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Non-isometric T-duality from gauged sigma models

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Local symmetries is one of the most successful themes in modern theoretical physics. Although they are usually associated to Lie algebras, a gradual increase of interest in more general situations where local symmetries are associated to groupoids and algebroids has taken place in recent years. On the other hand, dualities is another persistently interesting theme in modern physics. One of the most prominent examples is provided by target space duality in string theory. The latter, Abelian or not, is usually associated to the presence of isometries, which is however a very restrictive assumption. In this contribution we discuss some recent advances located at the intersection of the above two themes. Focusing on bosonic string sigma models we discuss certain gauged versions where (a) the invariance conditions on the background fields are much milder than the isometric case and (b) the gauge symmetry is generically associated to a Lie algebroid instead of just a Lie algebra. Furthermore we utilize such gauged sigma models to study the possibility of non-Abelian, non-isometric T-duality.

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1. Motivation and Introduction

The utmost importance of symmetry is modern theoretical physics is hard to overemphasize. The gauge principle, being the cornerstone of our theories describing fundamental interactions, has brought local symmetries to the forefront of attention since long ago. Conventionally, speaking of local symmetries what comes to mind is a Lie algebra and a set of Lie algebra-valued 1-forms called gauge fields that are introduced in the theory through minimal coupling. One question that comes to mind then is whether this is the most general setting, namely whether *every* local symmetry is associated to such a scenario. It is known that this is not the case (see for example the inspiring exposition in Ref. [3]). Given the unquestionable success of Lie-algebra-based gauge symmetries it is certainly worth studying such more general scenarios, based for example on the notion of a Lie algebroid. Moreover, a second part of this question refers to minimal coupling and whether it is always enough to guarantee the existence of a gauge theory. The investigation of these questions in the context of two-dimensional sigma models is the first motivation for the work presented here.

Another important notion related to symmetry is duality. Dualities are everywhere in physics (see for example the exposition in Ref. [4]) and this is certainly the case in string theory, where they are very profound properties that teach us several conceptual lessons about the theory. One of the prime examples is T-duality which identifies string backgrounds associated to different target spaces. Conventionally, speaking of T-duality what comes to mind is a target space with one or more isometric directions.[1] This is however extremely restrictive. Certainly a randomly chosen string background has no isometries whatsoever. The immediate question is then whether this is the best we can do. Recall that one approach to T-duality goes through an "intermediate" gauge theory which on-shell reduces to one or a dual background [7, 8]. Here we will study such gauge theories even when no isometries are available. The relation among gauged sigma models and T-duality is therefore our second general motivation.

Finally, there is a third motivation that is worth mentioning although it is not going to be addressed in this work. It regards the so-called non-geometric string backgrounds, which often originate from T-dualities and their precise understanding is still a programme under way. One certain lesson of recent studies on this topic is that their description, be it from the target space viewpoint or the world sheet one, requires generalized geometric concepts. Here we are going to study gauged sigma models whose target space is essentially some generalized tangent bundle.

It is remarkable that there is a common mathematical theme underlying *all* three motivations above: the theory of groupoids and algebroids. Indeed, (i) behind every local symmetry there is a groupoid or an algebroid [3], (ii) T-duality is mathematically described, under certain restrictions of course, as an isomorphism of (Courant) algebroids [9, 10], and (iii) generalized geometry is partially yet crucially based on algebroid theory [11].

Based on the above motivations, we discuss a threefold generalization of the traditional picture of Lie algebra-based, minimally coupled 2D gauge theories with background fields satisfying strong invariance conditions. In particular, the first fold of the generalization is to replace the Lie algebra $\mathfrak{g}$ by a Lie algebroid[2] over the target space M. A Lie algebroid is an interesting mathematical structure that merges two very important notions for physics, namely algebras and vector

[1] An interesting exception is Poisson-Lie T-duality [5, 6].

[2] A more general case where the algebroid is required to be just *almost* Lie is discussed elsewhere [2].

bundles. At first approximation one could think of it as a generalization of an algebra such that the structure constants are not constant anymore but are instead replaced by *structure functions*. One particularly illuminating way to see this is to study the famous Cartan problem, which is also relevant in many physical contexts, ranging from the vierbein formulation of gravity to string theory reductions. The problem can be stated as follows (see for example the excellent lectures notes [12], which we follow closely here): given two sets of functions defined locally on $\mathbb{R}^n$, say C^a_{bc} and e^i_a, where the indices range as $a = 1,\dots,r$ and $i = 1,\dots n$, we are asked to determine (i) a manifold M, (ii) a coframe $\{e^a\}$ on the manifold, namely a basis of its cotangent bundle T^*M, and (iii) a local coordinate system $X : M \to \mathbb{R}^n$, such that the following two equations hold:

$$\mathrm{d}e^a = -\tfrac{1}{2}C^a_{bc}(X)e^b \wedge e^c \ , \tag{1.1}$$

$$\mathrm{d}X^i = e^i_a(X)e^a \ . \tag{1.2}$$

Asking when this problem has solutions, it is immediately observed that there are two necessary conditions obtained by taking the exterior derivatives of the above two equations,

$$C^a_{e[b}C^e_{cd]} = e^i_{[b}\partial_i C^a_{cd]} \ , \tag{1.3}$$

$$2e^j_{[b}\partial_j e^i_{c]} = C^a_{bc}e^i_a \ , \tag{1.4}$$

where antisymmetrizations are taken with weight. A very familiar special case (e.g. from the vielbein formulation of gravity) corresponds to contant functions C^a_{bc}. Then the condition (1.3) becomes the usual Jacobi identity for Lie algebras, and the condition (1.4) becomes the closure of the algebra of the vectors e_a dual to the coframe. Then one lands in the Lie algebra case and (1.1) becomes the Maurer-Cartan equation. However, in general the functions C^a_{bc} are not constant, in which case the solution to the Cartan problem is associated to a Lie algebroid. One can reach a good definition by using the necessary conditions (1.3) and (1.4). The second condition indicates that we have to replace the Lie algebra $\mathfrak{g}$ by a vector bundle L, which is equipped with a bracket such that its sections e_a close under it with the structure functions C^a_{bc}:

$$[e_a, e_b]_L = C^c_{ab}(X)e_c \ . \tag{1.5}$$

In order to associate this to (1.4), an additional ingredient is a map from the vector bundle L to the tangent bundle TM, i.e. a rule that assigns a vector field to every section of L, $\rho : L \to TM$. Then Eq. (1.4) simply says that this map is a homomorphism:

$$(1.4) \quad \Leftrightarrow \quad \rho([e_a, e_b]_L) = [\rho(e_a), \rho(e_b)] \ . \tag{1.6}$$

Moreover, the other necessary condition acquires a simple explanation too:

$$(1.3) \quad \Leftrightarrow \quad [e_a, [e_b, e_c]_L]_L + \text{(cyclic permutations)} = 0 \ ; \tag{1.7}$$

in other words it becomes the Jacobi identity for the bracket on L, which is thus a Lie bracket. The above three ingredients, the vector bundle L with a Lie bracket and a homomorphism ρ to the tangent bundle of M, define a Lie algebroid. Certainly Lie algebras are included in this definition, simply by taking M to be just a point. For a list of examples we refer e.g. to [12].

It should now be fairly obvious what the first generalization amounts to. The idea is to replace the Lie algebra $\mathfrak{g}$ by a Lie algebroid L such that the "generalized local symmetry" is generated by vector fields that close under the Lie bracket with some X-dependent structure functions.

Let us now turn to 2D sigma models and explain what the second and third folds of the generalization are. Recall that the common bosonic sector of string theory includes the background fields (g,B,Φ), namely a target space metric, the Kalb-Ramond 2-form and the scalar dilaton. The theory is described by a non-linear sigma model whose source is a 2D world sheet Σ and its target a manifold M, and the dynamical fields are the scalar components of the corresponding map $X=(X^i):\Sigma\to M$. The corresponding action functional is

$$S = \int_\Sigma \tfrac{1}{2}g_{ij}(X)\mathrm{d}X^i\wedge *\mathrm{d}X^j + \int_{\hat{\Sigma}} \tfrac{1}{6}H_{ijk}(X)\mathrm{d}X^i\wedge\mathrm{d}X^j\wedge\mathrm{d}X^k + \alpha' S_{\text{dilaton}} \,, \tag{1.8}$$

where $*$ is the Hodge operator on the world sheet ($*^2=\mp 1$ for Euclidean and Lorentzian signature respectively) and H is a Wess-Zumino term, only locally exact such that $H=\mathrm{d}B$, living on an open membrane world volume $\hat{\Sigma}$ whose boundary is the world sheet Σ. The dilaton coupling involves the world sheet curvature scalar and it is of next order in α'. In this work it will therefore be ignored.

Suppose now that we are given a set of vector fields $\rho_a=\rho_a^i(X)\frac{\partial}{\partial X^i}$ which generate the following global symmetry

$$\delta_\varepsilon X^i = \rho^i_a(X)\varepsilon^a \,, \tag{1.9}$$

for rigid transformation parameters ε^a. Then the action (1.8) is invariant under this symmetry provided that the following two conditions hold:

$$\mathscr{L}_{\rho_a} g = 0 \,, \tag{1.10}$$
$$\iota_{\rho_a} H = \mathrm{d}\theta_a \,, \tag{1.11}$$

for some 1-form θ_a. Eq. (1.10) means that the vector fields ρ_a are Killing, namely they generate isometries for the metric g. Then this global symmetry can be promoted to local one by introducing Lie algebra-valued 1-forms $A=(A^a)$ (gauge fields), allowing the parameters ε^a to depend on world sheet coordinates σ^μ and also allowing the gauge fields to transform appropriately under such gauge transformations, in particular as follows:

$$\delta_\varepsilon A^a = \mathrm{d}\varepsilon^a + C^a_{bc}A^b\varepsilon^c \,, \tag{1.12}$$

with C^a_{bc} here the structure constants of the Lie algebra. This was studied in detail in [13, 14] and revisited recently in [15–17]. It can already be invoked from these works that minimal coupling is not enough to yield the correct gauged action functional. Thus the generalization to non-minimally coupled gauge fields is already present (and necessary) at this level and will be even more transparent in the body of this work. Moreover, already in the isometric case there are additional constraints that have to be satisfied in order for the A-extended action functional to be gauge invariant. This will also become transparent in the main text.

Most importantly, it should be stressed that the invariance conditions (1.10) and (1.11) are extremely restrictive. Indeed an arbitrary choice of background fields is unlikely to satisfy them. Thus it would be important to be able to write down gauged action functionals that still permit *local* symmetries of the type (1.9) but without having to satisfy such restrictive conditions. Clearly this is not possible at the global level; however, starting from the inspiring work [18, 19], it turns out to be possible at the level of gauge symmetries. In other words, in an inversion of the usual spirit, we do not start with a global symmetry and then promote it to a local one; instead we search for local symmetries directly, given a candidate action functional that is likely to realize a desired symmetry. This is the third and most crucial generalization of the usual approach. We will see that upon an appropriate choice of gauge transformation for the 1-forms A, the invariance conditions are replaced by extremely milder ones, certainly such that

$$\mathscr{L}_{\rho_a} g \neq 0 , \tag{1.13}$$

$$\iota_{\rho_a} H - \mathrm{d}\theta_a \neq 0 , \tag{1.14}$$

which allow for gauge theories without isometry.

It should be stressed that the three above generalizations are to some extent independent. Indeed, it is possible for example to drop isometry while keeping either minimal coupling or Lie algebra-valued gauge fields or both.

Returning to the second motivation, that is T-duality, the generalization described above has direct consequences. One can now view the gauged action functionals as intermediate gauge theories in a Buscher-like approach to T-duality. In the present case the associated vector fields ρ_a are neither required to be Abelian nor Killing. Thus, introducing Lagrange multipliers in the action functional so as to reduce the additional degrees of freedom introduced by the gauge fields A, it is possible to determine two different reduced models upon integrating out different fields in the action. Then these two resulting sigma models are in a sense dual to each other and correspond to backgrounds with different target space. At least at the classical level one can say that this is a version of non-Abelian and non-isometric T-duality.

It is natural to worry whether the above possibility can really be realized in non-trivial examples. We will see that at the classical level this is indeed possible and we will discuss some particular cases. It is not known whether examples surviving at the quantum level exist, but this is an interesting open question which we leave for future investigations.

2. Beyond the standard gauging in 2D sigma models

As explained in the Introduction, the starting point is the action functional with target space metric g and Wess-Zumino term,

$$S = \int_\Sigma \tfrac{1}{2} g_{ij}(X)\mathrm{d}X^i \wedge *\mathrm{d}X^j + \int_{\hat{\Sigma}} \tfrac{1}{6} H_{ijk}(X)\mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \, . \tag{2.1}$$

The difference to the standard case is that we do not assume *any* global symmetry for S. Our guiding principle is instead the following: We require the existence of a gauge extension of the action functional S such that it is invariant under the *local* symmetry

$$\delta_\varepsilon X^i = \rho_a^i(X)\varepsilon^a(\sigma) , \tag{2.2}$$

without any a priori assumptions on the metric g and the Wess-Zumino term H.

Let us now be more specific. We extend the action S by adding gauge fields taking values in some Lie algebroid L

$$L \xrightarrow{\rho} TM$$

instead of a Lie algebra. This means that if e_a is a local basis of sections of L, then

$$A = A^a e_a \ .$$

This basis satisfies Eq. (1.5) and by the map ρ one gets a set of vector fields $\rho_a = \rho(e_a)$ which satisfy a similar non-Abelian relation for the Lie bracket with the same structure functions:

$$[\rho_a, \rho_b] = C^c_{ab}(X)\rho_c \ . \tag{2.3}$$

These gauge fields represent additional would-be dynamical degrees of freedom in the theory. Since we do not wish to have extra degrees of freedom in the end, we require that their field strength vanishes. This can be either implemented as an additional constraint (see e.g. [17]) or taken care of by adding Lagrange multipliers in the action. Here we follow the second approach, which is closer in spirit to the traditional duality formulations of Buscher [7] and Duff [20]. Moreover, in order to build on the analogy to these formulations, we call these additional scalar fields $\widetilde{X}_a$ (instead of η_a, which was the notation used in Ref. [1]). Thus the general form of the candidate gauged action we propose is

$$\begin{aligned} S_{\text{gauged}} \ = \ & \int_\Sigma \tfrac{1}{2} g_{ij} DX^i \wedge * DX^j + \int_{\hat{\Sigma}} \tfrac{1}{6} H_{ijk} \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k - \\ & - \int_\Sigma (\theta_a + \mathrm{d}\widetilde{X}_a) \wedge A^a + \int_{\Sigma_2} \tfrac{1}{2} (\iota_{\rho_{[a}} \theta_{b]} + C^c_{ab}(X) \widetilde{X}_c) A^a \wedge A^b - \int_{\Sigma_2} \omega^a_{bi} \widetilde{X}_a A^b \wedge DX^i \ . \end{aligned} \tag{2.4}$$

Some comments and clarifications are in order here. First of all, the world sheet covariant derivative D is defined as

$$DX^i = \mathrm{d}X^i - \rho^i_a(X) A^a \ , \tag{2.5}$$

as usual. This means that at the level of the kinetic sector we have used minimal coupling of the gauge fields.[3] However, as already mentioned, minimal coupling does not work for the Wess-Zumino term and this is obvious from the above form of the action, where A-dependent terms appear without the involvement of the covariant derivative. Secondly, θ_a is an 1-form as in the Introduction, namely it can be expanded as $\theta_a = \theta_{ai}(X)\mathrm{d}X^i$. Finally, the presence of the last term in S_{gauged} is certainly puzzling at first sight. The involvement of the Lagrange multipliers in this term indicates that it is related to the field strength of the gauge fields. Moreover, it involves some yet undefined parameters ω^a_{bi}. The explanation of these puzzles is one of our main purposes below.

Up to now we have at hand the action S_{gauged} and the gauge transformation (2.2) of the fields X^i, under which we would like it to be invariant. For this to work we have to determine appropriate gauge transformations for the gauge fields A as well as for the scalar fields $\widetilde{X}_a$. Guided by the corresponding transformation in the standard Lie algebraic and isometric case given by (1.12), we write the gauge transformation for A as

$$\delta_\varepsilon A^a = \mathrm{d}\varepsilon^a + C^a_{bc}(X) A^b \varepsilon^c + \omega^a_{bi} \varepsilon^b DX^i \ . \tag{2.6}$$

[3] For a discussion on the generalization to non-minimal kinetic coupling we refer to [2].

Note that the modification is both in the X-dependence of the structure functions C^a_{bc} as well as in the presence of an additional term. This does not yet specify the gauge transformation; it just parametrizes our up to this moment ignorance of the correct transformation such that the action is gauge invariant. Indeed, the parameters ω^a_{bi} still remain unspecified.[4] What remains is the gauge transformation for $\widetilde{X}_a$. This can be determined either the hard way, namely using an arbitrary ansatz, or it can be invoked by extending known results from the standard case (e.g. from [16]) in a simple way. In any case the result is

$$\delta_\varepsilon \widetilde{X}_a = -\iota_{\rho_{(a}}\theta_{b)}\varepsilon^b - (C^c_{ab}(X) - 2\rho^i_{[a}\omega^c_{b]i})\widetilde{X}_c\varepsilon^b \ . \tag{2.7}$$

The cautious reader must have already noticed the repeated appearance of the combination

$$T^c_{ab} = C^c_{ab}(X) - 2\rho^i_{[a}\omega^c_{b]i} \tag{2.8}$$

and the fact that the transition from the standard case to the non-standard one partially goes through a substitution

$$C^c_{ab} \to T^c_{ab}(X) \ .$$

This is certainly not an accident. A similar combination was found already in Ref. [21]. As explained there, although none of C^a_{bc} and ω^a_{bi} transform as tensors, the combination T^c_{ab} does. The geometric meaning of those objects is then the following. The parameters ω^a_{bi} are coefficients of a connection 1-form $\omega^a_b = \omega^a_{bi}\mathrm{d}X^i$ on the Lie algebroid L, namely there is an connection ∇ on L such that

$$\nabla e_a = \omega^b_a \otimes e_b \ . \tag{2.9}$$

This induces also a connection $\nabla_{\rho(\cdot)}$ by means of the map ρ whose torsion T is exactly the one with components as in (2.8). The curvature 2-form of the connection 1-form ω^a_b may be defined the usual way by the formula

$$R^a_b = \mathrm{d}\omega^a_b + \omega^a_c \wedge \omega^c_b \ . \tag{2.10}$$

These geometric explanations also shed light to a puzzle encountered previously in relation to the last term of the action S_{gauged}. Given the extended gauge transformation of the gauge field A^a, its field strength should be defined as

$$F^a = \mathrm{d}A^a + \tfrac{1}{2}C^a_{bc}A^b \wedge A^c - \omega^a_{bi}A^b \wedge DX^i \ . \tag{2.11}$$

This is necessary so that F^a has a chance to be covariant. We will not discuss further this field strength here, since we do not wish to add dynamics for the gauge fields. However it should now be clear that collecting all the terms in S_{gauged} which contain a Lagrange multiplier, one obtains $\widetilde{X}_aF^a$, as required. In particular, the last term is absolutely necessary for this to work.[5]

Let us recapitulate. We have written an action S_{gauged} which includes the scalar fields X^i, the gauge fields A^a and the extra scalar fields $\widetilde{X}_a$, and we know their infinitesimal gauge transformations

[4] In [2] an additional 2D-admissible term proportional to the Hodge dual of DX^i was added. Its presence allows for more freedom on the one hand and also yields the relation to generalized geometry more transparent. However here we keep the discussion in its simplest possible form.

[5] It is mentioned once more that this necessity is only valid in the present approach; one could avoid introducing Lagrange multipliers altogether and use constraints instead.

in terms of the structure functions $C^a_{bc}(X)$ and the coefficients ω^a_{bi} of a connection on L. Thus it is now a straightforward task to examine under which conditions the action is gauge invariant. Before we do so, let us mention that in the Lie-algebraic and isometric case there are two invariance conditions that g and H have to satisfy and two additional constraints. We will see that in the present case the count of conditions and constraints will be the same, with the profit of having milder conditions.

Indeed, direct variation of the action S_{gauged} reveals that gauge invariance is guaranteed provided that the background fields satisfy

$$\mathscr{L}_{\rho_a} g = \omega^b_a \vee \iota_{\rho_b} g \ , \qquad (2.12)$$
$$\iota_{\rho_a} H = \mathrm{d}\theta_a + \theta_b \wedge \omega^b_a - \widetilde{X}_b R^b_a \ , \qquad (2.13)$$

where $\vee$ is defined as the symmetric product[6] $\mathrm{d}X^i \vee \mathrm{d}X^j = \mathrm{d}X^i \otimes \mathrm{d}X^j + \mathrm{d}X^j \otimes \mathrm{d}X^i$. In component form, Eq. (2.12) is then written as

$$(\mathscr{L}_{\rho_a} g)_{ij} = \omega^b_{ai}\rho^k_b g_{jk} + \omega^b_{aj}\rho^k_b g_{ik} \ .$$

It is already evident that these conditions allow for non-isometric directions. Indeed for non-vanishing ω^a_b the Lie derivative of the metric is not any more zero. Thus we have derived the explicit expressions for the right hand sides of Eqs. (1.13) and (1.14) advertised in the Introduction.

As anticipated, the above invariance conditions for the background fields are not the end of the story as far as gauge invariance is concerned. In direct analogy to the standard case there are two additional constraints,

$$\mathscr{L}_{\rho_{[a}} \theta_{b]} = C^d_{ab}\theta_d - \iota_{\rho_d}\theta_{[a}\omega^d_{b]} - \iota_{\rho_{[a}}\omega^d_{b]}\theta_d - D^c_{ab}\widetilde{X}_c \ , \qquad (2.14)$$
$$\tfrac{1}{3}\iota_{\rho_a}\iota_{\rho_b}\iota_{\rho_c} H = \iota_{\rho_{[a}} C^d_{bc]}\theta_d - 2\iota_{\rho_{[a}}\omega^d_b\iota_{\rho_{c]}}\theta_d - 2\tilde{D}^d_{abc}\widetilde{X}_d \ , \qquad (2.15)$$

where we used the following definitions

$$D^e_{ab} = \mathrm{d}C^e_{ab} + C^c_{ab}\omega^e_c + 2C^e_{d[a}\omega^d_{b]} + 2\iota_{\rho_d}\omega^e_{[b}\omega^d_{a]} + 2\mathscr{L}_{\rho_{[b}}\omega^e_{a]} + \iota_{\rho_{[a}}R^e_{b]} \ , \qquad (2.16)$$
$$\tilde{D}^e_{abc} = \iota_{\rho_{[a}}\iota_{\rho_b}R^e_{c]} \ . \qquad (2.17)$$

In this derivation we used the Jacobi identity (1.3) for the Lie algebroid L.

It is useful to cross-check that all above expressions are consistent with previously known results in the Lie-algebraic and isometric limit. Essentially this is obvious, since in the limits $C^a_{bc}(X) \to C^a_{bc}$ and $\omega^a_{bi} \to 0$, the invariance conditions become identical to (1.10) and (1.11), while the two additional constraints reduce to

$$(2.14) \quad \xrightarrow{\omega^a_b\to 0,\ C^a_{bc}(X)\to C^a_{bc}} \quad \mathscr{L}_{\rho_{[a}}\theta_{b]} = C^d_{ab}\theta_d \ , \qquad (2.18)$$
$$(2.15) \quad \xrightarrow{\omega^a_b\to 0,\ C^a_{bc}(X)\to C^a_{bc}} \quad \tfrac{1}{3}\iota_{\rho_a}\iota_{\rho_b}\iota_{\rho_c}H = \iota_{\rho_{[a}}C^d_{bc]}\theta_d \ , \qquad (2.19)$$

which are identical to the ones found e.g. in [16]. However, in general the conditions we derived are much milder and have the potential to yield gauged actions for vastly more initial backgrounds. We discuss whether this potential can be fulfilled later on.

[6]In completely analogous way to the familiar antisymmetric (wedge) product $\mathrm{d}X^i \wedge \mathrm{d}X^j = \mathrm{d}X^i \otimes \mathrm{d}X^j - \mathrm{d}X^j \otimes \mathrm{d}X^i$.

3. Non-Abelian, non-isometric T-duality

One of the most direct applications of the type of 2D gauge theories described in the previous section is T-duality in string theory. Recall that Buscher's procedure, which leads to the derivation of the widely used T-duality rules for background fields, involves a 2D gauge theory with Lagrange multipliers as an intermedium between two string theories on different target spaces. The two dual backgrounds are obtained upon integration of different fields in the theory. In particular, the integration of the Lagrange multipliers leads back to the original background, while the integration of the gauge fields returns a different background whose target space coordinates are essentially the Lagrange multipliers in the action. This procedure, be it Abelian [7] or non-Abelian [13, 22], always assumes isometric directions from the beginning.

The essence of our approach here is that isometries are neither an assumption nor a resulting requirement for a meaningful, gauge invariant action functional with a local symmetry generated by the vector fields ρ_a. Thus it is obvious what the next step in our analysis should be. Starting with the action S_{gauged} we should first integrate out the Lagrange multipliers to confirm that the original action is recovered and then we should integrate the gauge fields and determine the new action. The calculational details are explained in Ref. [1]; here we emphasize the final results.

In order to integrate the Lagrange multipliers $\widetilde{X}_a$ we vary the action S_{gauged} with respect to them and derive the field equation

$$F^a = \mathrm{d}A^a + \tfrac{1}{2}C^a_{bc}(X)A^b\wedge A^c - \omega^a_{bi}A^b\wedge DX^i = 0\ . \tag{3.1}$$

This is indeed expected; it means that the non-Abelian gauge fields A^a are pure gauge. We may fix the gauge on-shell, for example with the simplest choice being just $A^a = 0$, a common choice in the literature (cf. [8]). Then the action reduces to (2.1), which is the original action for g and H.

On the other hand, integrating the gauge fields A^a is slightly more involved. First the action is varied with respect to them and the resulting field equation is

$$*\rho^*_a - \xi_a = G_{ab}*A^b + D_{ab}A^b\ , \tag{3.2}$$

where the following definitions were used:

$$G_{ab} = \rho^i_a g_{ij}\rho^j_b\ , \tag{3.3}$$
$$D_{ab} = \iota_{\rho_{[a}}\theta_{b]} - T^c_{ab}\widetilde{X}_c\ , \tag{3.4}$$

and

$$\xi_a = \theta_a + \mathrm{d}\widetilde{X}_a + \omega^b_a\widetilde{X}_b\ , \tag{3.5}$$
$$\rho^*_a = g_{ij}\rho^i_a\mathrm{d}X^j\ . \tag{3.6}$$

Now in order to eliminate the gauge fields from the action, it is required to solve the field equation (3.2) for A^a. Since this equation involves the differentials $\mathrm{d}X^i$ and $\mathrm{d}\widetilde{X}_a$, as well as their Hodge duals, we anticipate that in general A^a will contain all four corresponding terms. Thus we are naturally led to make the following ansatz:

$$A^a = M^{ab}\rho^*_b + N^{ab}\xi_b + P^{ab}*\rho^*_b + Q^{ab}*\xi_b\ , \tag{3.7}$$

where M,N,P and Q are to be determined. This is essentially the same trick one uses to derive dual models in the standard approaches of Refs. [7, 20]. Inserting this ansatz in the relevant field equation, one ends up with a linear system of equations which is solved by

$$\begin{aligned} Q &= -(G-DG^{-1}D)^{-1} \ , \\ M &= -Q \ , \\ N &= -G^{-1}DQ \ , \\ P &= G^{-1}DQ \ , \end{aligned} \tag{3.8}$$

where G and D are the matrices corresponding to the definitions (3.3) and (3.4). These expressions are not surprising if one compares to similar results in Ref. [20].

The last step is to insert the result for A^a in the action S_{gauged}. This then leads to the dual action functional

$$S_{\text{dual}} = \int_\Sigma \Big(\tfrac{1}{2}(G-DG^{-1}D)^{ab} e_a\wedge *e_b - \tfrac{1}{2}\big(G^{-1}D(G-DG^{-1}D)^{-1}\big)^{ab} e_a\wedge e_b \Big) \ , \tag{3.9}$$

where we defined the 1-forms

$$e_a = \mathrm{d}\widetilde{X}_a + \theta_a - (\omega^b_{ai}\widetilde{X}_b + (G^{-1}D)^b_a\rho^k_b g_{ki})\mathrm{d}X^i \ . \tag{3.10}$$

At the *classical* level this is the dual action functional from which a new metric and a new Kalb-Ramond field (or 3-form H) can be read off. It is observed that the coframe defined by e_a mixes the original scalar fields X^i with the new ones $\widetilde{X}_a$.[7] Formally this is the case in the standard approach as well, however there it is clear that at the end of the day the two sets are disentangled. This is not obvious in the present case, however we will study some particular examples below to investigate the possibilities and comment accordingly.

4. Some simple cases

As mentioned in the Introduction, at this stage one might worry whether *any* non-trivial example realizing the above results exists. In other words whether non-zero parameters ω^a_{bi} can be found such that all the invariance conditions and constraints that make S_{gauged} consistent are satisfied. Here we discuss some examples where the procedure indeed works. We already note that they are just toy models and they do not correspond to true string backgrounds at the quantum level. Whether new dual string backgrounds exist remains an open question that requires a careful analysis which will be performed elsewhere. However, certain toy models are useful and often provide valuable hints.

Abelian$\oplus$Non-isometric. First we discuss a simple example where the vector fields ρ_a are Abelian, thus there are no structure functions (or, for that matter, even constants) involved; however not all of them generate isometries for the metric.

[7] This can be related to the embedding of the string world sheet in a higher-dimensional geometry. It is useful to note that such embeddings also appear for example in [23, 24] from a different perspective but still in the context of sigma models.

Let us be more specific. Consider the metric

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2 - x^1\mathrm{d}x^3)^2 + (\mathrm{d}x^3)^2. \tag{4.1}$$

This is the metric of the well-known 3D Heisenberg nilmanifold. This is obtained as the quotient of the real 3D Heisenberg group $\mathrm{H}(3;\mathbb{R})$ by its integer counterpart $\mathrm{H}(3;\mathbb{Z})$. In more geometric terms it gives rise to a non-trivial 2-torus fibration over a base circle, as it can be easily seen by taking the basis of 1-forms

$$e^1 = \mathrm{d}x^1 \ , \quad e^2 = \mathrm{d}x^2 - x^1\mathrm{d}x^3 \ , \quad e^3 = \mathrm{d}x^3 \ , \tag{4.2}$$

and asking it to be globally well-defined. This leads to the identifications

$$(x^1,x^2,x^3) \sim (x^1,x^2+2\pi R,x^3) \sim (x^1,x^2,x^3+2\pi R) \sim (x^1+2\pi R,x^2+2\pi Rx^3,x^3) \ , \tag{4.3}$$

where for simplicity we assumed equal radii for the three circles. It is observed that a torus $T^2_{(x^2,x^3)}$ is fibered over the circle $S^1_{(x^1)}$. For this reason, this manifold is sometimes called twisted torus in the physics literature. The 1-forms (4.2) satisfy the Maurer-Cartan equation

$$\mathrm{d}e^2 = -C^2_{13}e^1\wedge e^3 \ , \quad C^2_{13} = 1 \ . \tag{4.4}$$

The non-vanishing C^2_{13} is often referred to as geometric flux in the context of string compactifications.

In this example we would like to determine the gauge theory S_{gauged} and the dual sigma model in the case of vanishing Wess-Zumino term, namely $H = 0$, and for the choice of vector fields $\rho_a = (\partial_1, \partial_2)$, which obviously commute. In other words, we would simply like to dualize along the directions x^1 and x^2. Is that possible? First we ask whether the chosen vector fields are Killing. The second one, ρ_2, indeed is. In fact, choosing to dualize along only this vector field produces a dual sigma model with constant Wess-Zumino term H_{123} and target space a 3-torus. However the first vector field is not Killing; it satisfies

$$\mathscr{L}_{\rho_1}g = -\mathrm{d}x^2\otimes\mathrm{d}x^3 - \mathrm{d}x^3\otimes\mathrm{d}x^2 + 2x^1\mathrm{d}x^3\otimes\mathrm{d}x^3 \ . \tag{4.5}$$

This already indicates what ω^a_{bi} should be in order to compensate for the non-vanishing right hand side of the Lie derivative. The invariance condition (2.12) is solved with

$$\omega^2_{13} = -1 \ . \tag{4.6}$$

Note that all the rest of ω^a_{bi} are vanishing, and there is no property among indices that relates any other to the single non-vanishing component. It is clear that for this solution it holds that $R^a_b = 0$ and thus the second invariance condition (2.13) may be simply solved with the choice $\theta_a = 0$. Simple inspection of the constraints (2.14) and (2.15) shows that they are satisfied. This means that we indeed have a consistent gauged sigma model with two gauge fields A^1 and A^2 and two Lagrange multipliers $\widetilde{X}_1$ and $\widetilde{X}_2$. Integrating out the latter and gauge fixing we obtain the sigma model

$$S = \int_\Sigma \tfrac{1}{2}\delta_{ab}e^a\wedge *e^b \ , \tag{4.7}$$

which is precisely the purely geometric sigma model with target the Heisenberg nilmanifold. However, integrating out the gauge fields through their field equations,

$$\begin{aligned} A^1 &= \mathrm{d}X^1 + *(\mathrm{d}\widetilde{X}_1 - \widetilde{X}_2\mathrm{d}X^3) , \\ A^2 &= \mathrm{d}X^2 - X^1\mathrm{d}X^3 + *\mathrm{d}\widetilde{X}_2 , \end{aligned}$$

we are led to the following dual model:

$$S_{\text{dual}} = \int_\Sigma \Big(\tfrac{1}{2}(\mathrm{d}\widetilde{X}_1 - \widetilde{X}_2\mathrm{d}X^3)\wedge *(\mathrm{d}\widetilde{X}_1 - \widetilde{X}_2\mathrm{d}X^3) + \tfrac{1}{2}\mathrm{d}\widetilde{X}_2\wedge *\mathrm{d}\widetilde{X}_2 + \tfrac{1}{2}\mathrm{d}X^3\wedge *\mathrm{d}X^3 \Big) , \tag{4.8}$$

up to total derivatives. This comprises a coframe

$$e_1 = \mathrm{d}\widetilde{X}_1 - \widetilde{X}_2\mathrm{d}X^3 , \quad e_2 = \mathrm{d}\widetilde{X}_2 , \quad e^3 = \mathrm{d}X^3 , \tag{4.9}$$

which satisfies

$$\mathrm{d}e_1 = -C^2_{13}e_2\wedge e^3 .$$

The result has a flavour of self-duality. This is expected though, in view of the fact that if one first T-dualizes with respect to ρ_2, thus reaching the case of the 3-torus with H flux, ρ_1 in this intermediate situation is now a Killing vector. Then the T-duality along ρ_1 leads again to a Heisenberg nilmanifold. At the level of the B field, a gauge transformation is needed in this intermediate step. This may be summarized as follows

$$\begin{tikzcd} & H_{123} \arrow[loop above, "\delta B"] \arrow[d, "\mathrm{T}^{\text{iso}}_{\partial_1}"] \\ C^2_{13} \arrow[ur, bend left, "\mathrm{T}^{\text{iso}}_{\partial_2}"] \arrow[r, "\mathrm{T}^{\text{non-iso}}_{(\partial_1,\partial_2)}"] & C^1_{23} \end{tikzcd} \tag{4.10}$$

with the diagram being commutative. Thus this example represents a case where the non-isometric approach acts as a short-cut in reproducing an otherwise known isometric duality chain.

Non-Abelian⊕Non-isometric. A more involved example, discussed already in Ref. [1], starts with the same manifold M as above, but this time with a choice of non-Abelian vector fields ρ_a. The most obvious option for such a set are the vector fields dual to the 1-forms e^a. These are given by

$$\rho_a = (\partial_1, \partial_2, \partial_3 + x^1\partial_2) .$$

Note that the first two are the same as before, but the added one is such that $[\rho_1,\rho_3] = C^2_{13}\rho_2$, thus they satisfy the 3D Heisenberg algebra. As before, ρ_2 is Killing but ρ_1 and ρ_3 are not; ρ_3 satisfies

$$\mathscr{L}_{\rho_3}g = \mathrm{d}x^1\otimes\mathrm{d}x^2 + \mathrm{d}x^2\otimes\mathrm{d}x^1 - x^1\mathrm{d}x^1\otimes\mathrm{d}x^3 - x^1\mathrm{d}x^3\otimes\mathrm{d}x^1 . \tag{4.11}$$

Once more there is no Wess-Zumino term and θ_a are taken to be zero. Now there are three gauge fields A^1,A^2 and A^3 and three associated Lagrange multipliers. The non-vanishing ω^a_{bi} coefficients that guarantee that all the invariance conditions and constraints are solved now are [8]

$$\omega^2_{31} = -\omega^2_{13} = 1 . \tag{4.12}$$

[8] A numerical mistake in Ref. [1], which propagated in the ensuing Eqs. (4.13) and (4.14) is corrected here. This led to a somewhat obscure interpretation of the dual action in [1], which is now fully clarified.

Integrating out the Lagrange multipliers one arrives again at the action (4.7), while the integration of the gauge fields leads first to the field equations

$$\begin{aligned} A^1 &= \mathrm{d}X^1 - \tfrac{\widetilde{X}_2}{1+(\widetilde{X}_2)^2}\mathrm{d}\widetilde{X}_3 - \tfrac{1}{1+(\widetilde{X}_2)^2} * \mathrm{d}\widetilde{X}_1 \ , \\ A^2 &= \mathrm{d}X^2 - X^1\mathrm{d}X^3 - *\mathrm{d}\widetilde{X}_2 \ , \\ A^3 &= \mathrm{d}X^3 + \tfrac{\widetilde{X}_2}{1+(\widetilde{X}_2)^2}\mathrm{d}\widetilde{X}_1 - \tfrac{1}{1+(\widetilde{X}_2)^2} * \mathrm{d}\widetilde{X}_3 \ , \end{aligned} \tag{4.13}$$

and upon substitution in S_{gauged} to the dual action

$$S_{\text{dual}} = \frac{1}{2}\int_\Sigma \left(\mathrm{d}\widetilde{X}_2 \wedge *\mathrm{d}\widetilde{X}_2 + \frac{1}{1+(\widetilde{X}_2)^2}(\mathrm{d}\widetilde{X}_1 \wedge *\mathrm{d}\widetilde{X}_1 + \mathrm{d}\widetilde{X}_3 \wedge *\mathrm{d}\widetilde{X}_3) + \frac{2\widetilde{X}_2}{1+(\widetilde{X}_2)^2}\mathrm{d}\widetilde{X}_1 \wedge \mathrm{d}\widetilde{X}_3 \right) \ , \tag{4.14}$$

up to total derivatives. This action exhibits a structure identical to that of a T-fold [25,26], namely a non-geometric Q flux background. In fact, although in a less expected fashion than in the previous example, we encounter again a commutative diagram:

$$\begin{tikzcd} H_{123} \arrow[loop above, "\delta B"] \arrow[r, "\mathrm{T}^{\text{iso}}_{\partial_1}"] & C^1_{23} \arrow[d, "\mathrm{T}^{\text{iso}}_{\partial_3}"] \\ C^2_{13} \arrow[u, "\mathrm{T}^{\text{iso}}_{\partial_2}"] \arrow[r, "\mathrm{T}^{\text{non-iso}}_{\rho_a}"] & Q^{13}_2 \end{tikzcd} \tag{4.15}$$

The commutativity of this diagram is less expected because the Killing vector fields for the isometric route are not all the same with the ones in the non-isometric route—in particular the third one is ∂_3 and $\rho_3 = \partial_3 + x^1\partial_2$ respectively. Thus in this example we encounter a less obvious non-isometric short cut for an isometric duality chain. Of course the real challenge would be to perform a T-duality for a case that is completely out of the realm of standard methods. We mention such a possibility through the following example.

A note on isometries broken only by the Wess-Zumino term. Let us briefly refer to another interesting option arising in the context of non-isometric T-duality. Suppose we have a set of Abelian vector fields ρ_a, thus $C^a_{bc} = 0$, a metric for which these vector fields are all Killing, and a non-vanishing Wess-Zumino term given by H. We encounter now the possibility that although the vector fields generate would-be isometries, these isometries are broken by the Wess-Zumino term. Thus, in case $\omega^a_{bi} = 0$ we face a serious obstacle: although the invariance conditions (2.12) and (2.13) are satisfied (at least for some choice of θ_a), the constraint (2.15), reduced now to (2.19), cannot be satisfied. This was also noticed in Ref. [16] and it is related to the problem of finding a triple T-dual of the torus with H flux. Although we are not going to solve this problem here, we now indicate a possible direction for its potential solution. Although the Lie derivative of the metric is zero, (2.12) does not mean that ω^a_{bi} has to vanish. Instead it just means that

$$\rho^k_b \omega^b_{a(i} g_{j)k} = 0 \ ,$$

which is much milder. For example, for the simple case of $\rho_a = \delta_a^i \partial_i$ and $g_{ij} = \delta_{ij}$, it reduces to

$$\omega_{a(ij)} = 0 \, ,$$

which is solved by any set of coefficients antisymmetric in the indices involved in the above symmetrization. The point is that the previously lethal constraint now reads as in Eq. (2.15) and its right hand side is not any more necessarily zero. This allows for the possibility of solving the constraints for such cases too, previously impossible. We plan to report on this issue in a future publication.

5. Take-home messages

The main messages of this work may be summarized as follows

- Given background fields g and B, there exist classically consistent gauged 2D sigma models of maps $X = (X^i) : \Sigma \to M$ whose gauge symmetry is generated by vector fields that do not necessarily generate isometries.
- These gauged sigma models can act as intermediate gauge theories to study candidate T-dual string backgrounds beyond the realm of isometry.
- Non-trivial toy models that realize such non-Abelian and non-isometric T-duality do exist.

Although these results are certainly encouraging it is equally useful to keep in mind the limitations of the approach presented here. Some of these are the following: (α) The analysis is limited to the classical level. It is not yet clear whether our results survive quantization, (β) in relation to the above, we were only able at this stage to verify that the approach works non-trivially in toy models and not in true, conformal string backgrounds, (γ) the dilaton was simply ignored, (δ) the fully worked-out examples where our approach indeed works are so far just short cuts for results that can be obtained by the standard method. A merit test for our approach would be, for instance, the proper derivation of the triple T-dual of a torus threaded by H flux. Further work is required in order to answer questions posed by these remarks.

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