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Working with Nonassociative Geometry and Field Theory

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We review aspects of our formalism for differential geometry on noncommutative and nonassociative spaces which arise from cochain twist deformation quantization of manifolds. We work in the simplest setting of trivial vector bundles and flush out the details of our approach providing explicit expressions for all bimodule operations, and for connections and curvature. As applications, we describe the constructions of physically viable action functionals for Yang-Mills theory and Einstein-Cartan gravity on noncommutative and nonassociative spaces, as first steps towards more elaborate models relevant to non-geometric flux deformations of geometry in closed string theory.

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1. Introduction and summary

Recent advances in understanding flux compactifications of string theory have suggested that non-geometric frames are related to noncommutative and nonassociative deformations of spacetime geometry [11, 21, 16, 1, 9]; as these flux deformations of geometry are probed by *closed* strings, they have a much better potential for providing an effective target space description of quantum gravity than previous appearances of noncommutative geometry in string theory. In the standard T-duality orbit $\mathsf{H} \to \mathsf{f} \to \mathsf{Q} \to \mathsf{R}$ relating geometric and non-geometric fluxes, Q-flux backgrounds experience a noncommutative but strictly associative deformation while the purely non-geometric R-flux backgrounds witness a noncommutative and nonassociative geometry. Nonassociativity in this setting can be encoded by certain triproducts of fields on configuration space predicted by off-shell amplitudes in conformal field theory [12] and in double field theory [13], or by nonassociative $\star$-products from deformation quantization of twisted Poisson structures in the phase space formulation of nonassociative R-space [24, 4, 25]; the equivalence between these two approaches was demonstrated and extended in [3]. A general treatment of nonassociative $\star$-products in this context can be found in [20] (see also the contribution of V. Kupriyanov to these proceedings). Reviews of noncommutativity and nonassociativity in non-geometric closed string theory can be found in [22, 27, 26, 10] (see also the contributions of P. Schupp and I. Bakas to these proceedings).

The cochain twist deformation quantization techniques originally developed by [25] were motivated by the search for a systematic way to generalize notions of differential geometry to such non-geometric backgrounds, and in particular to construct nonassociative deformations of field theory and ultimately gravity (see also [3]); this approach is different in spirit to the nonassociative twist deformation of the geometric f-flux frame considered in [18], which does not seem to be of relevance for non-geometric string theory, nor does it agree with the string theory inspired nonassociative torus bundles of [14, 19] which reproduce the classical limit only up to Morita equivalence. Physically consistent models with novel properties in the context of quantum mechanics were constructed in [25] using this formalism, and of Euclidean scalar quantum field theory in [23]. To extend these considerations to more complicated field theories, a general systematic formalism was developed in [6, 7] for differential geometry on noncommutative and nonassociative spaces internal to the representation category of any quasi-Hopf algebra, generalizing and extending earlier work [15, 8, 2]. This is the starting point for the present contribution.

The purpose of this contribution is to unpack and make explicit the somewhat abstract categorical constructions of [6, 7] in a less formal language that we hope will be palatable to a larger audience. We focus on the special case of most physical relevance: the cochain twist quantization of a classical manifold; this construction is reviewed in Section 2. The formalism is powerful enough to capture the cases of constant non-geometric fluxes as well as non-constant ones such as those which arise in the flux formulation of double field theory [13]; in fact, our constructions in the remainder of this paper are completely general and can be applied to a much broader framework without specific reference to string theory. We further restrict to trivial vector bundles over these noncommutative and nonassociative spaces with diagonal action of the pertinent Hopf algebra of symmetries of the non-geometric background. This simplification enables us to give very explicit "local" descriptions of the noncommutative and nonassociative geometry while still retaining generic features and indicating how the general formalism of [6, 7] may be applied to constructions of physically viable field theories; in particular, we give concrete realizations of the pertinent bimodule operations for homomorphism bundles. In Section 3 we apply this framework to obtain explicit expressions for connections and their curvatures on noncommutative and nonassociative vector bundles. As a starting point for building more elaborate models describing the low-energy effective dynamics of closed strings in non-geometric backgrounds, in Section 4 we demonstrate how to apply our formalism to the constructions of physically sensible action functionals for Yang-Mills theory and Einstein-Cartan gravity on noncommutative and nonassociative spaces.

2. Nonassociative spaces and vector bundles

2.1 Spaces

We briefly review how a classical manifold may be deformed into a noncommutative and nonassociative space by using cochain twist deformation techniques. Recall that associated to any manifold M is the Lie algebra $\mathrm{Vec}(M)$ of vector fields on M (with Lie bracket $[\cdot,\cdot]$ given by the vector field commutator), which plays the role of the infinitesimal diffeomorphisms of M. This Lie algebra gives rise to a Hopf algebra $U\mathrm{Vec}(M)$, the universal enveloping algebra of $\mathrm{Vec}(M)$, which is characterized as follows: As an algebra, $U\mathrm{Vec}(M)$ is the free unital algebra generated by $\mathrm{Vec}(M)$ modulo the relations $v\,w - w\,v = [v,w]$, for all $v,w \in \mathrm{Vec}(M)$. The coproduct Δ, counit ε and antipode S on $U\mathrm{Vec}(M)$ are defined on generators by

$$\Delta(v) = v\otimes 1 + 1\otimes v~, \quad \Delta(1) = 1\otimes 1~, \tag{2.1a}$$

$$\varepsilon(v) = 0~, \quad \varepsilon(1) = 1~, \tag{2.1b}$$

$$S(v) = -v~, \quad S(1) = 1~, \tag{2.1c}$$

for all $v \in \mathrm{Vec}(M)$. The maps Δ and ε are extended as algebra homomorphisms and S as an anti-algebra homomorphism to all of $U\mathrm{Vec}(M)$.

In the following we fix a choice of sub-Hopf algebra $H \subseteq U\mathrm{Vec}(M)$, which we shall interpret as the symmetries of M along which we want to perform the deformation quantization. See Examples 2.1 and 2.2 below for typical choices.

Let us denote by $A := C^\infty(M)$ the algebra of complex-valued smooth functions on M. The action of vector fields on A as derivations can be extended to an H-action $\triangleright : H\otimes A \to A$, which preserves the product and unit in A, i.e.

$$h\triangleright(a\,b) = \big(h_{(1)}\triangleright a\big)\,\big(h_{(2)}\triangleright b\big)~, \quad h\triangleright 1 = \varepsilon(h)\,1~, \tag{2.2}$$

for all $h \in H$ and $a,b \in A$. Here we have used the Sweedler notation $\Delta(h) = h_{(1)}\otimes h_{(2)}$ (with summations understood) to abbreviate the coproduct. In technical terms (2.2) states that A is an H-module algebra.

The commutative and associative algebra A can be deformed by using a cochain twist F of H into a noncommutative and nonassociative algebra $A_\star$. Recall that a cochain twist is an invertible element $F = F^{(1)}\otimes F^{(2)} \in H\otimes H$ (with summations understood) satisfying the normalization condition

$$\varepsilon(F^{(1)})\,F^{(2)} = 1 = F^{(1)}\,\varepsilon(F^{(2)})~. \tag{2.3a}$$

As a consequence, the inverse twist $F^{-1} = F^{(-1)}\otimes F^{(-2)} \in H\otimes H$ (with summations understood) satisfies a similar normalization condition

$$\varepsilon(F^{(-1)})\,F^{(-2)} = 1 = F^{(-1)}\,\varepsilon(F^{(-2)})~. \tag{2.3b}$$

Given any cochain twist $F \in H\otimes H$, we can deform the Hopf algebra H into a *quasi*-Hopf algebra H_F: As algebras, H_F is the same as H and also the counit of H_F agrees with that of H, i.e. $\varepsilon_F = \varepsilon$. However, the coproduct, quasi-antipode and associator in H_F are deformed according to

$$\Delta_F(\,\cdot\,) := F\,\Delta(\,\cdot\,)\,F^{-1}~, \tag{2.4a}$$

$$S_F := S~, \quad \alpha_F := S(F^{(-1)})\,\alpha\,F^{(-2)}~, \quad \beta_F := F^{(1)}\,\beta\,S(F^{(2)})~, \tag{2.4b}$$

$$\phi_F := (1\otimes F)\,(\mathrm{id}_H\otimes\Delta)(F)\,\phi\,(\Delta\otimes\mathrm{id}_H)(F^{-1})\,(F^{-1}\otimes 1)~, \tag{2.4c}$$

where $\alpha = 1 = \beta$ and $\phi = 1\otimes 1\otimes 1$ in the original Hopf algebra H.

The cochain twist F can be used to deform the product μ in the algebra A to a noncommutative and nonassociative $\star$-product

$$\mu_\star := \mu \circ F^{-1} \ . \tag{2.5}$$

We denote the resulting noncommutative and nonassociative algebra by $A_\star$ and abbreviate the $\star$-product as $a \star b := \mu_\star(a \otimes b)$, for $a, b \in A_\star$. In the spirit of noncommutative geometry, we interpret the algebra $A_\star$ as (the algebra of functions on) a noncommutative and nonassociative space.

By construction, the original H-action $\triangleright : H \otimes A \to A$ induces an H_F-action $\triangleright : H_F \otimes A_\star \to A_\star$, which preserves the product and unit in $A_\star$, i.e.

$$h \triangleright (a \star b) = \big(h_{(1)_F} \triangleright a\big) \star \big(h_{(2)_F} \triangleright b\big) \ , \quad h \triangleright 1 = \varepsilon_F(h)\, 1 \ , \tag{2.6}$$

for all $h \in H_F$ and $a, b \in A_\star$. Here we have used the Sweedler notation $\Delta_F(h) = h_{(1)_F} \otimes h_{(2)_F}$ (with summations understood) to abbreviate the deformed coproduct. In technical terms (2.6) states that $A_\star$ is an H_F-module algebra.

It is important to observe that the noncommutativity of $A_\star$ is controlled by the triangular R-matrix

$$R_F = F_{21}\, R\, F^{-1} = R_F^{(1)} \otimes R_F^{(2)} \tag{2.7}$$

in $H_F \otimes H_F$, where $R = 1 \otimes 1$ in the original Hopf algebra H and $F_{21} = F^{(2)} \otimes F^{(1)}$ is the twist with flipped legs. Explicitly, the $\star$-product is commutative up to the action of R_F, i.e.

$$a \star b = \big(R_F^{(2)} \triangleright b\big) \star \big(R_F^{(1)} \triangleright a\big) \ , \tag{2.8}$$

for all $a, b \in A_\star$. Similarly, the nonassociativity of $A_\star$ is controlled by the associator $\phi_F = \phi_F^{(1)} \otimes \phi_F^{(2)} \otimes \phi_F^{(3)}$ in $H_F \otimes H_F \otimes H_F$ given by (2.4c). Explicitly, the $\star$-product is associative up to the action of ϕ_F, i.e.

$$(a \star b) \star c = (\phi_F^{(1)} \triangleright a) \star \big((\phi_F^{(2)} \triangleright b) \star (\phi_F^{(3)} \triangleright c)\big) \ , \tag{2.9}$$

for all $a, b, c \in A_\star$.

Example 2.1. Let $M = \mathbb{R}^m$ and consider the Abelian cocycle twist (with summation over $i, j, \dots$ understood here and in the following)

$$F = \exp\big(-\tfrac{\mathrm{i}\hbar}{2}\, \Theta^{ij}\, P_i \otimes P_j\big) \tag{2.10}$$

based on the cocommutative Hopf algebra $H = U\mathfrak{g}$, where $\mathfrak{g}$ is the Abelian Lie algebra of infinitesimal translations $\{P_i : 1 \le i \le m\}$ and $\Theta = (\Theta^{ij})_{i,j=1}^m = (\mathrm{Q}^{ij}{}_k\, w^k)_{i,j=1}^m$ is an antisymmetric real-valued $m \times m$-matrix which arises from a constant non-geometric Q-flux of closed string theory [17, 5]. In this example we have

$$R_F = F^{-2} = \exp\big(\mathrm{i}\hbar\, \Theta^{ij}\, P_i \otimes P_j\big) \quad , \qquad \phi_F = 1 \otimes 1 \otimes 1 \ . \tag{2.11}$$

In particular $A_\star$ is strictly associative for this choice of twist.

Example 2.2. Let $M=\mathbb{R}^{2n}=\mathbb{R}^n\times\mathbb{R}^n$ and consider the non-Abelian cochain twist

$$F=\exp\Big(-\tfrac{\mathrm{i}\hbar}{2}\big(\tfrac{1}{4}\,\mathsf{R}^{ijk}\,(M_{ij}\otimes P_k-P_i\otimes M_{jk})+P_i\otimes\tilde{P}^i-\tilde{P}^i\otimes P_i\big)\Big) \tag{2.12}$$

based on the cocommutative Hopf algebra $H=U\mathfrak{g}$, where $\mathfrak{g}$ is the non-Abelian nilpotent Lie algebra of infinitesimal translations and Bopp shifts $\{P_i,\tilde{P}^i,M_{ij}:1\leq i<j\leq n\}$; the nontrivial Lie bracket relations are given by $[\tilde{P}^i,M_{jk}]=\delta^i{}_j\,P_k-\delta^i{}_k\,P_j$. Here $\mathsf{R}=(\mathsf{R}^{ijk})^n_{i,j,k=1}$ is a completely antisymmetric real-valued tensor of rank 3 which arises from a constant non-geometric R-flux of closed string theory [25]. In this example we have

$$R_F=F^{-2}\quad,\qquad\phi_F=\exp\big(\tfrac{\hbar^2}{2}\,\mathsf{R}^{ijk}\,P_i\otimes P_j\otimes P_k\big)\ . \tag{2.13}$$

In particular $A_\star$ is *not* strictly associative for this choice of twist.

2.2 Vector bundles

Given any (complex) vector bundle $E\to M$ over the manifold M, we can consider its smooth sections $\Gamma^\infty(E)$, which is a bimodule over $A=C^\infty(M)$ with respect to the usual pointwise module structures. To simplify our considerations in this paper, we assume that $E\to M$ is a trivial complex vector bundle of rank n, i.e. $E=M\times\mathbb{C}^n\to M$ with bundle projection given by projecting on the first factor. For a discussion of generic vector bundles see [6, 7].

The sections of a trivial vector bundle over M of rank n can be described by a free A-bimodule $V=A^n$. Elements $v\in V$ are thus given by column vectors with entries in A, i.e.

$$v=\begin{pmatrix}v^1\\ \vdots\\ v^n\end{pmatrix}\quad,\qquad v^i\in A\ ,\ \ i=1,\dots,n\ . \tag{2.14}$$

Alternatively, we can make use of the standard basis $\{e_i\}_{i=1}^n$ and write

$$v=e_i\,v^i\quad,\qquad v^i\in A\ ,\ \ i=1,\dots,n\ . \tag{2.15}$$

The left and right A-actions on V are given componentwise, i.e.

$$a\,v:=e_i\,(a\,v^i)\ , \tag{2.16a}$$

$$v\,a:=e_i\,(v^i\,a)\ , \tag{2.16b}$$

for all $a\in A$ and $v\in V$. Similarly, we equip V with a componentwise H-action $\triangleright:H\otimes V\to V$, i.e.

$$h\triangleright v:=e_i\,(h\triangleright v^i)\ , \tag{2.17}$$

for all $h\in H$ and $v\in V$. It follows that

$$h\triangleright e_i=\varepsilon(h)\,e_i\ , \tag{2.18}$$

for all $h \in H$ and $i = 1, \dots, n$, i.e. the basis $\{e_i\}_{i=1}^n$ is H-invariant. As a consequence of (2.2), we obtain further that

$$h \triangleright (a\, v) = \big(h_{(1)} \triangleright a\big)\, \big(h_{(2)} \triangleright v\big) \ , \tag{2.19a}$$

$$h \triangleright (v\, a) = \big(h_{(1)} \triangleright v\big)\, \big(h_{(2)} \triangleright a\big) \ , \tag{2.19b}$$

for all $a \in A$, $v \in V$ and $h \in H$. In technical terms (2.19) states that V is an H-module bimodule over the H-module algebra A.

We have explained how a twist $F \in H \otimes H$ can be used to deform the Hopf algebra H to a quasi-Hopf algebra H_F, and the commutative and associative algebra A to a noncommutative and nonassociative algebra $A_\star$. Similarly, we can deform V into an H_F-module $A_\star$-bimodule $V_\star$ by introducing the H_F and $A_\star$-actions

$$h \triangleright v := e_i\, (h \triangleright v^i) \ , \tag{2.20a}$$

$$a \star v := e_i\, (a \star v^i) \ , \tag{2.20b}$$

$$v \star a := e_i\, (v^i \star a) \ , \tag{2.20c}$$

for all $h \in H_F$, $a \in A_\star$ and $v \in V_\star$. One easily verifies the compatibility conditions between the H_F and $A_\star$-actions

$$h \triangleright (a \star v) = \big(h_{(1)_F} \triangleright a\big) \star \big(h_{(2)_F} \triangleright v\big) \ , \tag{2.21a}$$

$$h \triangleright (v \star a) = \big(h_{(1)_F} \triangleright v\big) \star \big(h_{(2)_F} \triangleright a\big) \ , \tag{2.21b}$$

for all $h \in H_F$, $a \in A_\star$ and $v \in V_\star$. In the spirit of noncommutative geometry, we interpret $V_\star$ as (the module of sections of) a vector bundle over $A_\star$.

Noncommutativity of the $A_\star$-bimodule structure is controlled as in (2.8) by the R-matrix R_F, i.e.

$$a \star v = \big(R_F^{(2)} \triangleright v\big) \star \big(R_F^{(1)} \triangleright a\big) \ , \tag{2.22a}$$

$$v \star a = \big(R_F^{(2)} \triangleright a\big) \star \big(R_F^{(1)} \triangleright v\big) \ , \tag{2.22b}$$

for all $a \in A_\star$ and $v \in V_\star$, while nonassociativity is controlled as in (2.9) by the associator ϕ_F, i.e.

$$(a \star b) \star v = (\phi_F^{(1)} \triangleright a) \star \big((\phi_F^{(2)} \triangleright b) \star (\phi_F^{(3)} \triangleright v)\big) \ , \tag{2.23a}$$

$$v \star (a \star b) = \big((\phi_F^{(-1)} \triangleright v) \star (\phi_F^{(-2)} \triangleright a)\big) \star (\phi_F^{(-3)} \triangleright b) \ , \tag{2.23b}$$

for all $a, b \in A_\star$ and $v \in V_\star$. Here we have denoted the components of the inverse associator by $\phi_F^{-1} = \phi_F^{(-1)} \otimes \phi_F^{(-2)} \otimes \phi_F^{(-3)}$ (with summations understood).

2.3 Homomorphism bundles

Many interesting objects in differential geometry are described by maps between vector bundles. For example, a metric is a map $g : TM \to T^*M$ from the tangent bundle to the cotangent bundle, while the curvature of a connection on a vector bundle $E \to M$ is a map $E \to E \otimes \bigwedge^2 T^*M$. Recall that vector bundle maps between two vector bundles $E \to M$ and $E' \to M$ can be equivalently described by sections of the homomorphism bundle $\mathrm{hom}(E,E') \to M$. The module of sections $\Gamma^\infty(\mathrm{hom}(E,E'))$ of the homomorphism bundle is isomorphic (as a $C^\infty(M)$-bimodule) to the module of right module maps $\mathrm{hom}_{C^\infty(M)}(\Gamma^\infty(E),\Gamma^\infty(E'))$; the latter are linear maps $L : \Gamma^\infty(E) \to \Gamma^\infty(E')$ which satisfy additionally the right $C^\infty(M)$-linearity condition

$$L(v\,a) = L(v)\,a \;, \tag{2.24}$$

for all $v \in \Gamma^\infty(E)$ and $a \in C^\infty(M)$.

Our goal now is to describe the analog of homomorphism bundles in our noncommutative and nonassociative framework. Given two modules $V_\star = A_\star^n$ and $W_\star = A_\star^m$, we first consider the vector space of linear maps $\mathrm{hom}_F(V_\star,W_\star)$ from $V_\star$ to $W_\star$. This vector space comes together with a natural H_F-action $\triangleright : H_F \otimes \mathrm{hom}_F(V_\star,W_\star) \to \mathrm{hom}_F(V_\star,W_\star)$ given by the adjoint action

$$h \triangleright L := \big(h_{(1)_F} \triangleright \cdot\,\big) \circ L \circ \big(S_F(h_{(2)_F}) \triangleright \cdot\,\big) \;, \tag{2.25}$$

for all $h \in H_F$ and $L \in \mathrm{hom}_F(V_\star,W_\star)$. It is important to stress that we *do not* require the linear maps $L : V_\star \to W_\star$ to preserve the H_F-action. As explained in [6, Section 1], this would lead to an overly rigid framework for studying noncommutative and nonassociative geometry.

The standard operations of evaluating linear maps $\mathrm{hom}_F(V_\star,W_\star)$ on elements in $V_\star$ and composing or tensoring linear maps with each other are in general not compatible with the H_F-action given in (2.25). In particular, for generic cochain twists F we have the *non-equality*

$$h \triangleright \big(L(v)\big) \neq \big(h_{(1)_F} \triangleright L\big)\big(h_{(2)_F} \triangleright v\big) \;, \tag{2.26}$$

for some $h \in H$, $L \in \mathrm{hom}_F(V_\star,W_\star)$ and $v \in V_\star$. Using internal homomorphism techniques from category theory, one can show that there exist deformations of the evaluation, composition and tensor product operations which are compatible with the H_F-actions [6]. We denote these by

$$\mathrm{ev}_F : \mathrm{hom}_F(V_\star,W_\star) \otimes_\star V_\star \longrightarrow W_\star \;, \tag{2.27a}$$

$$\bullet_F : \mathrm{hom}_F(W_\star,X_\star) \otimes_\star \mathrm{hom}_F(V_\star,W_\star) \longrightarrow \mathrm{hom}_F(V_\star,X_\star) \;, \tag{2.27b}$$

$$\circledast_F : \mathrm{hom}_F(V_\star,X_\star) \otimes_\star \mathrm{hom}_F(W_\star,Y_\star) \longrightarrow \mathrm{hom}_F(V_\star \otimes_\star W_\star, X_\star \otimes_\star Y_\star) \;, \tag{2.27c}$$

and refer to [6, 7] for further details. The $\star$-tensor product $V_\star \otimes_\star W_\star$ is the ordinary tensor product of vector spaces equipped with the H_F-action

$$h \triangleright (v \otimes_\star w) = \big(h_{(1)_F} \triangleright v\big) \otimes_\star \big(h_{(2)_F} \triangleright w\big) \;, \tag{2.28}$$

for all $h \in H_F$, $v \in V_\star$ and $w \in W_\star$. For the example of the evaluation ev_F, compatibility with the H_F-actions means that

$$h \triangleright \mathrm{ev}_F(L \otimes_\star v) = \mathrm{ev}_F\big((h_{(1)_F} \triangleright L) \otimes_\star (h_{(2)_F} \triangleright v)\big) \;, \tag{2.29}$$

for all $h \in H$, $L \in \mathrm{hom}_F(V_\star,W_\star)$ and $v \in V_\star$, which resolves the problem encountered in (2.26).

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The H_F-compatible version of the right A-linearity condition (2.24) is given by the weak right $A_\star$-linearity condition

$$\mathrm{ev}_F\big(L\otimes_\star(v\star a)\big)=\mathrm{ev}_F\big((\phi_F^{(-1)}\triangleright L)\otimes_\star(\phi_F^{(-2)}\triangleright v)\big)\star(\phi_F^{(-3)}\triangleright a)\ , \tag{2.30}$$

for all $v\in V_\star$ and $a\in A_\star$. We denote by $\mathrm{hom}_{A_\star}(V_\star,W_\star)$ the vector space of all linear maps $L\in\mathrm{hom}_F(V_\star,W_\star)$ which satisfy the condition (2.30). It can be shown that $\mathrm{hom}_{A_\star}(V_\star,W_\star)$ is an H_F-module $A_\star$-bimodule, and hence a noncommutative and nonassociative vector bundle in its own right. We interpret $\mathrm{hom}_{A_\star}(V_\star,W_\star)$ as (the module of sections of) the homomorphism bundle from $V_\star$ to $W_\star$. Moreover, the operations (2.27) induce to

$$\mathrm{ev}_{A_\star}:\mathrm{hom}_{A_\star}(V_\star,W_\star)\otimes_{A_\star}V_\star\longrightarrow W_\star\ , \tag{2.31a}$$

$$\bullet_{A_\star}:\mathrm{hom}_{A_\star}(W_\star,X_\star)\otimes_{A_\star}\mathrm{hom}_{A_\star}(V_\star,W_\star)\longrightarrow\mathrm{hom}_{A_\star}(V_\star,X_\star)\ , \tag{2.31b}$$

$$\circledast_{A_\star}:\mathrm{hom}_{A_\star}(V_\star,X_\star)\otimes_{A_\star}\mathrm{hom}_{A_\star}(W_\star,Y_\star)\longrightarrow\mathrm{hom}_{A_\star}(V_\star\otimes_{A_\star}W_\star,X_\star\otimes_{A_\star}Y_\star)\ , \tag{2.31c}$$

where $\otimes_{A_\star}$ denotes the tensor product over $A_\star$. Explicitly, $V_\star\otimes_{A_\star}W_\star$ is the quotient of $V_\star\otimes_\star W_\star$ by the relations

$$(v\star a)\otimes_\star w=(\phi_F^{(1)}\triangleright v)\otimes_\star\big((\phi_F^{(2)}\triangleright a)\star(\phi_F^{(3)}\triangleright w)\big)\ , \tag{2.32}$$

for all $a\in A_\star$, $v\in V_\star$ and $w\in W_\star$.

As $V_\star=A_\star^n$ and $W_\star=A_\star^m$ are by assumption free $A_\star$-bimodules (as are $X_\star$ and $Y_\star$), we can make use of the corresponding bases $\{e_i\}_{i=1}^n$ and $\{e_j\}_{j=1}^m$ to find simple expressions for the homomorphisms $\mathrm{hom}_{A_\star}(V_\star,W_\star)$, and in particular the operations (2.31). In the following, we shall denote (with an abuse of notation) all bases by the same symbols.

Evaluation: Because of the weak right $A_\star$-linearity condition (2.30), any $L\in\mathrm{hom}_{A_\star}(V_\star,W_\star)$ is specified by its evaluation on the basis $\{e_i\}_{i=1}^n$ of $V_\star$. Using also the basis $\{e_j\}_{j=1}^m$ of $W_\star$, we have the expansion

$$\mathrm{ev}_{A_\star}(L\otimes_{A_\star}e_i)=e_j\,L^j{}_i\ , \tag{2.33}$$

which allows us to characterize L in terms of an $m\times n$-matrix with coefficients given by $L^j{}_i\in A_\star$. Hence we have established an isomorphism of vector spaces

$$\mathrm{hom}_{A_\star}(V_\star,W_\star)\longrightarrow A_\star^{m\times n}\ ,\quad L\longmapsto(L^j{}_i)\ , \tag{2.34}$$

which assigns to any L its matrix representation. The evaluation of $L\in\mathrm{hom}_{A_\star}(V_\star,W_\star)$ on a generic element $v=e_i\,v^i\in V_\star$ can then be expressed as

$$\begin{aligned}\mathrm{ev}_{A_\star}(L\otimes_{A_\star}v)&=\mathrm{ev}_{A_\star}(L\otimes_{A_\star}(e_i\,v^i))\\&=\mathrm{ev}_{A_\star}\big((\phi_F^{(-1)}\triangleright L)\otimes_{A_\star}(\phi_F^{(-2)}\triangleright e_i)\big)\star(\phi_F^{(-3)}\triangleright v^i)\\&=\mathrm{ev}_{A_\star}\big(L\otimes_{A_\star}e_i\big)\star v^i\\&=(e_j\,L^j{}_i)\star v^i=e_j\,(L^j{}_i\star v^i)\ .\end{aligned} \tag{2.35}$$

In the second step we have used (2.30) and $e_i\, v^i = e_i \star v^i$, which follows from H_F-invariance of the basis and normalization of the twist. The third step follows by using again H_F-invariance of the basis and also normalization of the associator.

Because the evaluation operation is compatible with the H_F-actions, it follows that

$$\mathrm{ev}_{A_\star}\big((h\triangleright L)\otimes_{A_\star} e_i\big) = \mathrm{ev}_{A_\star}\big(h\triangleright (L\otimes_{A_\star} e_i)\big) = h\triangleright \mathrm{ev}_{A_\star}\big(L\otimes_{A_\star} e_i\big) = e_j\,\big(h\triangleright L^j{}_i\big) \ , \tag{2.36}$$

for all $h\in H_F$ and $L\in \mathrm{hom}_{A_\star}(V_\star,W_\star)$, where in the first step we have used again H_F-invariance of the basis. It follows that, by equipping $A_\star^{m\times n}$ with the componentwise H_F-action, the isomorphism (2.34) is an isomorphism of H_F-modules. By equipping $A_\star^{m\times n}$ further with the componentwise $A_\star$-bimodule structure, the map (2.34) is an isomorphism of H_F-module $A_\star$-bimodules.

Composition: Given $V_\star = A_\star^n$, $W_\star = A_\star^m$ and $X_\star = A_\star^l$, one can show by similar calculations that the composition $L'\bullet_{A_\star} L\in \mathrm{hom}_{A_\star}(V_\star,X_\star)$ of any $L\in\mathrm{hom}_{A_\star}(V_\star,W_\star)$ and $L'\in\mathrm{hom}_{A_\star}(W_\star,X_\star)$ is given by the components

$$\mathrm{ev}_{A_\star}\big((L'\bullet_{A_\star} L)\otimes_{A_\star} e_i\big) = e_k\,\big(L'^k{}_j\star L^j{}_i\big) \ . \tag{2.37}$$

Hence the isomorphism (2.34) sends the composition operation $\bullet_{A_\star}$ to the $\star$-matrix product

$$\star : A_\star^{l\times m}\otimes_{A_\star} A_\star^{m\times n}\longrightarrow A_\star^{l\times n} \ , \qquad (L'^k{}_j)\otimes_{A_\star}(L^j{}_i)\longmapsto (L'^k{}_j\star L^j{}_i) \ . \tag{2.38}$$

In the special case where $V_\star = W_\star = X_\star$, it follows that the endomorphism algebra $\mathrm{end}_{A_\star}(V_\star) := \mathrm{hom}_{A_\star}(V_\star,V_\star)$ (with product $\bullet_{A_\star}$) is isomorphic to the $\star$-matrix product algebra $A_\star^{n\times n}$.

Tensor product: Given $V_\star = A_\star^n$, $W_\star = A_\star^m$, $X_\star = A_\star^l$ and $Y_\star = A_\star^p$, one can show by similar calculations that the tensor product $L'\circledast_{A_\star} L\in\mathrm{hom}_{A_\star}(V_\star\otimes_{A_\star} W_\star, X_\star\otimes_{A_\star} Y_\star)$ of any $L\in\mathrm{hom}_{A_\star}(V_\star,X_\star)$ and $L'\in\mathrm{hom}_{A_\star}(W_\star,Y_\star)$ is given by the components

$$\mathrm{ev}_{A_\star}\big((L\circledast_{A_\star} L')\otimes_{A_\star}(e_i\otimes_{A_\star} e_j)\big) = (e_k\otimes_{A_\star} e_r)\,(L^k{}_i\star L'^r{}_j) \ . \tag{2.39}$$

Hence the isomorphism (2.34) sends the tensor product operation $\circledast_{A_\star}$ to the $\star$-outer product

$$\circledast : A_\star^{l\times n}\otimes_{A_\star} A_\star^{p\times m}\longrightarrow A_\star^{(l\,p)\times(n m)} \ , \qquad (L^k{}_i)\otimes_{A_\star}(L'^r{}_j)\longmapsto (L^k{}_i\star L'^r{}_j) \ . \tag{2.40}$$

2.4 Form-valued homomorphism bundles

As we shall see in more detail in the next sections, many homomorphisms in differential geometry are valued in the exterior algebra of differential forms $\Omega^\sharp_\star$ on $A_\star$, i.e. they are maps $L\in\mathrm{hom}_{A_\star}(V_\star, W_\star\otimes_{A_\star}\Omega^\sharp_\star)$ for some modules $V_\star$ and $W_\star$. The differential forms $\Omega^\sharp_\star$ on $A_\star$ are obtained by twisting, with respect to the cochain twist $F\in H\otimes H$, the differential forms $\Omega^\sharp(M)$ on the underlying classical manifold M: As vector spaces $\Omega^\sharp_\star = \Omega^\sharp(M)$, while the product on $\Omega^\sharp_\star$ is given by the $\star$-exterior product

$$\wedge_\star := \wedge\circ F^{-1} : \Omega^p_\star\otimes_\star\Omega^q_\star\longrightarrow\Omega^{p+q}_\star \ . \tag{2.41}$$

The relevant H-action on $\Omega^\sharp(M)$ is given by the Lie derivative of vector fields on forms. Similarly to (2.8), the (graded) noncommutativity of the $\star$-exterior product is controlled by the R-matrix,

$$\omega \wedge_\star \omega' = (-1)^{|\omega|\,|\omega'|} \left(R_F^{(2)} \triangleright \omega'\right) \wedge_\star \left(R_F^{(1)} \triangleright \omega\right) , \tag{2.42}$$

for all homogeneous forms $\omega, \omega' \in \Omega^\sharp_\star$. Nonassociativity is controlled as in (2.9) by the associator

$$\left(\omega \wedge_\star \omega'\right) \wedge_\star \omega'' = (\phi_F^{(1)} \triangleright \omega) \wedge_\star \left((\phi_F^{(2)} \triangleright \omega') \wedge_\star (\phi_F^{(3)} \triangleright \omega'')\right) , \tag{2.43}$$

for all $\omega, \omega', \omega'' \in \Omega^\sharp_\star$. The differential

$$\mathrm{d} : \Omega^p_\star \longrightarrow \Omega^{p+1}_\star \tag{2.44}$$

on $\Omega^\sharp_\star$ is given by the ordinary de Rham exterior derivative and it satisfies the graded Leibniz rule

$$\mathrm{d}(\omega \wedge_\star \omega') = \mathrm{d}\omega \wedge_\star \omega' + (-1)^{|\omega|}\, \omega \wedge_\star \mathrm{d}\omega' , \tag{2.45}$$

for all homogeneous forms $\omega, \omega' \in \Omega^\sharp_\star$.

Because $\Omega^\sharp_\star$ is a graded H_F-module algebra and not only an H_F-module $A_\star$-bimodule, the modules of homomorphisms $\hom_{A_\star}(V_\star, W_\star \otimes_{A_\star} \Omega^\sharp_\star)$ may be equipped with additional structures, which we shall now briefly describe. For this, we introduce the notation

$$V^\sharp_\star := V_\star \otimes_{A_\star} \Omega^\sharp_\star \tag{2.46}$$

to denote the tensor product of the module $V_\star$ with the module of differential forms $\Omega^\sharp_\star$. A generic element in $V^\sharp_\star$ is of the form $e_i \otimes_{A_\star} \omega^i$, where $\omega^i \in \Omega^\sharp_\star$. Notice that $V^\sharp_\star$ is a graded module, with $V^p_\star = V_\star \otimes_{A_\star} \Omega^p_\star$. Because $\Omega^\sharp_\star$ is a graded H_F-module algebra, $V^\sharp_\star$ is moreover a graded H_F-module $\Omega^\sharp_\star$-bimodule with left and right $\Omega^\sharp_\star$-action given by the $\star$-exterior product, i.e.

$$(e_i \otimes_{A_\star} \omega^i) \wedge_\star \omega' := e_i \otimes_{A_\star} \left(\omega^i \wedge_\star \omega'\right) , \tag{2.47a}$$

$$\omega' \wedge_\star (e_i \otimes_{A_\star} \omega^i) := e_i \otimes_{A_\star} \left(\omega' \wedge_\star \omega^i\right) , \tag{2.47b}$$

for all $\omega^i, \omega' \in \Omega^\sharp_\star$. (Notice that this definition uses H_F-invariance of the basis e_i.)

We shall now show that the module of homomorphisms $\hom_{A_\star}(V_\star, W_\star \otimes_{A_\star} \Omega^\sharp_\star)$ is isomorphic (as an H_F-module $A_\star$-bimodule) to the module $\hom_{\Omega^\sharp_\star}(V^\sharp_\star, W^\sharp_\star)$ of weak right $\Omega^\sharp_\star$-linear maps, which is characterized by the condition (compare with (2.30))

$$\mathrm{ev}_F\Big(L \otimes_\star \big((e_i \otimes_{A_\star} \omega^i) \wedge_\star \omega'\big)\Big) = \mathrm{ev}_F\Big((\phi_F^{(-1)} \triangleright L) \otimes_\star \big(e_i \otimes_{A_\star} (\phi_F^{(-2)} \triangleright \omega^i)\big)\Big) \wedge_\star (\phi_F^{(-3)} \triangleright \omega') , \tag{2.48}$$

for all $\omega^i, \omega' \in \Omega^\sharp_\star$. In fact, following the same arguments as before, we use the bases of $V_\star = A^n_\star$ and $W_\star = A^m_\star$ to show that there is an isomorphism of H_F-module $\Omega^\sharp_\star$-bimodules

$$\hom_{\Omega^\sharp_\star}(V^\sharp_\star, W^\sharp_\star) \longrightarrow {\Omega^\sharp_\star}^{m\times n} , \quad L \longmapsto (L^j{}_i) . \tag{2.49}$$

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The matrix coefficients are defined by

$$\mathrm{ev}_F\big(L\otimes_\star(e_i\otimes_{A_\star}1)\big)=e_j\otimes_{A_\star}L^j{}_i\ , \tag{2.50}$$

where $1\in A_\star\subseteq\Omega^\sharp_\star$ is the unit element. Any element $L\in\hom_{A_\star}(V_\star,W_\star\otimes_{A_\star}\Omega^\sharp_\star)$ has exactly the same expansion in the bases of $V_\star$ and $W_\star$, hence we can define an isomorphism

$$(\,\cdot\,)^\sharp:\hom_{A_\star}(V_\star,W_\star\otimes_{A_\star}\Omega^\sharp_\star)\longrightarrow\hom_{\Omega^\sharp_\star}(V^\sharp_\star,W^\sharp_\star) \tag{2.51}$$

by going via the matrix representations.

Given $V_\star=A^n_\star$, $W_\star=A^m_\star$ and $X_\star=A^l_\star$, we use the isomorphisms (2.51) and (2.49) to define a composition operation

$$\bullet_{A_\star}:\hom_{A_\star}(W_\star,X_\star\otimes_{A_\star}\Omega^\sharp_\star)\otimes_{A_\star}\hom_{A_\star}(V_\star,W_\star\otimes_{A_\star}\Omega^\sharp_\star)\longrightarrow\hom_{A_\star}(V_\star,X_\star\otimes_{A_\star}\Omega^\sharp_\star) \tag{2.52a}$$

in terms of the $\wedge_\star$-matrix product

$$\wedge_\star:\Omega^{\sharp\,l\times m}_\star\otimes_{A_\star}\Omega^{\sharp\,m\times n}_\star\longrightarrow\Omega^{\sharp\,l\times n}_\star\ ,\quad (L'^k{}_j)\otimes_{A_\star}(L^j{}_i)\longmapsto(L'^k{}_j\wedge_\star L^j{}_i)\ . \tag{2.52b}$$

Given $V_\star=A^n_\star$, $W_\star=A^m_\star$, $X_\star=A^l_\star$ and $Y_\star=A^p_\star$, we define a tensor product operation

$$\circledast_{A_\star}:\hom_{A_\star}(V_\star,X_\star\otimes_{A_\star}\Omega^\sharp_\star)\otimes_{A_\star}\hom_{A_\star}(W_\star,Y_\star\otimes_{A_\star}\Omega^\sharp_\star)\longrightarrow\hom_{A_\star}\big(V_\star\otimes_{A_\star}W_\star,(X_\star\otimes_{A_\star}Y_\star)\otimes_{A_\star}\Omega^\sharp_\star\big) \tag{2.53a}$$

in terms of the $\wedge_\star$-outer product

$$\circledast:\Omega^{\sharp\,l\times n}_\star\otimes_{A_\star}\Omega^{\sharp\,p\times m}_\star\longrightarrow\Omega^{\sharp\,(l\,p)\times(nm)}_\star\ ,\quad (L^k{}_i)\otimes_{A_\star}(L'^r{}_j)\longmapsto(L^k{}_i\wedge_\star L'^r{}_j)\ . \tag{2.53b}$$

These operations generalize (2.38) and (2.40) to form-valued homomorphisms.

3. Nonassociative connections and curvature

3.1 Connections

A nonassociative connection on a module $V_\star$ is a linear map $\nabla\in\hom_F(V_\star,V_\star\otimes_{A_\star}\Omega^1_\star)$ which satisfies the Leibniz rule

$$\mathrm{ev}_F\big(\nabla\otimes_\star(v\star a)\big)=\mathrm{ev}_F\big((\phi_F^{(-1)}\triangleright\nabla)\otimes_\star(\phi_F^{(-2)}\triangleright v)\big)\star(\phi_F^{(-3)}\triangleright a)+v\otimes_{A_\star}\mathrm{d}a\ , \tag{3.1}$$

for all $v\in V_\star$ and $a\in A_\star$, where d is the exterior derivative of the differential calculus $\Omega^\sharp_\star$. We denote the space of connections on $V_\star$ by $\mathrm{con}_F(V_\star)$ and note that it is an affine space over the module of homomorphisms $\hom_{A_\star}(V_\star,V_\star\otimes_{A_\star}\Omega^1_\star)$.

As $V_\star=A^n_\star$ is by assumption a free $A_\star$-bimodule, we can describe any connection $\nabla\in\mathrm{con}_F(V_\star)$ in terms of its coefficients $\Gamma^j{}_i\in\Omega^1_\star$ defined by

$$\mathrm{ev}_F(\nabla\otimes_\star e_i)=:e_j\otimes_{A_\star}\Gamma^j{}_i\ . \tag{3.2}$$

Using (3.1), after a short calculation we obtain

$$\mathrm{ev}_F(\nabla \otimes_\star v) = e_i \otimes_{A_\star} \big(\mathrm{d}v^i + \Gamma^i{}_j \star v^j\big) \ , \tag{3.3}$$

for all $v = e_i\, v^i \in V_\star$.

As $\mathrm{con}_F(V_\star) \subseteq \mathrm{hom}_F(V_\star, V_\star \otimes_{A_\star} \Omega^1_\star)$ is an affine subspace, we can act with any $h \in H_F$ on a connection ∇ and obtain an element $h \triangleright \nabla \in \mathrm{hom}_F(V_\star, V_\star \otimes_{A_\star} \Omega^1_\star)$, which however in general does not lie in $\mathrm{con}_F(V_\star)$: In contrast to the Leibniz rule (3.3), $h \triangleright \nabla$ satisfies

$$\mathrm{ev}_F\big((h\triangleright\nabla)\otimes_\star v\big) = e_i \otimes_{A_\star} \big(\varepsilon_F(h)\,\mathrm{d}v^i + (h\triangleright\Gamma^i{}_j)\star v^j\big) \ , \tag{3.4}$$

for all $v = e_i\, v^i \in V_\star$. In particular, if $h \in H_F$ satisfies $\varepsilon_F(h) = 1$ then $h\triangleright\nabla \in \mathrm{con}_F(V_\star)$, while if $\varepsilon_F(h) = 0$ then $h\triangleright\nabla \in \mathrm{hom}_{A_\star}(V_\star, V_\star \otimes_{A_\star} \Omega^1_\star)$.

Similarly to the case of homomorphisms (2.51), we can lift connections $\nabla \in \mathrm{con}_F(V_\star)$ to linear maps $\nabla^\sharp \in \mathrm{end}_F(V_\star^\sharp)$, which then satisfy the condition

$$\mathrm{ev}_F\big(\nabla^\sharp \otimes_\star (e_i \otimes_{A_\star} \omega^i)\big) = e_i \otimes_{A_\star} \big(\mathrm{d}\omega^i + \Gamma^i{}_j \wedge_\star \omega^j\big) \ , \tag{3.5}$$

for all $\omega^i \in \Omega^\sharp_\star$. Notice that (3.5) implies the graded Leibniz rule

$$\mathrm{ev}_F\big(\nabla^\sharp \otimes_\star (s\wedge_\star \omega')\big) = \mathrm{ev}_F\big((\phi_F^{(-1)}\triangleright\nabla^\sharp)\otimes_\star(\phi_F^{(-2)}\triangleright s)\big)\wedge_\star(\phi_F^{(-3)}\triangleright\omega') + (-1)^{|s|}\, s\wedge_\star \mathrm{d}\omega' \ , \tag{3.6}$$

for all homogeneous forms $s = e_i \otimes_{A_\star} \omega^i \in V_\star^\sharp$ and $\omega' \in \Omega^\sharp_\star$.

3.2 Connections on tensor products

Given $V_\star = A^n_\star$ and $W_\star = A^m_\star$, together with connections $\nabla \in \mathrm{con}_F(V_\star)$ and $\nabla' \in \mathrm{con}_F(W_\star)$, we can construct a connection on $V_\star \otimes_{A_\star} W_\star$ by taking their sum $\nabla \boxplus_F \nabla'$, see [7, Section 4.2] for details. In terms of the coefficients $\Gamma^k{}_i, \Gamma'^l{}_j \in \Omega^1_\star$, the sum of connections takes a simple form and it is specified by the coefficients

$$\mathrm{ev}_F\big((\nabla\boxplus_F\nabla')\otimes_\star(e_i\otimes_{A_\star} e_j)\big) = (e_k\otimes_{A_\star} e_l)\otimes_{A_\star}\big(\Gamma^k{}_i\,\delta^l{}_j + \delta^k{}_i\,\Gamma'^l{}_j\big) \ . \tag{3.7}$$

On a generic element $v\otimes_{A_\star} w = e_i\otimes_{A_\star} e_j\,(v^i\star w^j) \in V_\star\otimes_{A_\star} W_\star$, the sum of connections acts as

$$\begin{aligned}\mathrm{ev}_F\big((\nabla\boxplus_F\nabla')\otimes_\star(v\otimes_{A_\star} w)\big) &= \\ &(e_k\otimes_{A_\star} e_l)\otimes_{A_\star}\big(\mathrm{d}(v^k\star w^l) + \Gamma^k{}_i\wedge_\star(v^i\star w^l) + \Gamma'^l{}_j\wedge_\star(v^k\star w^j)\big) \ .\end{aligned} \tag{3.8}$$

The sum of connections can be consistently extended to tensor products of finitely many modules by inductively using (3.7). For example, given $V_\star = A^n_\star$, $W_\star = A^m_\star$ and $X_\star = A^l_\star$, together with connections $\nabla \in \mathrm{con}_F(V_\star)$, $\nabla' \in \mathrm{con}_F(W_\star)$ and $\nabla'' \in \mathrm{con}_F(X_\star)$, then $(\nabla\boxplus_F\nabla')\boxplus_F\nabla'' \in \mathrm{con}_F\big((V\otimes_{A_\star} W_\star)\otimes_{A_\star} X_\star\big)$ is specified by the connection coefficients

$$\begin{aligned}\mathrm{ev}_F\Big(\big((\nabla\boxplus_F\nabla')\boxplus_F\nabla''\big)\otimes_\star\big((e_i\otimes_{A_\star} e_j)\otimes_{A_\star} e_k\big)\Big) &= \\ &\big((e_{i'}\otimes_{A_\star} e_{j'})\otimes_{A_\star} e_{k'}\big)\otimes_{A_\star}\big(\Gamma^{i'}{}_i\,\delta^{j'}{}_j\,\delta^{k'}{}_k + \delta^{i'}{}_i\,\Gamma'^{j'}{}_j\,\delta^{k'}{}_k + \delta^{i'}{}_i\,\delta^{j'}{}_j\,\Gamma''^{k'}{}_k\big) \ .\end{aligned} \tag{3.9}$$

Moreover, $(\nabla\boxplus_F\nabla')\boxplus_F\nabla''$ and $\nabla\boxplus_F(\nabla'\boxplus_F\nabla'')$ are related by adjoining the associator

$$(\nabla\boxplus_F\nabla')\boxplus_F\nabla'' = \phi_F^{-1}\circ\big(\nabla\boxplus_F(\nabla'\boxplus_F\nabla'')\big)\circ\phi_F \ . \tag{3.10}$$

3.3 Connections on homomorphism bundles

Given $V_\star = A_\star^n$ and $W_\star = A_\star^m$, together with connections $\nabla \in \mathrm{con}_F(V_\star)$ and $\nabla' \in \mathrm{con}_F(W_\star)$, we can construct a connection on $\mathrm{hom}_{A_\star}(V_\star, W_\star)$ by taking their adjoint $\mathrm{ad}_{\bullet_F}(\nabla', \nabla)$, see [7, Section 4.3] for details. In terms of the coefficients $\Gamma^k{}_i, \Gamma'^l{}_j \in \Omega^1_\star$, the adjoint connection takes a simple form: Denoting by $\{e_j{}^i\}$ the basis of $\mathrm{hom}_{A_\star}(V_\star, W_\star)$ given by the isomorphism (2.34) and the standard basis of $A_\star^{m\times n}$, the coefficients of $\mathrm{ad}_{\bullet_F}(\nabla', \nabla)$ are given by

$$\mathrm{ev}_F\big(\mathrm{ad}_{\bullet_F}(\nabla', \nabla) \otimes_\star e_j{}^i\big) = e_{j'}{}^{i'} \otimes_{A_\star} \big(\Gamma'^{j'}{}_j\, \delta^i{}_{i'} - \delta^{j'}{}_j\, \Gamma^i{}_{i'}\big)\ . \tag{3.11}$$

On a generic element $L = e_j{}^i L^j{}_i \in \mathrm{hom}_{A_\star}(V_\star, W_\star)$, the adjoint connection acts as

$$\mathrm{ev}_F\big(\mathrm{ad}_{\bullet_F}(\nabla', \nabla) \otimes_\star L\big) = e_{j'}{}^{i'} \otimes_{A_\star} \big(\mathrm{d}L^{j'}{}_{i'} + \Gamma'^{j'}{}_j \star L^j{}_{i'} - (R_F^{(2)} \triangleright L^{j'}{}_i) \star (R_F^{(1)} \triangleright \Gamma^i{}_{i'})\big)\ , \tag{3.12}$$

where in the last term we have used the R-matrix to rearrange the term $\Gamma^i{}_{i'} \star L^{j'}{}_i$ so that $\star$-matrix multiplication is obvious.

The adjoint connection extends to form-valued homomorphisms $L \in \mathrm{hom}_{A_\star}(V_\star, W_\star \otimes_{A_\star} \Omega^\sharp_\star)$. The resulting expression

$$\mathrm{ev}_F\big(\mathrm{ad}_{\bullet_F}(\nabla', \nabla) \otimes_\star L\big) = e_{j'}{}^{i'} \otimes_{A_\star} \big(\mathrm{d}L^{j'}{}_{i'} + \Gamma'^{j'}{}_j \wedge_\star L^j{}_{i'} - (-1)^{|L|}\, (R_F^{(2)} \triangleright L^{j'}{}_i) \wedge_\star (R_F^{(1)} \triangleright \Gamma^i{}_{i'})\big) \tag{3.13}$$

is very similar to (3.12) whereby we simply replace $\star$-products by $\wedge_\star$-products and include a degree-dependent sign factor in front of the last term.

As an important example, let us consider the dual module $V_\star^\vee := \mathrm{hom}_{A_\star}(V_\star, A_\star)$ of $V_\star = A_\star^n$. Following the notations used above, we denote the basis of the dual module by $\{e^i\}$, i.e. with an upper index. A generic element in $V_\star^\vee$ is thus of the form $L = e^i L_i$ with $L_i \in A_\star$. Given now any connection $\nabla \in \mathrm{con}_F(V_\star)$, we can use the differential $\mathrm{d} : A_\star \to \Omega^1_\star$ as a connection on $A_\star$, and define a connection on $V_\star^\vee$ by taking the adjoint connection $\nabla^\vee := \mathrm{ad}_{\bullet_F}(\mathrm{d}, \nabla)$. Because d does not have any nontrivial connection coefficients, the general expression (3.12) implies that the dual connection acts on $L = e^i L_i \in V_\star^\vee$ as

$$\mathrm{ev}_F\big(\nabla^\vee \otimes_\star L\big) = e^{i'} \otimes_{A_\star} \big(\mathrm{d}L_{i'} - (R_F^{(2)} \triangleright L_i) \star (R_F^{(1)} \triangleright \Gamma^i{}_{i'})\big)\ . \tag{3.14}$$

3.4 Curvature

The curvature of a connection $\nabla \in \mathrm{con}_F(V_\star)$ is given by the graded R-matrix commutator

$$R(\nabla) := \tfrac{1}{2}\, [\nabla^\sharp, \nabla^\sharp]_F := \tfrac{1}{2}\, \big(\nabla^\sharp \bullet_F \nabla^\sharp + (R_F^{(2)} \triangleright \nabla^\sharp) \bullet_F (R_F^{(1)} \triangleright \nabla^\sharp)\big) \tag{3.15}$$

of its lift $\nabla^\sharp \in \mathrm{end}_F(V_\star^\sharp)$ defined in (3.5). Due to the graded Leibniz rule (3.6), it follows that $R(\nabla) \in \mathrm{hom}_{A_\star}(V_\star, V_\star \otimes_{A_\star} \Omega^2_\star)$ is a homomorphism valued in 2-forms. The coefficients of the curvature are given by

$$\mathrm{ev}_{A_\star}\big(R(\nabla) \otimes_{A_\star} e_i\big) = e_j \otimes_{A_\star} R^j{}_i = e_j \otimes_{A_\star} \big(\mathrm{d}\Gamma^j{}_i + \tfrac{1}{2}\, [\Gamma, \Gamma]_\star{}^j{}_i\big)\ , \tag{3.16a}$$

where

$$[\Gamma\,,\Gamma]_{\star}{}^{j}{}_{i} := \Gamma^{j}{}_{k} \wedge_{\star} \Gamma^{k}{}_{i} + \big(R_F^{(2)} \triangleright \Gamma^{j}{}_{k}\big) \wedge_{\star} \big(R_F^{(1)} \triangleright \Gamma^{k}{}_{i}\big)\ . \tag{3.16b}$$

On the sum of connections $\nabla \in \mathrm{con}_F(V_\star)$ and $\nabla' \in \mathrm{con}_F(W_\star)$, the curvature $R(\nabla \boxplus_F \nabla')$ has the desired additive behavior

$$\mathrm{ev}_{A_\star}\big(R(\nabla \boxplus_F \nabla') \otimes_{A_\star} (e_i \otimes_{A_\star} e_j)\big) = (e_k \otimes_{A_\star} e_l) \otimes_{A_\star} \big(R^{k}{}_{i}\, \delta^{l}{}_{j} + \delta^{k}{}_{i}\, R'^{\,l}{}_{j}\big)\ . \tag{3.17}$$

The Bianchi tensor of a connection $\nabla \in \mathrm{con}_F(V_\star)$ is defined by acting with the adjoint connection on the curvature using (3.13) to get

$$\mathrm{Bianchi}(\nabla) := \mathrm{ev}_F\big(\mathrm{ad}_{\bullet F}(\nabla,\nabla) \otimes_\star R(\nabla)\big)\ . \tag{3.18}$$

By definition, it follows that $\mathrm{Bianchi}(\nabla) \in \hom_{A_\star}(V_\star, V_\star \otimes_{A_\star} \Omega^3_\star)$ is a homomorphism valued in 3-forms. Using (3.13) we find

$$\mathrm{ev}_{A_\star}\big(\mathrm{Bianchi}(\nabla) \otimes_{A_\star} e_i\big) = e_j \otimes_{A_\star} \mathrm{Bianchi}^{j}{}_{i} = e_j \otimes_{A_\star} \big(\mathrm{d}R^{j}{}_{i} + [\Gamma, R]_{\star}{}^{j}{}_{i}\big)\ , \tag{3.19a}$$

where

$$[\Gamma, R]_{\star}{}^{j}{}_{i} := \Gamma^{j}{}_{k} \wedge_\star R^{k}{}_{i} - \big(R_F^{(2)} \triangleright R^{j}{}_{k}\big) \wedge_\star \big(R_F^{(1)} \triangleright \Gamma^{k}{}_{i}\big)\ . \tag{3.19b}$$

An interesting consequence of the noncommutativity and nonassociativity of $A_\star$ (which is controlled by the R-matrix and associator) is that in general the Bianchi tensor does not vanish, i.e. the Bianchi identity is generally violated. However, for trivial R-matrix and associator we recover the usual Bianchi identity in classical differential geometry for any connection ∇.

4. Nonassociative field theory

4.1 Yang-Mills theory

Let M be an oriented m-dimensional manifold equipped with an H-invariant Riemannian or Lorentzian metric. Then the classical Hodge operator $*_M : \Omega^p(M) \to \Omega^{m-p}(M)$ is H-equivariant, i.e. $*_M \circ (h\triangleright \cdot\,) = (h\triangleright \cdot\,) \circ *_M$ for all $h \in H$. We equip the deformed differential forms with the same Hodge operator, leading to an H_F-equivariant map

$$*_M : \Omega^p_\star \longrightarrow \Omega^{m-p}_\star\ . \tag{4.1}$$

Given any module $V_\star = A^n_\star$ and any connection $\nabla \in \mathrm{con}_F(V_\star)$, let $\mathscr{L}(\nabla) \in \hom_{A_\star}(V_\star, V_\star \otimes_{A_\star} \Omega^m_\star)$ be the homomorphism valued in top-forms which is given by the components

$$\mathscr{L}^{j}{}_{i} = \tfrac{1}{2}\, F^{j}{}_{k} \wedge_\star *_M F^{k}{}_{i}\ , \tag{4.2}$$

where as usual we denote the curvature of a gauge connection by $F^{j}{}_{i} = \mathrm{d}\Gamma^{j}{}_{i} + \frac{1}{2}\,[\Gamma\,,\Gamma]_{\star}{}^{j}{}_{i}$. The action functional for Yang-Mills gauge theory is given by tracing and integrating $\mathscr{L}(\nabla)$, i.e.

$$S_{\mathrm{YM}}(\nabla) := \int_M \mathrm{Tr}\big(\mathscr{L}(\nabla)\big) = \frac{1}{2} \int_M F^{j}{}_{k} \wedge_\star *_M F^{k}{}_{j}\ . \tag{4.3}$$

We shall now show that, under certain natural conditions on the twist $F \in H \otimes H$ and the connection ∇, the Yang-Mills action (4.3) is real-valued.

The first condition is that F is Hermitean, i.e. it defines a Hermitean star-product on $A_\star$. This means that $(a \star b)^* = b^* \star a^*$, where * denotes the involution given by pointwise complex conjugation of functions on M. This is clearly the case for Examples 2.1 and 2.2. We extend the involution * on $A_\star$ to a graded involution on the differential forms $\Omega^\sharp_\star$ by setting

$$(\omega \wedge_\star \omega')^* = (-1)^{|\omega|\,|\omega'|}\ \omega'^* \wedge_\star \omega^* \ , \quad (\mathrm{d}\omega)^* = \mathrm{d}\omega^* \ , \tag{4.4}$$

for all homogeneous forms $\omega, \omega' \in \Omega^\sharp_\star$.

The second condition is that ∇ is unitary, i.e. the corresponding connection coefficients satisfy

$$\Gamma^j{}_i{}^* = -\Gamma^i{}_j \ . \tag{4.5}$$

Using (3.16) one easily shows that the curvature of a unitary connection is an anti-Hermitean matrix, i.e.

$$F^j{}_i{}^* = -F^i{}_j \ . \tag{4.6}$$

The third condition is the graded 2-cyclicity property

$$\int_M \omega \wedge_\star \omega' = (-1)^{|\omega|\,|\omega'|} \int_M \omega' \wedge_\star \omega \ , \tag{4.7}$$

for all homogeneous forms $\omega, \omega' \in \Omega^\sharp_\star$. This property holds for Abelian twists, as in Example 2.1, and also for the nonassociative deformation of Example 2.2, see [25].

The first two conditions imply that the complex conjugate of the action (4.3) can be simplified as

$$S_{\mathrm{YM}}(\nabla)^* = \frac{1}{2} \int_M \big(F^j{}_k \wedge_\star *_M F^k{}_j\big)^* = \frac{1}{2} \int_M *_M F^k{}_j{}^* \wedge_\star F^j{}_k{}^* = \frac{1}{2} \int_M *_M F^j{}_k \wedge_\star F^k{}_j \ , \tag{4.8}$$

where in the second step we have also used compatibility between the Hodge operator and the complex conjugation involution. The third condition then implies that we can interchange the two terms in the last equality of (4.8), and hence find that the noncommutative and nonassociative Yang-Mills action is real, i.e.

$$S_{\mathrm{YM}}(\nabla)^* = S_{\mathrm{YM}}(\nabla) \ . \tag{4.9}$$

In particular, the noncommutative and nonassociative Yang-Mills action (4.3) is real-valued for all unitary connections in Examples 2.1 and 2.2.

4.2 Einstein-Cartan gravity

The field content of Einstein-Cartan gravity is a spin connection ∇ and a vielbein field E. Let M be an oriented m-dimensional manifold which admits a trivial Dirac spinor bundle

$$S = M \times \mathbb{C}^{2^{\lfloor \frac{m}{2} \rfloor}} \longrightarrow M \ . \tag{4.10}$$

We denote the module of sections of the spinor bundle by $V := \Gamma^\infty(S) = A^{2^{\lfloor \frac{m}{2} \rfloor}}$.

Without loss of generality, here we can take $H = U\mathrm{Vec}(M)$ to be the Hopf algebra of all infinitesimal diffeomorphisms of M. Then given any cochain twist $F \in H \otimes H$, we twist $A = C^\infty(M)$ to a noncommutative and nonassociative algebra $A_\star$ and V to an H_F-module $A_\star$-bimodule $V_\star = A_\star^{2^{\lfloor \frac{m}{2} \rfloor}}$.

A spin connection on $V_\star$ is a connection $\nabla \in \mathrm{con}_F(V_\star)$ for which the coefficients take the special form

$$\Gamma^j{}_i = \tfrac{1}{4}\, \omega^{ab}\, \gamma_{ab}{}^j{}_i \,, \tag{4.11}$$

where $\omega^{ab} \in \Omega^1_\star$ is antisymmetric in ab and $\gamma_{ab} = \frac{1}{2}\,[\gamma_a, \gamma_b]$ is given by the commutator of the gamma-matrices γ_a; here the indices $a, b, \dots$ run from 1 to m, the dimension of M, while $i, j, \dots$ run from 1 to $2^{\lfloor \frac{m}{2} \rfloor}$, the rank of the Dirac spinor bundle S. The curvature (3.16) of a spin connection can be computed with some standard gamma-matrix algebra and it reads as

$$R^j{}_i = \tfrac{1}{4}\, R^{ab}\, \gamma_{ab}{}^j{}_i = \tfrac{1}{4}\, \big(\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge_\star \omega^{cb}\big)\, \gamma_{ab}{}^j{}_i \,, \tag{4.12}$$

where the c-index was lowered by the flat metric η_{ab}.

A vielbein is a homomorphism $E \in \mathrm{hom}_{A_\star}(V_\star, V_\star \otimes_{A_\star} \Omega^1_\star)$ valued in 1-forms for which the coefficients take the special form

$$E^j{}_i = E^a\, \gamma_a{}^j{}_i \,, \tag{4.13}$$

where $E^a \in \Omega^1_\star$.

Let us assume for the moment that the dimension m of M is even. We propose the noncommutative and nonassociative generalization of the Einstein-Cartan action functional given by

$$S^{\mathrm{even}}_{\mathrm{EC}}(\nabla, E) := \int_M \Big(E_{\mathrm{left}}^{a_1 \cdots a_{\frac{m}{2}-1}} \wedge_\star R^{a_{\frac{m}{2}}\, a_{\frac{m}{2}+1}}\Big) \wedge_\star E_{\mathrm{right}}^{a_{\frac{m}{2}+2} \cdots a_m}\ \varepsilon_{a_1 \cdots a_m} \ , \tag{4.14}$$

where $\varepsilon_{a_1 \cdots a_m}$ is the antisymmetric tensor and

$$E^{a_1 \cdots a_k}_{\mathrm{left}} := \big(\cdots\big((E^{[a_1} \wedge_\star E^{a_2}) \wedge_\star E^{a_3}\big) \cdots\big) \wedge_\star E^{a_k]} \,, \tag{4.15a}$$

$$E^{a_1 \cdots a_k}_{\mathrm{right}} := E^{[a_1} \wedge_\star \big(\cdots \big(E^{a_{k-2}} \wedge_\star (E^{a_{k-1}} \wedge_\star E^{a_k]})\big) \cdots\big) \,, \tag{4.15b}$$

is the $\wedge_\star$-product of k vielbeins in $\Omega^k_\star$ with special bracketing conventions and totally antisymmetrized (with weight 1) in the indices $a_1 \cdots a_k$. This choice of bracketing allows us to show that the Einstein-Cartan action (4.14) is real-valued, under similar assumptions as for the Yang-Mills action.

Let us now assume that the twist F is Hermitean and further demand the reality conditions

$$\omega^{ab\,*} = -\omega^{ba} = \omega^{ab} \ , \quad E^{a\,*} = E^a \ , \tag{4.16}$$

for the spin connection and vielbein. As a consequence, we obtain

$$R^{ab\,*} = -R^{ba} = R^{ab} \ , \quad E^{a_1 \cdots a_k\,*}_{\mathrm{left}} = E^{a_1 \cdots a_k}_{\mathrm{right}} \ . \tag{4.17}$$

The complex conjugate of the action (4.14) can now be simplified as

$$S^{\rm even}_{\rm EC}(\nabla,E)^* = (-1)^{\frac{m}{2}-1}\int_M E_{\rm left}^{a_{\frac{m}{2}+2}\cdots a_m}\wedge_\star\Big(R^{a_{\frac{m}{2}}a_{\frac{m}{2}+1}}\wedge_\star E_{\rm right}^{a_1\cdots a_{\frac{m}{2}-1}}\Big)\,\varepsilon_{a_1\cdots a_m}$$
$$= \int_M E_{\rm left}^{a_1\cdots a_{\frac{m}{2}-1}}\wedge_\star\Big(R^{a_{\frac{m}{2}}a_{\frac{m}{2}+1}}\wedge_\star E_{\rm right}^{a_{\frac{m}{2}+2}\cdots a_m}\Big)\,\varepsilon_{a_1\cdots a_m}\ , \tag{4.18}$$

where the sign factor in the first equality is due to (4.4). In the second equality we have reordered the indices of $\varepsilon_{a_1\cdots a_m}$ by using its total antisymmetry property.

We further assume the 3-cyclicity property

$$\int_M (\omega\wedge_\star\omega')\wedge_\star\omega'' = \int_M \omega\wedge_\star(\omega'\wedge_\star\omega'')\ , \tag{4.19}$$

for all $\omega,\omega',\omega''\in\Omega^\sharp_\star$. This property obviously holds for Abelian twists as in Example 2.1, because they give strictly associative deformations. For the nonassociative deformation of Example 2.2 the 3-cyclicity property is shown in [25]. We can then rebracket the expression after the last equality of (4.18) and find that the noncommutative and nonassociative Einstein-Cartan action in even dimensions (4.14) is real, i.e.

$$S^{\rm even}_{\rm EC}(\nabla,E)^* = S^{\rm even}_{\rm EC}(\nabla,E)\ . \tag{4.20}$$

In the case of an odd-dimensional manifold M, one way to obtain a real-valued Einstein-Cartan action functional is to modify (4.14) as

$$S^{\rm odd}_{\rm EC}(\nabla,E) := \frac{1}{2}\int_M \Big(E_{\rm left}^{a_1\cdots a_{\frac{m-1}{2}-1}}\wedge_\star R^{a_{\frac{m-1}{2}}a_{\frac{m-1}{2}+1}}\Big)\wedge_\star E_{\rm right}^{a_{\frac{m-1}{2}+2}\cdots a_m}\ \varepsilon_{a_1\cdots a_m}$$
$$+\frac{1}{2}\int_M \Big(E_{\rm left}^{a_1\cdots a_{\frac{m-1}{2}}}\wedge_\star R^{a_{\frac{m-1}{2}+1}a_{\frac{m-1}{2}+2}}\Big)\wedge_\star E_{\rm right}^{a_{\frac{m-1}{2}+3}\cdots a_m}\ \varepsilon_{a_1\cdots a_m}\ , \tag{4.21}$$

where in the first line the form degree of $E_{\rm right}$ is larger by 1 than the form degree of $E_{\rm left}$ and vice versa in the second line. Under the same assumptions as in the even-dimensional case, one can show that the action (4.21) is real-valued, i.e.

$$S^{\rm odd}_{\rm EC}(\nabla,E)^* = S^{\rm odd}_{\rm EC}(\nabla,E)\ . \tag{4.22}$$

In fact, the second term in (4.21) is the conjugate of the first term and vice versa.

In particular, the noncommutative and nonassociative Einstein-Cartan gravity action in even dimensions (4.14) and in odd dimensions (4.21) is real-valued in Examples 2.1 and 2.2.

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