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Number theoretic tools in perturbative quantum field theory

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Feynman amplitudes in perturbative quantum field theory are being expressed in terms of an algebra of functions, extending the familiar logarithms, and associated numbers - *periods*. The study of these functions (including hyperlogarithms) and numbers (like the multiple zeta values), that dates back to Leibniz and Euler, has attracted anew the interest of algebraic geometers and number theorists during the last decades. The two originally independent developments are recently coming together in an unlikely confluence between particle physics and what were regarded as the most abstruse branches of pure mathematics.

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1. Introduction

The part of fundamental physics that does have applications to high-energy physics is good old perturbative *quantum field theory* (QFT) - just as it used to be with QED nearly 70 years ago. With the fast growing number and complexity of higher order Feynman graphs and with the advent of modern computers one could imagine a rather dull future for aspiring mathematical physicists in this field. Indeed, I witnessed a brave young Italian telling theorists at a CERN seminar to just fill the entries of his perfect computer program rather than polluting the field with their confused ideas. Happily, things developed differently.

A remarkable prologue to the recent development is provided by the analytic calculations of the anomalous magnetic moment of the electron. It involves prominently the multiple zeta values (MZVs) or rather their close relatives, the Euler alternating (phi-) series – see Sect. 2 below.

These numbers appear as values of the *hyperlogarithms*

$$Li_{n_1,\ldots,n_d}(z_1,\ldots,z_d) = \sum_{1\leq k_1<\ldots<k_d} \frac{z_1^{k_1}\ldots z_d^{k_d}}{k_1^{n_1}\ldots k_d^{n_d}} \tag{1.1}$$

at algebraic arguments (so far, at roots of unity). An integral representation for the first non-elementary function of this family, the dilogarithm $Li_2(z)$, was introduced in a letter of Leibniz (to Johann Bernoulli) in 1696 and was studied by Euler and many others (see [Z] for a beautiful modern review and more historical references). The resurgence of polylogarithms in pure mathematics, anticipated by 19 century work of Kummer and Poincaré and a 20 century contribution by Lappo-Danilevsky, was prepared by the work of Chen [C, B09] on iterated path integrals. David Broadhurst was a pioneer in the systematic study of MZV in QFT (his influential papers with Dirk Kreimer [BK] contain earlier references; see also [B10, B16]). In the work on position space renormalization, initiated by the late Raymond Stora, [NST], they appear as residues of regularized primitively divergent Feynman integrals. More generally, it was demonstrated in [BW] that the values of Feynman amplitudes for rational ratios of dimensionful arguments are *periods* in the sense of [KZ]. The notes [Zh] of one of the many conferences dedicated to this topic, entitled *Polylogarithms as a Bridge between Number Theory and Particle Physics*, contain a bibliography of some 394 entries.

This is an active field for both physicists and mathematicians. Rather than using ready tools/formulas from "classical mathematics" we are working separately and together on (different sides of) the same problems.

After introducing in Sects. 3, 4 the algebras of hyperlogarithms and MZVs we briefly sketch (in Sect. 5) the status of one such problem: the result of [B1, B2] describing the weight spaces of (motivic) MZVs.

For a more comprehensive (33 pages long) exposition, including sections on residues of primitively divergent amplitudes, on single-valued hyperlogarithms and a historical survey - see [T1].

2. From Euler's alternating series to the electron magnetic moment

The idle curiosity of mathematicians in the late 17th and a good part of 18th centuries and the development of renormalization theory in quantum electrodynamics (QED) (triggered by precision measurements in the wake of World War II) started happily coming together during the second half of XX century.

Euler's interest in the zeta function and its alternating companion $\phi(s)$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad \left(= (1-2^{1-s})\,\zeta(s) \text{ for } \Re(s) > 1\right) \tag{2.1}$$

was triggered by the *Basel problem* [W] (posed by Pietro Mengoli in mid 17 century): to find a closed form expression for $\zeta(2)$. Euler discovered the non-trivial answer, $\zeta(2) = \frac{\pi^2}{6}$, in 1734 and ten years later found an expression for all $\zeta(2n)$, $n = 1,2,\ldots$, as a rational multiple of π^{2n} - see (4.6) below. Euler tried to extend the result to odd zeta values but it did not work [D12]. (We still have no proof that $\frac{\zeta(3)}{\pi^3}$ is irrational.) Trying to find polynomial relations among zeta and phi values Euler was led by the *stuffle product*

$$\zeta(m)\,\zeta(n) = \zeta(m,n) + \zeta(n,m) + \zeta(n+m);\, \phi(m)\phi(n) = \phi(m,n) + \phi(n,m) + \zeta(m+n), \tag{2.2}$$

to the concept of double zeta and double phi values

$$\zeta(m,n) = \sum_{0<k<\ell} \frac{1}{k^m \ell^n},\ \phi(m,n) = \sum_{0<k<\ell} \frac{(-1)^{k+\ell}}{k^m \ell^n}. \tag{2.3}$$

The alternating series $\phi(s)$ (2.1) (alias the Dirichlet *eta function*) provide faster convergence in a larger domain. While $\zeta(s)$ has a pole for $s = 1$, we have

$$\phi(1) = \ln 2. \tag{2.4}$$

At this point we skip a century and turn to physics. The electron magnetic dipole moment μ is expressed in terms of its charge e, mass m, and spin s by

$$\mu = g\frac{e}{2m}s,\ s = \frac{\hbar}{2} \tag{2.5}$$

where the Dirac equation gives for the *g-factor* (or *gyromagnetic ratio*) $g = 2$. A 1947 measurement (using Zeeman splitting in Ga atoms) showed that g is slightly (by a quantity of order 10^{-3}) bigger than 2. This motivated Schwinger to apply the recently developed (by Tomonaga and himself) method of renoramlization to calculate the first QED correction (of order $\alpha = \frac{e^2}{4\pi\hbar c}$) - in accord with experiment. In the language of Feynman graphs, developed around the same time, Schwinger's calculation corresponds to computing a single triangular diagram. The Feynman(-Dyson) rules provide a significant simplification which opens the way to higher order calculations: to each Feynman propagator - i.e., to each internal line correspond two terms in the original Tomonaga-Schwinger formalism (thus giving 2^L terms for a graph with L lines). Nevertheless, it took nine years after the 1948 Schwinger's calculation of the order α term in the anomalous magnetic moment of the electron[1], a_e, before the α^2 term was computed *analytically* by Peterman (and independently by Sommerfield) - correcting the computer aided calculation of Karplus and Kroll of 1950. It is given

[1]A firsthand account of the saga of $g-2$ is given in [K] where the reader will also find a comprehensive list of relevant references; for an entertaining lighter discussion - see [H].

by a sum of seven two loop graphs (colorfully depicted in [H]), expressed as a rational linear combination of the Euler phi function. The first two terms in the expansion of a_e in powers of the fine structure constant α read (cf. [Sch]):

$$a_e = \frac{g-2}{2} = \sum_{n\geq 1} A_n \left(\frac{\alpha}{\pi}\right)^n = \frac{1}{2}\frac{\alpha}{\pi} + (\phi(3) - 6\phi(1)\phi(2) + \frac{197}{2^4 3^2})\left(\frac{\alpha}{\pi}\right)^2 \ (+...). \tag{2.6}$$

The same weight three combination, $\phi(3) - 6\phi(1)\phi(2)$, appears in the second order of the Lamb shift calculation, [LPR]. Increasing the accuracy with a couple of decimal points is a career consuming enterprise for both experimenters and theorists. Hans Dehmelt and his group at the University of Washington started their work on the penning trap involving isolated electrons (with given names, like family pets ... [H]) in 1958 and only completed it by 1987. The next, third order calculation of the anomalous magnetic moment had to wait for almost forty years after the work of Peterman and Sommerfield. Two competing - and helping each other - theoretical assaults on the problem began in 1969. Toichiro Kinoshita[2] started a computation, developing by 1974 a method of numerical renormalization, adaptable to automation – completed by 1995. Ettore Remiddi, joined at the final stretch by Stefano Laporta, *calculated analytically the seventy two three-loop graphs* (also depicted in [H]), finishing a year later, [LR]. Their result can be again expressed as a linear combination of (multiple) phi values - of overall weight upto five - with rational coefficients:

$$\left(\frac{\alpha}{\pi}\right)^3 : A_3 = \frac{2}{3^2}(83\,\phi(2)\,\phi(3) - 43\,\phi(5)) - \frac{50}{3}\phi(1,3) + \frac{13}{5}\phi(2)^2 \tag{2.7}$$
$$\frac{278}{3}\left(\frac{\phi(3)}{3^2} - 12\,\phi(1)\,\phi(2)\right) + \frac{34202}{3^3\,5}\phi(2) + \frac{28259}{2^5\,3^4}.$$

The next round of the "tennis match between theory and experiment" (to quote [H]) involved the 20 years effort of the Harvard group of Gabrielse to increase 15 times the experimental accuracy (completed in 2008), matched by the numerical work of (the approaching 90 veteran) Kinoshita and his group who computed the 891 4-loop graphs and estimated the contribution of the 12 672 5-loop ones by 2012. The outcomes agree within one part in a trillion!

Eqs. (2.6) (2.7) provide exact analytic results for the coefficients A_2, A_3 of the $\frac{\alpha}{\pi}$ expansion - with no theoretical/numerical uncertainty involved - and they are confirmed by experiment with unprecedented accuracy. One is tempted to place these formulas among what Galileo's alter ego Salviati elevates to "those few which the human intellect does understand, I believe that its knowledge equals the Divine in objective certainty" [G].

Developments in the last two decades are raising hope that Feynman amplitudes in the Standard Model of particle physics can be computed exactly order by order as rational linear combination in a basis of special functions ("master integrals") and numbers (cf. the TASI lectures [D] and references therein).

3. Differential graded algebra of hyperlogarithms

The hyperlogarithms are not just a list of useful special functions. They have a rich algebraic structure and satisfy *unipotent differential equations*, [B09, B1].

[2] About the early years of this hero of the a_e-calculations - see his own account [K16].

Let $\sigma_0 = 0, \sigma_1, ..., \sigma_N$ be distinct complex numbers corresponding to an alphabet $X = \{e_0, ..., e_N\}$. Let X^* be the set of words w in this alphabet including the empty word $\emptyset$. The hyperlogarithm $L_w(z)$ is an iterated integral [C, B09] defined recursively in any simply connected open subset U of the punctured complex plane $\mathbb{C} \setminus \Sigma$, $\Sigma = \{\sigma_0, ..., \sigma_N\}$ by the differential equations[3]

$$\frac{d}{dz} L_{w\sigma}(z) = \frac{L_w(z)}{z-\sigma}, \quad \sigma \in \Sigma, \tag{3.1}$$

and the initial conditions

$$L_w(0) = 0 \quad \text{for} \quad w \neq 0^n (= 0 \dots 0), \quad L_{0^n}(z) = \frac{(\ln z)^n}{n!}, \; L_\emptyset = 1. \tag{3.2}$$

There is a correspondence between iterated integrals and multiple power series:

$$(-1)^d L_{\sigma_1 0^{n_1-1} \dots \sigma_d 0^{n_d-1}}(z) = Li_{n_1,...,n_d}\left(\frac{\sigma_2}{\sigma_1}, ..., \frac{\sigma_d}{\sigma_{d-1}}, \frac{z}{\sigma_d}\right) \tag{3.3}$$

where $Li_{n_1,...,n_d}$ is given by the d-fold series (1.1). More generally, we have

$$(-1)^d L_{0^{n_0}\sigma_1 0^{n_1-1} \dots \sigma_d 0^{n_d-1}}(z) = \sum_{\substack{k_0 \geq 0 k_i \geq n_i, 1 \leq i \leq d \\ k_0 + \dots + k_d = n_0 + \dots + n_d}} (-1)^{k_0+n_0} \prod_{i=1}^{d} \binom{k_i - 1}{n_i - 1} L_{0^{k_0}}(z) \, L_{k_1 - k_r}\left(\frac{\sigma_2}{\sigma_1}, ..., \frac{\sigma_d}{\sigma_{d-1}}, \frac{z}{\sigma_d}\right). \tag{3.4}$$

In particular, $L_{01}(z) = Li_2(z) - \ln z Li_1(z) = Li_2(z) + \ln z \ln(1-z)$. The number of letters $|w| = n_0 + ... + n_d$ of a word w defines its *weight*, while the number d of non zero letters is its *depth*. We observe that the product $L_w L_{w'}$ of two hyperlogarithms of weights $|w|, |w'|$ and depths d, d' can be expanded in hyperlogarithms of weight $|w| + |w'|$ and depth $d + d'$ (as the product of simplices can be expanded into a sum of higher dimensional simplices). This observation can be formalized as follows. The set X^* of words is naturally equipped with a commutative *shuffle product* $w ⧢ w'$ defined recursively by

$$\emptyset ⧢ w = w (= w ⧢ \emptyset), \quad au ⧢ bv = a(u ⧢ bv) + b(au ⧢ v) \tag{3.5}$$

where u, v, w are (arbitrary) words while a, b are letters (note that the empty word $\emptyset$ is *not* a letter). We denote by

$$\mathscr{O}_\Sigma = \mathbb{C}\left[z, \left(\frac{1}{z-\sigma_i}\right)_{i=1,...,N}\right] \tag{3.6}$$

the ring of regular functions on $\mathbb{C} \setminus \Sigma$. Extending by $\mathscr{O}_\Sigma$ linearity the correspondence $w \to L_w$ one proves that it defines a homomorphism of shuffle algebras $\mathscr{O}_\Sigma \otimes \mathbb{C}(X) \to \mathscr{L}_\Sigma$ where $\mathscr{L}_\Sigma$ is the $\mathscr{O}_\Sigma$ span of $L_w, w \in X^*$. The commutativity of the shuffle product is made obvious by the identity

$$L_{u ⧢ v} = L_u L_v (= L_v L_u). \tag{3.7}$$

If the shuffle relations are suggested by the expansion of products of iterated integrals, the product of series expansions of type (1.1) suggests the (also commutative) *stuffle product*. Rather than

[3] We use, following [B1, S], concatenation to the right. Other authors, [D], use the opposite convention. This also concerns the definition of coproduct (5.10) below.

giving a cumbersome general definition we shall illustrate the rule on the simple example of the product of depth one and depth two factors (cf. [D]):

$$
\begin{aligned}
Li_{n_1,n_2}(z_1,z_2)Li_{n_3}(z_3) = Li_{n_1,n_2,n_3}(z_1,z_2,z_3)+\\
Li_{n_1,n_3,n_2}(z_1,z_3,z_2)+Li_{n_3,n_1,n_2}(z_3,z_1,z_2)+\\
Li_{n_1,n_2+n_3}(z_1,z_2z_3)+Li_{n_1+n_3,n_2}(z_1z_3,z_2).
\end{aligned}
\tag{3.8}
$$

We observe the that the multiple polylogarithms of one variable (with $z_1 = ... = z_{d-1} = 1$), considered in [S], span a shuffle but not a stuffle algebra. As seen from the above example the stuffle product also respects the weight but (in contrast to the shuffle product) only filters the depth: the depth of each term in the right hand side does not exceed the sum of depths of the factors in the left hand side (which is three in Eq. (3.8)).

It is convenient to rewrite the definition of hyperlogarithms in terms of a formal series $L(z)$ with values in the (free) tensor algebra $\mathbb{C}(X)$ (the complex vector space generated by all words in X^*) which satisfies the *Knizhnik-Zamoldchikov (KZ) equation*:

$$
L(z) := \sum_w L_w(z)w, \ \frac{d}{dz}L(z) = L(z)\sum_{i=0}^{N}\frac{e_i}{z-\sigma_i}. \tag{3.9}
$$

One assigns weight -1 to e_σ, so that $L(z)$ carries weight zero. If the index of the hyperlogarithm L_w is expressed by its (potential) singularities σ_i the word w which multiplies it in the series (3.9) should be written in terms of the corresponding (noncommuting) symbols e_i (thus justifying the apparent doubling of notation). In the special case when the alphabet X consists of just two letters e_0, e_1 corresponding to $\sigma_0 = 0, \sigma_1 = 1$, $L(z)$ is the generating function of the *classical multipolylogarithms* while its value at $z = 1, Z := L(1)$ is the generating function of MZVs. In these notations the *monodromy* of L around the points 0 and 1 is given by

$$
\begin{aligned}
\mathscr{M}_0 L(z) = e^{2\pi i e_0} L(z)\,, \quad \mathscr{M}_1 L(z) = Z\, e^{2\pi i e_1} Z^{-1} L(z),\\
Z = \sum_w \zeta_w w = 1+\zeta(2)(e_0e_1 - e_1e_0)+...,
\end{aligned}
\tag{3.10}
$$

so that $\mathscr{M}_0 L_{0^n}(z) = L_{0^n}(z) + 2\pi i L_{0^{(n-1)}}(z)$, $\mathscr{M}_1 Li_n(z) = Li_n(z) - 2\pi i L_{0^{(n-1)}}(z)$. (Formal power series starting with 1 are invertible so that Z^{-1} is well defined.)

Knowing the action of the monodromy M_{σ_i} around each singular point of a hyperlogarithm one can construct single valued hyperlogarithms in the tensor product of $\mathscr{L}_\Sigma$ with its complex conjugate [B]. A detailed survey of Brown's 2004 work on the (classical) *single-valued multiple polylogarithms* (SVMPs), $P_w(z)$ (with $w \in X^*$, a word in the two-letter alphabet) is provided in [S] (for a physicist oriented outlook - see [T, T1]). Their generating function $P_X(z)$ obeys the same KZ equation,

$$
\partial P_X(z) = P_X(z)(\frac{e_0}{z}+\frac{e_1}{z-1})\,,\ \partial := \frac{\partial}{\partial z}\,,\ (\bar\partial := \frac{\partial}{\partial \bar z}) \tag{3.11}
$$

as L (3.9). For small weights (or depth) we also have

$$
\bar\partial P_{aw}(z) = \frac{P_w(z)}{z-a} \,\text{for}\, |w| \le 1 \,\text{or}\, w = 0^n. \tag{3.12}
$$

This allows to readily compute all SVMPs of weight two and depth one:

$$P_{01} = L_{10}(\bar{z}) + L_{01}(z) + L_0(\bar{z})L_1(z) = Li_2(z) - Li_2(\bar{z}) + \ln \bar{z}z \ln(1-z),$$
$$P_{10} = L_{01}(\bar{z}) + L_{10}(z) + L_1(\bar{z})L_0(z) = Li_2(\bar{z}) - Li_2(z) + \ln \bar{z}z \ln(1-\bar{z}) \ . \tag{3.13}$$

They obey the shuffle relation $P_{01} + P_{10} = P_0 P_1 = \ln(z\bar{z})\ln((1-z)(1-\bar{z}))$ so that the only new weight two function is their difference,

$$P_{01} - P_{10} = 2(Li_2(z) - Li_2(\bar{z}) + \ln \bar{z}z \ln \frac{1-z}{1-\bar{z}}) = 4iD(z) \ , \tag{3.14}$$

proportional to the *Bloch-Wigner dilogarithm* (see the stimulating survey [Z]), $D(z) = \Im(Li_2(z) + \ln(1-z)\ln|z|)$. Remarkably, this function appears in the calculation of a primitively divergent (euclidean) position space Feynman amplitude $G(x_1,...,x_4)$ of the φ^4 theory with a single internal vertex:

$$\prod_{1\le i<j\le 4} x_{ij}^2 G(x_1,...,x_4) = x_{13}^2 x_{24}^2 \int \frac{d^4x}{\pi^2} \prod_{i=1}^{4} \frac{1}{(x-x_i)^2} = \frac{P_{01}(z) - P_{10}(z)}{z-\bar{z}}, \tag{3.15}$$

where $z, \bar{z}$ are determined by the (positive) conformal invariant crossratios

$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z} \ , \quad \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}). \tag{3.16}$$

In fact, any primitively divergent 4-point amplitude in the φ^4 theory is conformally invariant and can be expressed in terms of the variables (3.16) (and of SVMPs). They also appear in momentum space Feynman amplitudes (see [D]).

The weight of consecutive terms in the expansion of $L(z)$ (3.10) is the sum of the weights of hyperlogarithms and of the zeta factors. It makes therefore sense to proceed by reviewing the algebra of MZVs.

4. Formal multiple zeta values

We now turn to the alphabet X of two letters e_0, e_1 corresponding to $\sigma_0 = 0, \sigma_1 = 1$ and restrict the multiple polylogarithm (1.1) to a single variable:

$$Li_{n_1,...,n_d}(z) = \sum_{1\le k_1<...<k_d} \frac{z^{k_d}}{k_1^{n_1}...k_d^{n_d}}.$$

The MZV $\zeta(n_1,...,n_d)$ is then defined as its value at 1 whenever the corresponding series converges. Using also (3.3) we can write:

$$(-1)^d \zeta_{10^{n_1-1}...10^{n_d-1}} = \zeta(n_1,...,n_d) = \sum_{1\le k_1<...<k_d} \frac{1}{k_1^{n_1}...k_d^{n_d}} \text{ for } n_d > 1. \tag{4.1}$$

The convergent MZVs of a given weight satisfy a number of shuffle and stuffle identities. Looking for instance at the shuffle (sh) and the stuffle (st) products of two $-\zeta_{10} = \zeta(2)$ we find:

$$sh: \zeta_{10}^2 = 4\zeta_{1100} + 2\zeta_{1010} (= 4\zeta(1,3) + 2\zeta(2,2)); \ st: \zeta(2)^2 = 2\zeta(2,2) + \zeta(4);$$
$$\text{hence } \zeta(4) = 4\zeta(1,3) = \zeta(2)^2 - 2\zeta(2,2). \tag{4.2}$$

There are no non-zero convergent words of weight 1 and hence no shuffle or stuffle relations of weight 3. On the other hand, already Euler has discovered the equality $\zeta(1,2) = \zeta(3)$. Thus shuffle and stuffle relations among convergent words do not exhaust all known relations among MZVs of a given weight. Introducing the divergent zeta values which correspond to $n_d = 1$ we observe that they cancel in the difference between the shuffle and stuffle products $u ⧢ v - u * v$ of divergent words. For instance, at weight 3 we have

$$\zeta((1) ⧢ (2)) = 2\zeta(1,2) + \zeta(2,1); \, \zeta((1) * (2)) = \zeta(1,2) + \zeta(3) + \zeta(2,1). \tag{4.3}$$

Extending the homomorphism $w \to \zeta(w)$ as a homomorphism of both the shuffle and the stuffle algebras to divergent words, assuming, in particular, that $\zeta((1) ⧢ (2)) = \zeta((1) * (2)) = \zeta(1)\zeta(2)$ and taking the difference of the two equations (4.3), the divergent zeta's cancel and we recover the Euler relation. In fact, it suffices to add the difference of products with the divergent word (1),

$$\zeta((1) ⧢ w - (1) * w) = 0 \text{ for all convergent words } w, \tag{4.4}$$

to the shuffle and stuffle relations among convergent words in order to obtain all known relations among MZVs of a given weight. For $w = (n), n \geq 2$ (a word of depth 1), Eq. (4.4) gives

$$\zeta((1) ⧢ (n) - (1) * (n)) = \sum_{i=1}^{n-1} \zeta(i, n+1-i) - \zeta(n+1) = 0 \tag{4.5}$$

(another relation known to Euler). The discovery (and the proof) that

$$\zeta(2n) = -\frac{B_{2n}}{2(2n)!}(2\pi i)^{2n}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \, (-1)^{n-1} B_{2n} \in \mathbb{Q}_{>0}, \tag{4.6}$$

(B_n being the Bernoulli numbers), made Euler famous early on, [W].

We introduce following Leila Schneps [S11] the notion of a $\mathbb{Q}$-algebra $\mathscr{F}\mathscr{Z}$ of *formal MZVs* ζ^f which satisfy the relations:

$$\zeta^f(1) = 0, \, \zeta^f(u)\zeta^f(v) = \zeta^f(u ⧢ v) = \zeta^f(u * v), \, \zeta^f((1) ⧢ w - (1) * w) = 0. \tag{4.7}$$

The algebra $\mathscr{F}\mathscr{Z} = \bigoplus_n \mathscr{F}\mathscr{Z}_n$ is *weight graded* and

$$\begin{aligned} \mathscr{F}\mathscr{Z}_0 = \mathbb{Q}, \mathscr{F}\mathscr{Z}_1 = \{0\}, \mathscr{F}\mathscr{Z}_2 = \langle\zeta(2)\rangle, \mathscr{F}\mathscr{Z}_3 = \langle\zeta(3)\rangle, \mathscr{F}\mathscr{Z}_4 = \langle\zeta(4)\rangle, \\ \mathscr{F}\mathscr{Z}_5 = \langle\zeta(5), \zeta(2)\zeta(3)\rangle, \mathscr{F}\mathscr{Z}_6 = \langle\zeta(2)^3, \zeta(3)^2\rangle, \\ \mathscr{F}\mathscr{Z}_7 = \langle\zeta(7), \zeta(2)\zeta(5), \zeta(2)^2\zeta(3)\rangle, \end{aligned} \tag{4.8}$$

where $< x, y, ... >$ is the $\mathbb{Q}$ vector space spanned by $x, y, ...$ (and we have replaced ζ^f by ζ in the right hand side for short). Clearly, there is a surjection $\zeta^f \to \zeta$ of $\mathscr{F}\mathscr{Z}$ onto $\mathscr{Z}$. **The main conjecture in the theory of MZVs** is that *this surjection is an isomorphism of graded algebras*. This is a strong conjecture. If true it would imply that there is no linear relation among MZVs of different weights over the rationals; in particular, it would follow that all $\zeta(n)$ are irrational and linearly independent over the rationals. Actually, a less obvious statement is valid: such an isomorphism would imply that all MZVs are transcendental. Indeed, if a non-zero multiple zeta value is algebraic, then expanding out its minimal polynomial according to the shuffle relation

$\zeta(u)\zeta(v) = \zeta(u \sqcup\!\sqcup v)$ (starting with $\zeta^2(w)$) would give a linear combination of multiple zetas in different weights equal to zero, contradicting the weight grading. In fact, we only know that there are infinitely many linearly independent over $\mathbb{Q}$ odd zeta values (Ball and Rivoal, 2001) and that $\zeta(3)$ is irrational (Apéry, 1978). From now on, we shall follow the physicists' practice to treat this conjecture as true and to omit the f's in the notation for (formal) MZVs.

Examples: E1. In order to see that the space $\mathscr{Z}_4$ of weight four zeta values is 1-dimensional we should add to Eqs. (4.2) the relation (4.5) for $n = 3$ and its depth three counterpart:

$$\zeta((1) \sqcup\!\sqcup (1,2) - (1) * (1,2)) = \zeta(1,1,2) - \zeta(1,3) - \zeta(2,2) = 0. \tag{4.9}$$

This allows to express all zeta values of weight four as (positive) integer multiples of $\zeta(1,3)$ (see Eq. (B.8) of [T]).

E2. The shuffle and the stuffle products corresponding to $\zeta(2)\zeta(3)$ give two relations which combined with (4.5) for $n = 4$ allow to express the three double zeta values of weight five in terms of simple ones:

$$\zeta(1,4) = 2\zeta(5) - \zeta(2)\zeta(3),\ \zeta(2,3) = 3\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5),$$
$$\zeta(3,2) = \frac{9}{2}\zeta(5) - 2\zeta(2)\zeta(3). \tag{4.10}$$

One first needs a double zeta value, say $\zeta(3,5)$, in order to write a basis (of four elements) at weight eight (there being 60 relations among the 2^6 elements of $\mathscr{F}\mathscr{Z}_8$). It is natural to ask what is the dimension d_n of the space $\mathscr{F}\mathscr{Z}_n$ of (formal) MZVs of any given weight n and then to construct a basis of independent elements. These problems have only been solved for the so called *motivic MZV*. Here is a simple-minded version of their abstract construction.

5. Hopf algebra of motivic multiple zeta values

After the discovery that, for even n, $\zeta(n)$ is a rational multiple of π^n Euler calculated $\zeta(3)$ up to ten significant digits and convinced himself that it is not a rational multiple of π^3 (with a small denominator). Computers allow these days to increase the number of digits but not to prove algebraic (or even $\mathbb{Q}$-linear) independence. So, on the theoretical side, mathematicians proceeded to moving the problem elsewhere - managing in the last two decades to transform Gorthendieck's poetic vision of "motives" into a precise (and working!) mathematical tool.

Consider the *concatenation algebra*

$$\mathscr{C} = \mathbb{Q}\langle f_3, f_5, \ldots\rangle, \tag{5.1}$$

the free algebra over $\mathbb{Q}$ on the countable alphabet $\{f_3, f_5, \ldots.\}$ (see Example 21 in [Wa]). If we can identify the formal zeta values with the algebra

$$\mathscr{C}[f_2] = \mathscr{C} \otimes_{\mathbb{Q}} \mathbb{Q}[f_2], \tag{5.2}$$

which plays an important role in the theory of mixed Tate motives (see Sect. 3 of [B1]), we will be able to compute the dimension d_n of $\mathscr{Z}_n$ for any n. Indeed, the generating (or *Hibert-Poincaré*) series for the dimensions $d_n^{\mathscr{C}}$ of the weight n subspace of $\mathscr{C}$ is given by

$$\sum_{n\geq 0} d_n^{\mathscr{C}} t^n = \frac{1}{1-t^3-t^5-\dots} = \frac{1-t^2}{1-t^2-t^3} \tag{5.3}$$

while the corresponding series of the second factor $\mathbb{Q}[f_2]$ in (5.2) is $(1-t^2)^{-1}$. Multiplying the two we obtain - for the "motivic zeta values" - the dimensions d_n of the weight subspaces of $\mathscr{C}[f_2]$ conjectured by Don Zagier:

$$\sum_{n\geq 0} d_n t^n = \frac{1}{1-t^2-t^3}, d_0 = 1, d_1 = 0, d_2 = 1, d_{n+2} = d_n + d_{n-1}. \tag{5.4}$$

Here is a wonderful more detailed conjecture advanced by Broadhurst and Kreimer [BK] in 1997 whose motivic version is still occupying mathematicians.

Let $\mathscr{Z}_n^r$ be the linear span of $\zeta(n_1,\dots,n_k)$, $n_1+\dots+n_k = n$, $k \leq r$; we define $d_{n,r}$ as the dimension of the quotient space $\mathscr{Z}_n^r/\mathscr{Z}_n^{r-1}$. Broadhurst and Kreimer have advanced the following conjecture for the generating series of $d_{n,r}$ (based on experience with MZVs appearing in Feynman amplitudes):

$$D(X,Y) = \frac{1+\mathscr{E}(X)Y}{1-\mathscr{O}(X)Y+\mathscr{S}(X)Y^2(1-Y^2)} = \sum d_{n,r}X^nY^r. \tag{5.5}$$

Here $\mathscr{E}(X)$ and $\mathscr{O}(X)$ generate series of even and odd powers of X, respectively,

$$\mathscr{E}(X) = \frac{X^2}{1-X^2} = X^2+X^4+\dots, \mathscr{O}(X) = \frac{X^3}{1-X^2} = X^3+X^5+\dots, \tag{5.6}$$

while $\mathscr{S}(X)$ is the generating series for the dimensions of the spaces of *cusp modular forms* (see for background the physicists' oriented survey [Za]):

$$\mathscr{S}(X) = \frac{X^{12}}{(1-X^4)(1-X^6)}. \tag{5.7}$$

Setting in (5.5) $Y = 1$ we recover the Zagier conjecture (5.4) (proven for motivic MZVs). The ansatz (5.5) is presently only derived in the motivic case under additional assumptions (see [CGS]).

The concatenation algebra $\mathscr{C}$, identified with the quotient

$$\mathscr{C} = \mathscr{C}[f_2]/\mathbb{Q}[f_2], \tag{5.8}$$

can be equipped with a *Hopf algebra structure* (with f_i as primitive elements) with the *deconcatenation coproduct* $\Delta : \mathscr{C} \to \mathscr{C} \otimes \mathscr{C}$ given by

$$\Delta(f_{i_1}\dots f_{i_r}) = 1\otimes f_{i_1}\dots f_{i_r} + f_{i_1}\dots f_{i_r}\otimes 1 + \sum_{k=1}^{r-1} f_{i_1}\dots f_{i_k}\otimes f_{i_{k+1}}\dots f_{i_r}. \tag{5.9}$$

This coproduct can be extended to the trivial comodule $\mathscr{C}[f_2]$ (5.2) by setting

$$\Delta : \mathscr{C}[f_2] \to \mathscr{C}\otimes\mathscr{C}[f_2], \Delta(f_2) = 1\otimes f_2 \tag{5.10}$$

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(and assuming that f_2 commutes with f_{odd}). Remarkably, there appear to be a one-to-one (albeit non-canonical) correspondence between the bases of the weight spaces $\mathscr{Z}_n$ and $\mathscr{C}[f_2]_n$ as displayed in the following list ([B1], 3.4)

$$\begin{aligned}
&\langle\zeta(2)\rangle \leftrightarrow \langle f_2\rangle; \langle\zeta(3)\rangle \leftrightarrow \langle f_3\rangle; \langle\zeta(2)^2\rangle \leftrightarrow \langle f_2^2\rangle;\\
&\langle\zeta(5),\zeta(2)\zeta(3)\rangle \leftrightarrow \langle f_5, f_2f_3(=f_3f_2)\rangle; \langle\zeta(2)^3,\zeta(3)^2\rangle \leftrightarrow \langle f_2^3, f_3 ⧢ f_3\rangle;\\
&\langle\zeta(7),\zeta(2)\zeta(5),\zeta(2)^2\zeta(3)\rangle \leftrightarrow \langle f_7, f_2f_5, f_2^2f_3\rangle;\\
&\langle\zeta(2)^4,\zeta(2)\zeta(3)^2,\zeta(3)\zeta(5),\zeta(3,5)\rangle \leftrightarrow \langle f_2^4, f_3 ⧢ f_3 f_2, f_3 ⧢ f_5, f_5f_3\rangle.
\end{aligned} \tag{5.11}$$

There is a counterpart of Proposition 4.1 defining *motivic iterated integrals* whose Hopf algebra[4], [Gon], is non-canonically isomorphic to $\mathscr{C}[f_2]$. It allows to define a surjective *period map* $\mathscr{C}[f_2] \to \mathscr{Z}$ onto the algebra of real MZVs ([B1] Theorem 3.5). Since, on the other hand, $\mathscr{C}[f_2]$ satisfies the defining relations of the formal zeta values we have the surjections $\mathscr{F}\mathscr{Z} \to \mathscr{C}[f_2] \to \mathscr{Z}$. Our main conjecture would then mean that the two (surjective) maps are also injective and thus define isomorphisms of graded algebras. If true it would imply that the (infinite sequence of) numbers $\pi, \zeta(3), \zeta(5), \ldots$ are transcendental algebraically independent over the rationals (cf. [Wa]). It would also fix the dimension of the weight spaces $\mathscr{Z}_n$ to be equal to d_n (5.4). Presently, we only know that this is true for $n = 0,1,2,3,4$; in general, the above cited results prove that

$$dim\mathscr{Z}_n \le d_n\ , \quad dim\mathscr{Z}_n = d_n \quad \text{for} \quad n \le 4\ . \tag{5.12}$$

The validity of the above sharpened conjecture would imply, in particular, that $\zeta(2n+1)$ are primitive elements of the Hopf algebra of MZVs:

$$\Delta(\zeta(2n+1)) = \zeta(2n+1)\otimes 1 + 1\otimes\zeta(2n+1). \tag{5.13}$$

Eq. (4.6) precludes the possibility of extending this property to even zeta values. Indeed, it implies the relation $\zeta(2n) = b_n\zeta(2)^n, b_n = \frac{(24)^n|B_{2n}|}{2(2n)!}$ which is only compatible with $\Delta\zeta(2) = 1\otimes\zeta(2)$.

If for weights $n \le 7$ one can express all MZVs in terms of (products of) simple zeta values (of depth one) the last equation (5.11) shows that for $n \ge 8$ this is no longer possible. Brown [B2] has established that the *Hoffman elements* $\zeta(n_1,\ldots,n_d)$ with $n_i \in \{2,3\}$ form a basis of motivic zeta values for all n (see also [D12, Wa]).

* * *

The interplay between algebraic geometry, number theory and perturbative QFT is a young and vigorous subject and our survey is far from complete. We have not touched upon the application of cluster algebras to multileg on-shell Feynman amplitudes - see [GGSVV] for a remarkable first step in this direction. As hyperlogarithms and associated numbers do not suffice for expressing massive and higher order Feynman amplitudes, mathematicians and mathematical physicists are exploring their generalizations (including elliptic hyperlogarithms and modular forms) - for recent work and references see [ABW, BKV].

[4] Brown's definition which we follow differs from Goncharov's (adopted in [CGS]) in that the motivic $\zeta^m(2)$ is non-zero.

After this talk was presented a vigorous attempt has been made by Francis Brown [B15] (inspired by an ongoing study of φ^4 periods [PS]) to reveal structures common to *all* Feynman amplitudes - introducing the notion of a *cosmic Galois group of motivic periods* - a work opening a new chapter in the subject.

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