$ suggests that the Q^2, where it is stationary, will not be so different from $4(GeV)^2$.

In fact $B_H(Q^2) = -0.0257$

The standard form:

$$Ax^B(1-x)^C$$

implies that the decreasing at high x depends on the exponent C and gets faster at increasing x, while the Planck form, as soon as one can neglect the -1 in the denominator, has a more soft behaviour:

$$\exp{(\frac{-x}{\bar{x}})}$$

Since the gluon distribution in DIS has influence on the logaritmic scaling violation, a method to establish the degree of agreement of the Planck distribution with the experimental information obtained at HERA is to compare at $Q^2 = 4$:

$$\int_0^{0.2} xG(x)dx = 0.36$$
$$\int_{0.2}^1 xG(x)dx = 0.05$$

with the one implied by the Planck form:

$$\int_0^{0.2} \frac{A_g x}{[\exp{(x/\bar{x})} - 1]} = 0.34$$
$$\int_{0.2}^1 \frac{A_g x}{[\exp{(x/\bar{x})} - 1]} = 0.125$$

The agreement is good for :

$$\int_0^{0.2} xG(x)dx$$

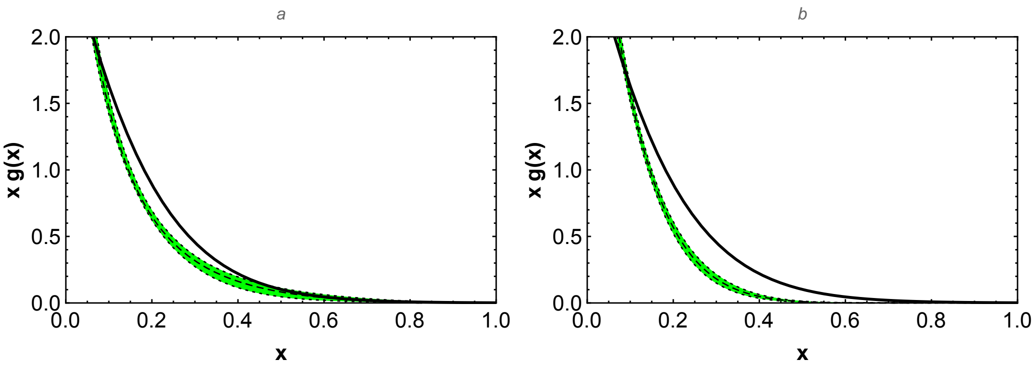

Figure 9: a)Distribution of Gluon (Solid line) in comparison to NNPDF (Dashed line) at $Q^2 = 4(GeV)^2$ [15], b) Distribution of Gluon (Solid line) in comparison to Hera (Dashed line) at $Q^2 = 4(GeV)^2$.

in the range, where most gluons are concentrated, while for $x > 0.2$ HERA gives a faster decrease.
Since for the fermion partons the decrease at high x is better described by the statistical distributions, it is legittimate to make the conjecture that the fast decrease at high x advocated by HERA is more a consequence of their parametrization than of the experimental evidence.

In fact the Planck expression agrees better with the "parametrization independent" NNPDF [34] than the one given by Hera. Indeed to adapt to the Procuste's bed of the standard parametrization the dramatic decrease x of the ratio $\frac{F_2^n(x)}{F_2^p(x)}$ in the region expected in the framework of the statistical approach they need to take P(x) for the u parton the form $(1+9.7x^2)$ and to take an exponent for C slightly larger than the one found for d. Also they have the strange property that the sea contribution has the smallest value for C, which makes it the dominant contribution in the limit $x \to 1$! We think the factors $(1-x)^C$ should be avoided, since to get vanishing parton distributions at $x = 1$ is not necessary, since the region around that value is not in the domain of deep inelastic scattering, where the parton description is appropriate. Indeed the Boltzmann factor exp $\frac{-x}{\bar{x}}$ strongly reduces the parton distributions at high x and the prediction that the ratios of the valence parton distributions in that region have a small variation and a x dependance fixed in the region of intermediate (x, $0.222, 0.446$), makes the statistical parametrization able to supply a reliable information in the high x region, where the experimental information is scarce.

7. CONCLUSION

The agreement with the Hera distributions with the form dictated by the quantum statistical approach for the fermion parton distributions is an impressive confirm of the validity of the proposal in [19] which has been improved with the extension to the transverse degrees of freedom [20] and with the consideration of the Melosh-Wigner [21] rotation. The similarity of the values of the parameters with the ones found in the previous work is another point in favor of the statistical approach. As long as the p_T dependance in the classical limit, neglecting the power dependance on x and with the gaussian approximation for the exponential we get the behaviour:

$$\sqrt{p_T}\exp\frac{-2p_T}{\mu\sqrt{\bar{x}}}$$

with an "effective temperature" $49MeV$, smaller than the range proposed in the paper by Cleymans, Lykasov, Sorin and Teryaev [35], $120-150MeV$, but the important quantum effect gives rise to a harder p_T distribution. The decrease at high x and the ratios between the different valence partons seem to be better described by the statistical distribution than by the standard distributions

$$Ax^B(1-x)^C$$

In fact the ratios change more fastly in the range: $(X_{d^\uparrow}, X_{u^\uparrow})$ = (0.222, 0.446) than above $X_{u^\uparrow}$.

An attractive feature of the statistical model is that the parameters are fixed by regions of x, where there is a large statistics and small systematic errors, small x for the two parameters needed for the diffractive term, the intermediate region (0.222, 0.446) for the ones associated to the valence partons, which fix both the high x Boltzmann behaviour proportional to $\exp\frac{-x}{\bar{x}}$ and the disentangling of the valence partons and their antiparticles. As long as for the gluons the difference at high x of the Planck form with the result by HERA may depend on their standard parametrization $Ax^B(1-x)^C$.

What we have shown may convince the reader to get his opinion on the important issue [36] : Statistics or standard parton distributions ? This is the question.

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