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Harmonic superspaces for $\mathcal{N} = (1,1)$ SYM theory

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This is an overview of the off-shell $\mathcal{N} = (1,0)$ and on-shell $\mathcal{N} = (1,1)$, $6D$ harmonic superspace formalism and its recent applications for the analysis of higher-dimension invariants in $\mathcal{N} = (1,1)$ SYM theory. The defining $\mathcal{N} = (1,1)$ SYM constraints are solved in terms of $\mathcal{N} = (1,0)$ harmonic superfields properly combined into the covariant $\mathcal{N} = (1,1)$ superfield strength. The latter is a convenient building block for the on-shell candidate counterterms and other invariants of $\mathcal{N} = (1,1)$ SYM. The invariants of the canonical dimensions $\underline{d = 6,8}$ and $\underline{d = 10}$ are explicitly presented. A crucial difference between the single- and double-trace $\underline{d = 10}$ invariants is pointed out. It could account for the absence of non-planar 3-loop divergencies in $\mathcal{N} = (1,1)$ SYM.

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1. Motivations and contents

For the last few years, there is a stable interest in the maximally extended (with 16 supercharges) supersymmetric gauge theories in diverse dimensions (see, e.g., [1]),

The renowned $\mathcal{N} = 4$, $4D$ SYM theory supplied the first example of an UV *finite* theory. Perhaps, it is also a *completely integrable* system [2]. The $\mathcal{N} = (1,1)$, $6D$ SYM is not renormalizable by formal power counting (the coupling constant is dimensionful) but it is also expected to possess various unique properties. In particular, it respects the so called “dual superconformal symmetry”, like its $4D$ counterpart [3]. It provides the effective theory descriptions of some particular low energy sectors of string theory, such as D5-brane dynamics. The full effective action of D5-brane, generalizing the $\mathcal{N} = (1,1)$ SYM action, was conjectured to be of non-abelian Born-Infeld type [4], [5]. The $\mathcal{N} = (1,1)$ SYM is anomaly free [6], as distinct from $\mathcal{N} = (1,0)$ SYM theory.

The $\mathcal{N} = (1,1)$ and $\mathcal{N} = (1,0)$ SYM theories can be regarded as toy models for $\mathcal{N} = 8$ supergravity and its some lower $\mathcal{N}$ descendants, which are also non-renormalizable by the formal counting.

The newest perturbative explicit calculations in $\mathcal{N} = (1,1)$ SYM show a lot of cancelations of the UV divergencies which cannot be predicted in advance. The theory is UV finite up to 2 loops, while at 3 loops only a single-trace (planar) counterterm of canonical dim 10 is required. The allowed double-trace (non-planar) counterterms do not appear [7], [8], [9]. Various arguments to explain why it happens were put forward in [10], [11], [12] [13], though the complete understanding is still lacking. This phenomenon implies the existence of some new non-renormalization theorems. As usual, to understand it in depth, the maximally supersymmetric off-shell formulations for $\mathcal{N} = (1,0)$ and $\mathcal{N} = (1,1)$ SYM theories are required.

The maximal off-shell supersymmetry that one can achieve in $6D$ is $\mathcal{N} = (1,0)$ supersymmetry. The most natural off-shell formulation of $\mathcal{N} = (1,0)$ SYM theory was given in harmonic $\mathcal{N} = (1,0), 6D$ superspace [14], [15] as a generalization of the $\mathcal{N} = 2, 4D$ harmonic superspace [16], [17]. This harmonic $6D$ formalism was further developed and applied in [18], [19], [20], [21] and [22].

The $\mathcal{N} = (1,1)$ SYM theory in the harmonic formulation can be schematically represented as a hybrid of two $\mathcal{N} = (1,0)$ theories, [$\mathcal{N} = (1,1)$ SYM] = [$\mathcal{N} = (1,0)$ SYM] + [$6D$ hypermultiplets] , with the second hidden on-shell $\mathcal{N} = (0,1)$ supersymmetry. How to construct higher-dimension $\mathcal{N} = (1,1)$ invariants in terms of $\mathcal{N} = (1,0)$ superfields?

One way is to follow the “brute-force” method. One starts with the appropriate dimension $\mathcal{N} = (1,0)$ SYM invariant and then completes it to $\mathcal{N} = (1,1)$ invariant by adding the proper hypermultiplet terms. This approach is rather cumbersome technically and actually works only for the lowest-order invariants. Nevertheless, based on this approach, in a recent paper [22] there was given a new proof of the 1- and 2-loop finiteness of $\mathcal{N} = (1,1)$ SYM theory by demonstrating the absence of $\mathcal{N} = (1,0)$ off-shell supersymmetric and gauge invariant counterterms of the canonical dimension $\underline{d = 6}$ and $\underline{d = 8}$, which would be non-vanishing on shell. However, starting with the invariants of dimension $\underline{d = 10}$, technical complications increase enormously.

The situation is simplified if one takes into account that for finding all admissible superfield counterterms it is enough to stay on the mass shell. One of the main results of [22] is the development of the new approach to constructing higher-dimension $\mathcal{N} = (1,1)$ invariants. It is based on the concept of the *on-shell* $\mathcal{N}=(1,1)$ harmonic superspace with the double set of the harmonic variables $u_i^\pm, u_A^\pm, i=1,2; A=1,2$ pioneered in [23]. The novel point of the construction in [22] is solving the $\mathcal{N}=(1,1)$ SYM constraints [25], [24] in terms of $\mathcal{N}=(1,0)$ superfields. Using these techniques, the $\underline{d=8}$ and $\underline{d=10}$ invariants were explicitly built in a simple way and an essential difference between the single- and double-trace $\underline{d=10}$ invariants was established. The present contribution provides a brief account of the $6D$ harmonic superspace methods, with the main focus on their recent uses in [22].

2. $6D$ superspaces and superfields

2.1 $6D$ superspaces

We start by listing various $\mathcal{N}=(1,0), 6D$ superspaces:

- The standard $\mathcal{N}=(1,0), 6D$ superspace is parametrized by the coordinates:

$$z=(x^M,\theta_i^a)\,,\quad M=0,\ldots,5\,,\ a=1,\ldots,4\,,\ i=1,2\,, \tag{2.1}$$

  where the Grassmann coordinates θ_i^a are pseudoreal.

- The harmonic $\mathcal{N}=(1,0), 6D$ superspace is obtained by adding $SU(2)$ harmonics to (2.1):

$$Z:=(z,u)=(x^M,\theta_i^a,u^{\pm i})\,,\quad u_i^-=(u_i^+)^*, u^{+i}u_i^-=1\,,\ u^{\pm i}\in SU(2)_R/U(1)\,. \tag{2.2}$$

- The *analytic* $\mathcal{N}=(1,0), 6D$ superspace is an invariant subspace of (2.2) with the halved number of Grassmann coordinates:

$$\zeta:=(x^M_{(\mathrm{an})},\theta^{+a},u^{\pm i})\subset Z\,,\quad x^M_{(\mathrm{an})}=x^M+\frac{i}{2}\theta^a_k\gamma^M_{ab}\theta^b_l u^{+k}u^{-l}\,,\quad \theta^{\pm a}=\theta^a_i u^{\pm i}\,. \tag{2.3}$$

In what follows, we will need the basic differential operators in the analytic basis of the harmonic superspace $Z_A:=(x^M_{(\mathrm{an})},\theta^{+a},u^{\pm i},\theta^{-a})$:

$$\begin{aligned}
&D^+_a=\partial_{-a}\,,\quad D^-_a=-\partial_{+a}-2i\theta^{-b}\partial_{ab}\,,\\
&D^0=u^{+i}\frac{\partial}{\partial u^{+i}}-u^{-i}\frac{\partial}{\partial u^{-i}}+\theta^{+a}\partial_{+a}-\theta^{-a}\partial_{-a}\\
&D^{++}=\partial^{++}+i\theta^{+a}\theta^{+b}\partial_{ab}+\theta^{+a}\partial_{-a}\,,\ D^{--}=\partial^{--}+i\theta^{-a}\theta^{-b}\partial_{ab}+\theta^{-a}\partial_{+a}\,,
\end{aligned} \tag{2.4}$$

where $\partial_{\pm a}\theta^{\pm b}=\delta^b_a$ and $\partial^{++}=u^{+i}\frac{\partial}{\partial u^{-i}},\quad \partial^{--}=u^{-i}\frac{\partial}{\partial u^{+i}}$.

2.2 Basic superfields

The basic object of $\mathcal{N}=(1,0)$ SYM theory is the analytic gauge $\mathcal{N}=(1,0)$ SYM connection $V^{++}(\zeta)$ covariantizing the analyticity-preserving harmonic derivative D^{++}:

$$\nabla^{++}=D^{++}+V^{++}\,,\quad \delta V^{++}=-\nabla^{++}\Lambda\,,\quad \Lambda=\Lambda(\zeta)\,. \tag{2.5}$$

The second harmonic (non-analytic) connection $V^{--}(Z)$ covariantizing D^{--},

$$\nabla^{--}=D^{--}+V^{--}\,,\quad \delta V^{--}=-\nabla^{--}\Lambda\,,$$

is related to V^{++} by the harmonic flatness condition

$$[\nabla^{++},\nabla^{--}]=D^0 \;\Rightarrow\; D^{++}V^{--}-D^{--}V^{++}+[V^{++},V^{--}]=0$$
$$\Rightarrow V^{--}=V^{--}(V^{++},u^{\pm})\,. \tag{2.6}$$

The off-shell field content of $\mathcal{N}=(1,0)$ SYM theory is revealed in the Wess-Zumino gauge:

$$V^{++} \;=\; \theta^{+a}\theta^{+b}A_{ab}+2(\theta^+)^3_a\lambda^{-a}-3(\theta^+)^4\mathscr{D}^{--}\,. \tag{2.7}$$

Here A_{ab} is the gauge field, $\lambda^{-a}=\lambda^{ai}u^-_i$ is the gaugino and $\mathscr{D}^{--}=\mathscr{D}^{ik}u^-_i u^-_k$, where $\mathscr{D}^{ik}=\mathscr{D}^{ki}$, are the auxiliary fields.

The $\mathcal{N}=(1,0)$ SYM covariant derivatives are given by the expressions

$$\nabla^-_a=[\nabla^{--},D^+_a]=D^-_a+\mathscr{A}^-_a\,,\qquad \nabla_{ab}=\frac{1}{2i}[D^+_a,\nabla^-_b]=\partial_{ab}+\mathscr{A}_{ab}\,,$$
$$\mathscr{A}^-_a(V)=-D^+_aV^{--}\,,\quad \mathscr{A}_{ab}(V)=\frac{i}{2}D^+_aD^+_bV^{--}\,,$$
$$[\nabla^{++},\nabla^-_a]=D^+_a\,,\quad [\nabla^{++},D^+_a]=[\nabla^{--},\nabla^-_a]=[\nabla^{\pm\pm},\nabla_{ab}]=0\,. \tag{2.8}$$

The covariant superfield strengths are defined as

$$[D^+_a,\nabla_{bc}]\;=\;\frac{i}{2}\varepsilon_{abcd}W^{+d}\,,\qquad [\nabla^-_a,\nabla_{bc}]\;=\;\frac{i}{2}\varepsilon_{abcd}W^{-d}\,,$$
$$W^{+a}=-\frac{1}{6}\varepsilon^{abcd}D^+_bD^+_cD^+_dV^{--}\,,\quad W^{-a}:=\nabla^{--}W^{+a}\,,$$
$$\nabla^{++}W^{+a}=\nabla^{--}W^{-a}\;=\;0\,,\quad \nabla^{++}W^{-a}=W^{+a}\,,$$
$$D^+_bW^{+a}=\delta^a_bF^{++}\,,\quad F^{++}=\frac{1}{4}D^+_aW^{+a}=(D^+)^4V^{--}\,,$$
$$\nabla^{++}F^{++}=0\,,\quad D^+_aF^{++}=0\,. \tag{2.9}$$

The hypermultiplet is accommodated by the analytic superfield $q^{+A}(\zeta)\,,(A=1,2)$, with the following component expansion:

$$q^{+A}(\zeta)=q^{iA}(x)u^+_i-\theta^{+a}\psi^A_a(x)+\text{ An infinite tail of auxiliary fields}\,. \tag{2.10}$$

2.3 $\mathcal{N}=(1,0)$ superfield actions

The $\mathcal{N}=(1,0)$ SYM action was constructed by Zupnik [15]:

$$S^{SYM}=\frac{1}{f^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{Tr}\int d^6x\,d^8\theta\,du_1\ldots du_n\frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u^+_1u^+_2)\ldots(u^+_nu^+_1)}\,,$$
$$\delta S^{SYM}=0\;\Rightarrow\;F^{++}=0\,. \tag{2.11}$$

Here, $(u^+_1u^+_2)^{-1},\ldots(u^+_nu^+_1)^{-1}$ are harmonic distributions [17].

The hypermultiplet action, with q^{+A} in adjoint of the gauge group, is written as

$$S^q=-\frac{1}{2f^2}\mathrm{Tr}\int d\zeta^{-4}q^{+A}\nabla^{++}q^+_A\,,\quad \nabla^{++}q^+_A=D^{++}q^+_A+[V^{++},q^+_A]\,,$$
$$\delta S^q=0\;\Rightarrow\;\nabla^{++}q^{+A}=0\,. \tag{2.12}$$

The $\mathcal{N}=(1,0)$ superfield form of the $\mathcal{N}=(1,1)$ SYM action ia a sum of the two superfield actions given above:

$$S^{(V+q)} = S^{SYM} + S^{q} = \frac{1}{f^2}\left(\int dZ\mathcal{L}^{\rm SYM} - \frac{1}{2}{\rm Tr}\int d\zeta^{-4} q^{+A}\nabla^{++}q^{+}_{A}\right),$$
$$\delta S^{(V+q)} = 0 \;\Rightarrow\; F^{++} + \frac{1}{2}[q^{+A}, q^{+}_{A}] = 0\,, \quad \nabla^{++}q^{+A} = 0\,. \tag{2.13}$$

It is invariant under the second hidden $\mathcal{N}=(0,1)$ supersymmetry:

$$\delta V^{++} = \varepsilon^{+A}q^{+}_{A}\,, \quad \delta q^{+A} = -(D^{+})^{4}(\varepsilon^{-}_{A}V^{--})\,, \quad \varepsilon^{\pm}_{A} = \varepsilon_{aA}\theta^{\pm a}\,. \tag{2.14}$$

These transformations have the correct closure with themselves and those of the manifest $\mathcal{N}=(1,0)$ supersymmetry only on shell.

3. Higher-dimensional $\mathcal{N}=(1,0)$ and $\mathcal{N}=(1,1)$ invariants

3.1 Dimension $\underline{d=6}$

In the pure $\mathcal{N}=(1,0)$ SYM the $\underline{d=6}$ invariant is defined uniquely [18]:

$$S^{(6)}_{SYM} = \frac{1}{2g^2}{\rm Tr}\int d\zeta^{-4}du\,\left(F^{++}\right)^{2} \;\sim\; {\rm Tr}\int d^{6}x[(\nabla^{M}F_{ML})^{2} + \ldots]. \tag{3.1}$$

It vanishes on shell, when $F^{++}=0$. Does its off-shell completion to an $\mathcal{N}=(1,1)$ invariant exist? The answer is negative, the only possibility one can achieve is an expression whose $\mathcal{N}=(0,1)$ variation vanishes *on-shell*. Using the results of [19], such an expression is defined uniquely, up to a real parameter

$$\mathcal{L}^{d=6} = \frac{1}{2g^2}{\rm Tr}\int du d\zeta^{-4}\left(F^{++} + \frac{1}{2}[q^{+A}, q^{+}_{A}]\right)\left(F^{++} + 2\beta[q^{+A}, q^{+}_{A}]\right). \tag{3.2}$$

But it vanishes on the full $\mathcal{N}=(1,1)$ SYM mass shell by itself! We have thus shown that the non-vanishing on-shell counterterms of the canonical dimension $\underline{d=6}$ are absent, and this proves the *one-loop finiteness* of $\mathcal{N}=(1,1)$ SYM theory.

3.2 Dimension $\underline{d=8}$

All $\mathcal{N}=(1,0)$ superfield terms of the canonical dimension $\underline{d=8}$ in the pure $\mathcal{N}=(1,0)$ SYM theory prove to vanish on the gauge fields mass shell, in accord with the old statement of ref. [24]. Can adding the hypermultiplet terms change this negative conclusion? Our analysis showed that there exist no $\mathcal{N}=(1,0)$ supersymmetric off-shell invariants of the dimension $\underline{d=8}$ which would respect the on-shell $\mathcal{N}=(1,1)$ invariance.

Surprisingly, the $\underline{d=8}$ superfield expression which is non-vanishing on shell and respects the on-shell $\mathcal{N}=(1,1)$ supersymmetry can be constructed by *giving up* the requirement of *off-shell* $\mathcal{N}=(1,0)$ supersymmetry.

An example of such an invariant in $\mathcal{N}=(1,0)$ SYM theory is very simple

$$\tilde{S}^{(8)}_{1} \sim {\rm Tr}\int d\zeta^{-4}\,\varepsilon_{abcd}W^{+a}W^{+b}W^{+c}W^{+d}\,. \tag{3.3}$$

Indeed, $D_a^+ W^{+b} = \delta_a^b F^{++}$, which vanishes on shell, where $F^{++}=0$. Thus, W^{+a} is an analytic superfield, when disregarding the terms proportional to the equations of motion, and the above action respects $\mathcal{N}=(1,0)$ supersymmetry on shell. Also, a double-trace on-shell invariant exists:

$$\tilde{S}_2^{(8)} \sim \int d\zeta^{-4}\, \varepsilon_{abcd} \mathrm{Tr}(W^{+a}W^{+b})\mathrm{Tr}(W^{+c}W^{+d})\,. \tag{3.4}$$

Do these invariants admit $\mathcal{N}=(1,1)$ completions? Yes, they do!

By varying the pure $\mathcal{N}=(1,0)$ SYM action (3.3) by the transformations of the second hidden $\mathcal{N}=(0,1)$ supersymmetry (2.14) and picking up the appropriate compensating hypermultiplet terms, after rather cumbersome computations we find

$$\begin{aligned}\mathcal{L}^{+4}_{(1,1)} = \mathrm{Tr}_{(S)}\Big\{&\frac{1}{4}\varepsilon_{abcd}W^{+a}W^{+b}W^{+c}W^{+d} + 3iq^{+A}\nabla_{ab}q^+_A W^{+a}W^{+b}\\ &- q^{+A}\nabla_{ab}q^+_A\, q^{+B}\nabla^{ab}q^+_B - W^{+a}[D^+_a q^-_A, q^+_B]q^{+A}q^{+B}\\ &- \frac{1}{2}[q^{+C}, q^+_C][q^-_A, q^+_B]q^{+A}q^{+B}\Big\}.\end{aligned} \tag{3.5}$$

Here, $\mathrm{Tr}_{(S)}$ stands for the *symmetrized* trace. This Lagrangian density is analytic, $D^+_a \mathcal{L}^{+4}_{(1,1)} = 0$, on the full shell $F^{++} + \frac{1}{2}[q^{+A}, q^+_A] = 0$, $\nabla^{++}q^{+A} = 0$, and is $\mathcal{N}=(1,1)$ supersymmetric on shell. Also, it is possible to extend the double-trace $\underline{d=8}$ invariant in a similar way.

Though the nontrivial on-shell $\underline{d=8}$ invariants exist, the perturbative expansion for the amplitudes in the $\mathcal{N}=(1,1)$ SYM theory does not involve divergences at the two-loop level. The matter is that these $\underline{d=8}$ invariants do *not* possess the full off-shell $\mathcal{N}=(1,0)$ supersymmetry which the physically relevant counterterms should obey. Indeed, we have at hand the harmonic off-shell $\mathcal{N}=(1,0)$ superfields. Given that, one can construct the $\mathcal{N}=(1,0)$ gauge-covariant supergraph technique, such that all the amplitudes and the counterterms would enjoy *off shell* $\mathcal{N}=(1,0)$ supersymmetry. Then, from the fact that such $\mathcal{N}=(1,0)$ off-shell $\underline{d=8}$ invariants cannot be constructed, it follows that $\mathcal{N}=(1,1)$ SYM theory is finite to two-loops.

4. $\mathcal{N}=(1,1)$ on-shell harmonic superspace

Despite the fact that the $\underline{d=8}$ terms mentioned above cannot appear as counterterms in $\mathcal{N}=(1,1)$ SYM theory, they can come out, e.g., as quantum corrections to the effective Wilsonian action. For the pure $\mathcal{N}=(1,0)$ SYM theory this was recently observed in [21]. It would be desirable to work out some simple and systematic way of constructing such higher-order on-shell $\mathcal{N}=(1,1)$ invariants. This becomes possible in the framework of the on-shell harmonic $\mathcal{N}=(1,1)$ superspace.

As the first step, extend the $\mathcal{N}=(1,0)$ superspace to the $\mathcal{N}=(1,1)$ one,

$$z = (x^{ab}, \theta^a_i) \;\Rightarrow\; \hat{z} = (x^{ab}, \theta^a_i, \hat{\theta}^A_a)\,. \tag{4.1}$$

Then we define the covariant spinor derivatives,

$$\nabla^i_a = \frac{\partial}{\partial\theta^a_i} - i\theta^{bi}\partial_{ab} + \mathcal{A}^i_a\,, \quad \hat{\nabla}^{aA} = \frac{\partial}{\partial\hat{\theta}_{Aa}} - i\hat{\theta}^A_b\partial^{ab} + \hat{\mathcal{A}}^{aA}\,. \tag{4.2}$$

The constraints defining the $\mathcal{N}=(1,1)$ SYM theory can now be written as follows [25], [24]:

$$\{\nabla^{(i}_{a},\nabla^{j)}_{b}\}=\{\hat{\nabla}^{a(A},\hat{\nabla}^{bB)}\}=0\,,\quad \{\nabla^{i}_{a},\hat{\nabla}^{bA}\}=\delta^{b}_{a}\phi^{iA}$$
$$\Rightarrow\quad \nabla^{(i}_{a}\phi^{j)A}=\hat{\nabla}^{a(A}\phi^{B)i}=0\quad(\text{By Bianchis})\,. \tag{4.3}$$

As the next step, we define the $\mathcal{N}=(1,1)$ harmonic superspace with the double set of harmonics [23]:

$$Z=(x^{ab},\theta^{a}_{i},u^{\pm}_{k})\ \Rightarrow\ \hat{Z}=(x^{ab},\theta^{a}_{i},\hat{\theta}^{A}_{b},u^{\pm}_{k},u^{\hat{\pm}}_{A})\,. \tag{4.4}$$

Then we pass to the analytic basis and choose the "hatted" spinor derivatives short, $\nabla^{\hat{+}a}=D^{\hat{+}a}=\frac{\partial}{\partial\theta^{\hat{-}}_{a}}$. The set of constraints in the ordinary $\mathcal{N}=(1,1)$ superspace amounts to the following set in the $\mathcal{N}=(1,1)$ harmonic one

$$\{\nabla^{+}_{a},\nabla^{+}_{b}\}=0\,,\quad \{D^{\hat{+}a},D^{\hat{+}b}\}=0\,,\quad \{\nabla^{+}_{a},D^{\hat{+}b}\}=\delta^{b}_{a}\phi^{+\hat{+}}\,,$$
$$[\nabla^{\hat{+}\hat{+}},\nabla^{+}_{a}]=0\,,\ [\tilde{\nabla}^{++},\nabla^{+}_{a}]=0\,,\ [\nabla^{\hat{+}\hat{+}},D^{a\hat{+}}]=0\,,\ [\tilde{\nabla}^{++},D^{a\hat{+}}]=0\,,$$
$$[\tilde{\nabla}^{++},\nabla^{\hat{+}\hat{+}}]=0\,. \tag{4.5}$$

Here

$$\nabla^{+}_{a}=D^{+}_{a}+\mathscr{A}^{+}_{a}(\hat{Z})\,,\ \tilde{\nabla}^{++}=D^{++}+\tilde{V}^{++}(\hat{\zeta})\,,\quad \nabla^{\hat{+}\hat{+}}=D^{\hat{+}\hat{+}}+V^{\hat{+}\hat{+}}(\hat{\zeta})\,,$$
$$\hat{\zeta}=(x^{ab}_{\text{an}},\theta^{\pm a},\theta^{\hat{+}}_{c},u^{\pm}_{i},u^{\hat{\pm}}_{A})\,. \tag{4.6}$$

5. Solving $\mathcal{N}=(1,1)$ SYM constraints through $\mathcal{N}=(1,0)$ superfields

The starting point of our analysis in [22] was to fix, using the $\Lambda(\hat{\zeta})$ gauge freedom, the WZ gauge for the second harmonic connection $V^{\hat{+}\hat{+}}(\hat{\zeta})$ as

$$V^{\hat{+}\hat{+}}=i\theta^{\hat{+}}_{a}\theta^{\hat{+}}_{b}\hat{\mathscr{A}}^{ab}+\varepsilon^{abcd}\theta^{\hat{+}}_{a}\theta^{\hat{+}}_{b}\theta^{\hat{+}}_{c}\varphi^{A}_{d}u^{\hat{-}}_{A}+\varepsilon^{abcd}\theta^{\hat{+}}_{a}\theta^{\hat{+}}_{b}\theta^{\hat{+}}_{c}\theta^{\hat{+}}_{d}\mathscr{D}^{AB}u^{\hat{-}}_{A}u^{\hat{-}}_{B}\,, \tag{5.1}$$

where $\hat{\mathscr{A}}^{ab},\varphi^{A}_{d}$ and $\mathscr{D}^{(AB)}$ are some $\mathcal{N}=(1,0)$ harmonic superfields, still arbitrary at this step.

Then the above constraints are reduced to some sets of harmonic equations which we have explicitly solved. The crucial point was the requirement that the vector $6D$ connections in the sectors of hatted and unhatted variables are identical to each other.

As the eventual result, we have obtained that the first harmonic connection V^{++} coincides precisely with the previous $\mathcal{N}=(1,0)$ one, $V^{++}=V^{++}(\zeta)$, while the dependence of all other geometric $\mathcal{N}=(1,1)$ objects on the "hatted" variables is strictly fixed

$$V^{\hat{+}\hat{+}}=i\theta^{\hat{+}}_{a}\theta^{\hat{+}}_{b}\mathscr{A}^{ab}-\frac{1}{3}\varepsilon^{abcd}\theta^{\hat{+}}_{a}\theta^{\hat{+}}_{b}\theta^{\hat{+}}_{c}D^{+}_{d}q^{-\hat{-}}+\frac{1}{8}\varepsilon^{abcd}\theta^{\hat{+}}_{a}\theta^{\hat{+}}_{b}\theta^{\hat{+}}_{c}\theta^{\hat{+}}_{d}[q^{+\hat{-}},q^{-\hat{-}}]$$
$$\phi^{+\hat{+}}=q^{+\hat{+}}-\theta^{\hat{+}}_{a}W^{+a}-i\theta^{\hat{+}}_{a}\theta^{\hat{+}}_{b}\nabla^{ab}q^{+\hat{-}}+\frac{1}{6}\varepsilon^{abcd}\theta^{\hat{+}}_{a}\theta^{\hat{+}}_{b}\theta^{\hat{+}}_{c}[D^{+}_{d}q^{-\hat{-}},q^{+\hat{-}}]$$
$$+\frac{1}{24}\varepsilon^{abcd}\theta^{\hat{+}}_{a}\theta^{\hat{+}}_{b}\theta^{\hat{+}}_{c}\theta^{\hat{+}}_{d}[q^{+\hat{-}},[q^{+\hat{-}},q^{-\hat{-}}]]\,. \tag{5.2}$$

PoS(CORFU2015)087

Here, $q^{+\hat{\pm}}=q^{+A}(\zeta)u^{\hat{\pm}}_A$, $q^{-\hat{\pm}}=q^{-A}(\zeta)u^{\hat{\pm}}_A$ and $W^{+a}, q^{\pm A}$ are just the $\mathcal{N}=(1,0)$ superfields explored previously. In the course of solving the constraints, there appeared the analyticity conditions for q^{+A}, as well as the full set of the superfield equations of motion

$$\nabla^{++}q^{+A}=0\,,\quad F^{++}=\frac{1}{4}D^+_aW^{+a}=-\frac{1}{2}[q^{+A},q^+_A]\,. \tag{5.3}$$

Also, the structure of the spinor covariant derivatives was fully fixed

$$\begin{aligned}\nabla^+_a&=D^+_a-\theta^{\hat{+}}_a q^{+\hat{-}}+\theta^{\hat{-}}_a\phi^{+\hat{+}}\,,\\ \nabla^-_a&=D^-_a-D^+_aV^{--}-\theta^{\hat{+}}_a q^{-\hat{-}}+\theta^{\hat{-}}_a\phi^{-\hat{+}}\,,\quad \phi^{-\hat{+}}=\nabla^{--}\phi^{+\hat{+}}\,.\end{aligned} \tag{5.4}$$

The basic advantage of using the constrained $\mathcal{N}=(1,1)$ strengths $\phi^{\pm\hat{+}}$ for constructing various invariants is their extremely simple transformation rules under the hidden $\mathcal{N}=(0,1)$ supersymmetry

$$\delta\phi^{\pm\hat{+}}=-\varepsilon^{\hat{+}}_a\frac{\partial}{\partial\theta^{\hat{+}}_a}\phi^{\pm\hat{+}}-2i\varepsilon^{\hat{-}}_a\theta^{\hat{+}}_b\partial^{ab}\phi^{\pm\hat{+}}-[\Lambda^{(comp)},\phi^{\pm\hat{+}}]\,, \tag{5.5}$$

where $\Lambda^{(comp)}$ is some common composite gauge parameter which does not contribute under the Tr symbol.

6. Invariants in $\mathcal{N}=(1,1)$ superspace

The previous single-trace $\underline{d=8}$ invariant Lagrangian (3.5) admits a simple rewriting in $\mathcal{N}=(1,1)$ superspace

$$S_{(1,1)}=\int d\zeta^{-4}\mathcal{L}^{+4}_{(1,1)}\,,\ \mathcal{L}^{+4}_{(1,1)}=-\mathrm{Tr}\frac{1}{4}\int d\hat{\zeta}^{-4}d\hat{u}\,(\phi^{+\hat{+}})^4,\quad d\hat{\zeta}^{-4}\sim(D^{\hat{-}})^4 \tag{6.1}$$

$$\delta\mathcal{L}^{+4}_{(1,1)}=-2i\partial^{ab}\mathrm{Tr}\int d\hat{\zeta}^{-4}d\hat{u}\left[\varepsilon^{\hat{-}}_a\theta^{\hat{+}}_b\frac{1}{4}(\phi^{+\hat{+}})^4\right].$$

The double-trace $\underline{d=8}$ invariant is given by

$$\hat{\mathcal{L}}^{+4}_{(1,1)}=-\frac{1}{4}\int d\hat{\zeta}^{-4}d\hat{u}\,\mathrm{Tr}\,(\phi^{+\hat{+}})^2\,\mathrm{Tr}\,(\phi^{+\hat{+}})^2. \tag{6.2}$$

It can also be easily rewritten in terms of $\mathcal{N}=(1,0)$ superfields.

Now it is easy to construct the single- and double-trace $\underline{d=10}$ invariants responsible for the candidate 3-loop counterterms

$$\begin{aligned}S^{(10)}_1&=\mathrm{Tr}\int dZd\hat{\zeta}^{-4}d\hat{u}\,(\phi^{+\hat{+}})^2(\phi^{-\hat{+}})^2,\quad \phi^{-\hat{+}}=\nabla^{--}\phi^{+\hat{+}}\,,\\ S^{(10)}_2&=-\int dZd\hat{\zeta}^{-4}d\hat{u}\,\mathrm{Tr}\Big(\phi^{+\hat{+}}\phi^{-\hat{+}}\Big)\,\mathrm{Tr}\Big(\phi^{+\hat{+}}\phi^{-\hat{+}}\Big)\,.\end{aligned} \tag{6.3}$$

These are $\mathcal{N}=(1,1)$ extensions of the $\mathcal{N}=(1,0)$ invariants $\sim\varepsilon_{abcd}\mathrm{Tr}\left(W^{+a}W^{-b}W^{+c}W^{-d}\right)$ and $\sim\varepsilon_{abcd}\,\mathrm{Tr}\,(W^{+a}W^{-b})\,\mathrm{Tr}\,(W^{+c}W^{-d})$.

It is notable that the single-trace $\underline{d = 10}$ invariant admits a representation as an integral over the *full* $\mathcal{N} = (1,1)$ superspace

$$S_1^{(10)} \sim \mathrm{Tr}\int dZ d\hat{Z} d\hat{u}\, \phi^{+\hat{+}}\phi^{-\hat{-}}, \quad \phi^{-\hat{-}} = \nabla^{\hat{-}\hat{-}}\phi^{-\hat{+}}, \tag{6.4}$$

with $d\hat{Z} \sim (D^{\hat{-}})^4 (D^{\hat{+}})^4$.

On the other hand, the double-trace $\underline{d = 10}$ invariant *cannot* be written as the full integral and so looks as being UV *protected*.

This could explain why in the perturbative calculations of the amplitudes in the $\mathcal{N} = (1,1)$ SYM single-trace 3-loop divergence is seen, while no double-trace structures at the same order were observed [7], [8], [9]. However, this does not seem to be like the standard non-renormalization theorems because the quantum calculation of $\mathcal{N} = (1,0)$ supergraphs should give some invariants in the off-shell $\mathcal{N} = (1,0)$ superspace, not in the on-shell $\mathcal{N} = (1,1)$ superspace. So the above property seems not enough to explain the absence of the double-trace divergences and some additional piece of reasoning is needed. One possibility is to generalize, to the $6D$ harmonic superspace approach, the so called algebraic renormalization method [26] the applications of which to SYM theories have been already initiated in [10].

Now there exist new methods in the $\mathcal{N} = (1,1), 6D$ SYM perturbative calculations based on the notion of the so called on-shell harmonic momentum superspace [27] (see also a recent work [28] and refs. therein). It also involves two sets of harmonic coordinates. Perhaps it is closely related to the x-space harmonic $\mathcal{N} = (1,1)$ superspace approach outlined above and would help to prove that all divergent quantum corrections to $\mathcal{N} = (1,1)$ SYM action arise just as integrals over the whole $\mathcal{N} = (1,1)$ harmonic superspace.

7. Summary and outlook

Basically following refs. [15], [18], [19] and a recent paper [22], the off-shell $\mathcal{N} = (1,0)$ and on-shell $\mathcal{N} = (1,1)$ harmonic superfield approaches were sketched and shown to provide the efficient tools of constructing higher-dimensional invariants in the $\mathcal{N} = (1,0)$ and $\mathcal{N} = (1,1)$ SYM theories. The $\mathcal{N} = (1,1)$ SYM constraints were solved in terms of harmonic $\mathcal{N} = (1,0)$ superfields. This allowed to explicitly construct the full set of the superfield dimensions $\underline{d = 8}$ and $\underline{d = 10}$ invariants possessing $\mathcal{N} = (1,1)$ on-shell supersymmetry.

All possible $\underline{d = 6}$ $\mathcal{N} = (1,1)$ invariants were demonstrated to be on-shell vanishing, thereby proving the UV finiteness of $\mathcal{N} = (1,1)$ SYM at one loop.

The off-shell $\underline{d = 8}$ invariants which would be non-vanishing on shell, are absent. The on-shell non-vanishing invariants (with both $\mathcal{N} = (1,0)$ and $\mathcal{N} = (0,1)$ supersymmetries being on-shell) are given by integrals over the analytic $\mathcal{N} = (1,0)$ subspace. Assuming that the $\mathcal{N} = (1,0)$ supergraphs yield integrals over the full $\mathcal{N} = (1,0)$ harmonic superspace, this means the absence of two-loop counterterms as well.

Two $\underline{d = 10}$ invariants were explicitly constructed as integrals over the whole $\mathcal{N} = (1,0)$ harmonic superspace. The single-trace invariant can be rewritten as an integral over the $\mathcal{N} = (1,1)$

superspace, while the double-trace one cannot. This property combined with an additional reasoning (e.g., based on the algebraic renormalization approach [26]) could explain why the double-trace invariant is UV protected.

Some further lines of development:

(a). To construct the next $d \geq 12$ invariants in the $\mathcal{N} = (1,1)$ SYM theory with the help of the on-shell $\mathcal{N} = (1,1)$ harmonic superspace techniques.

(b). To apply the same method for constructing the Born-Infeld action with the manifest off-shell $\mathcal{N} = (1,0)$ and hidden on-shell $\mathcal{N} = (0,1)$ supersymmetries. To check the hypothesis that such an action coincides with the full quantum effective action of the $\mathcal{N} = (1,1)$ SYM theory.

(c). To develop an analogous on-shell harmonic $\mathcal{N} = 4, 4D$ superspace approach to the $\mathcal{N} = 4, 4D$ SYM theory in the $\mathcal{N} = 2$ superfield formulation (by solving the $\mathcal{N} = 4$ SYM constraints in terms of $\mathcal{N} = 2$ superfields) and apply it to the problem of constructing the $\mathcal{N} = 4$ SYM effective action. The direct on-shell $\mathcal{N} = 2$ superfield approach was applied for this purpose in [29], [30], [31].

(d). Applications in supergravity? It is worth noting that the absence of the double-trace divergent structures in the 3-loop amplitude in $\mathcal{N} = (1,1)$ SYM theory is similar to the absence of analogous 3-loop and 4-loop divergences for the four-graviton amplitudes in $\mathcal{N} = 4, 4D$ and $\mathcal{N} = 5, 4D$ supergravities, respectively [32], [33], [34], [35]. So all these UV divergence cancelations could find a common explanation within the harmonic superspace approach supplemented with some extra algebraic arguments[1].

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References

[1] N. Seiberg, *Notes on theories with 16 supercharges*, Nucl. Phys. Proc. Suppl. **67** (1998) 158, hep-th/9705117.

[2] N. Beisert, *On Yangian Symmetry in Planar $\mathcal{N} = 4$ SYM*, In *Trieste 2010, Gribov-80 Memorial Volume*, pp. 413-438, arXiv:1004.5423 [hep-th].

[3] T. Dennen, Yu-tin Huang, *Dual Conformal Properties of Six-Dimensional Maximal Super Yang-Mills Amplitudes*, JHEP **1101** (2011) 140, arXiv:1010.5874 [hep-th].

[4] A.A. Tseytlin, *On non-abelian generalization of Born-Infeld action in string theory*, Nucl. Phys. B **501** (1997) 41-52, arXiv:hep-th/9701125.

[1] For a recent relevant discussion see [36].

[5] J.M. Drummond, P.J. Heslop, P.S. Howe, S.F. Kerstan, *Integral invariants in $\mathcal{N} = 4$ SYM and the effective action for coincident D-branes*, JHEP **0308** (2003) 016, `arXiv:hep-th/0305202`.

[6] P.H. Frampton, T.W. Kephart, *Explicit Evaluation of Anomalies in Higher Dimensions*, Phys. Rev. Lett. **50** (1983) 1343; Phys. Rev. Lett. **51** (1983) 232, Erratum;
B. Zumino, Y-S. Wu, A. Zee, *Chiral Anomalies, Higher Dimensions, and Differential Geometry*, Nucl. Phys. B **239** (1984) 477-507.

[7] Z. Bern, L.J. Dixon, V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, Phys. Rev. D **72** (2005) 085001, `arXiv:hep-th/0505205`.

[8] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson, R. Roiban, *The Complete Four-Loop Four-Point Amplitude in N=4 Super-Yang-Mills Theory*, Phys. Rev. D **82** (2010) 125040, `arXiv:1008.3327 [hep-th`.

[9] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson, R. Roiban, *Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes*, Phys. Rev. D **85** (2012) 105014, `arXiv:1201.5366 [hep-th]`.

[10] G. Bossard, P.S. Howe, U. Lindström, K.S. Stelle, L. Wulff, *Integral invariants in maximally supersymmetric Yang-Mills theories*, JHEP **1105** (2011) 021, `arXiv:1012.3142 [hep-th]`.

[11] N. Berkovits, M.B. Green, J.G. Russo, P. Vanhove, *Non-renormalization conditions for four-gluon scattering in supersymmetric string and field theory*, JHEP **0911** (2009) 063, `arXiv:0908.1923 [hep-th]`.

[12] J. Bjornsson, M.B. Green, *5 loops in 24/5 dimensions*, JHEP **1008** (2010) 132, `arXiv:1004.2692 [hep-th]`.

[13] J. Bjornsson, *Multi-loop amplitudes in maximally supersymmetric pure spinor field theory*, JHEP **1101** (2011) 002, `arXiv:1009.5906 [hep-th]`.

[14] P.S. Howe, K.S. Stelle, P.C. West, *$\mathcal{N} = 1, d = 6$ harmonic superspace*, Class. Quant. Grav. **2** (1985) 815.

[15] B.M. Zupnik, *Six-dimensional Supergauge Theories in the Harmonic Superspace*, Sov. J. Nucl. Phys. **44** (1986) 512 [Yad.Fiz. **44** (1986) 794-802];
B.M. Zupnik, *The Action of the Supersymmetric $\mathcal{N} = 2$ Gauge Theory in Harmonic Superspace*, Phys. Lett. B **183** (1987) 175-176.

[16] A.S. Galperin, E.A. Ivanov, S. Kalitzin, V.I. Ogievetsky, E.S. Sokatchev, *Unconstrained $\mathcal{N} = 2$ matter, Yang-Mills and supergravity theories in harmonic superspace*, Class. Quant. Grav. **1** (1984) 469-498.

[17] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky, E.S. Sokatchev, *Harmonic Superspace*, Cambridge University Press, 2001, 306 p.

[18] E.A. Ivanov, A.V. Smilga, B.M. Zupnik, *Renormalizable supersymmetric gauge theory in six dimensions*, Nucl. Phys. B **726** (2005) 131-148, `arXiv:hep-th/0505082`.

[19] E.A. Ivanov, A.V. Smilga, *Conformal properties of hypermultiplet actions in six dimensions*, Phys. Lett. B **637** (2006) 374-381, `arXiv:hep-th/0510273`.

[20] I.L. Buchbinder, N.G. Pletnev, *Construction of 6D supersymmetric field models in $\mathcal{N} = (1,0)$ harmonic superspace*, Nucl.Phys. B **892** (2015) 21-48, `arXiv:1411.1848 [hep-th]`.

[21] I.L. Buchbinder, N.G. Pletnev, *Leading low-energy effective action in the 6D hypermultiplet theory on a vector/tensor background*, Phys. Lett. B **744** (2015) 125-130, arXiv:1502.03257 [hep-th].

[22] G. Bossard, E. Ivanov, A. Smilga, *Ultraviolet behavior of 6D supersymmetric Yang-Mills theories and harmonic superspace*, JHEP **1512** (2015) 085, arXiv:1509.08027 [hep-th].

[23] G. Bossard, P.S. Howe, K.S. Stelle, *The Ultra-violet question in maximally supersymmetric field theories*, Gen. Rel. Grav. **41** (2009) 919, arXiv:0901.4661 [hep-th].

[24] P.S. Howe, K.S. Stelle, *Ultraviolet Divergences in Higher Dimensional Supersymmetric Yang-Mills Theories*, Phys. Lett. B **137** (1984) 175-180.

[25] P.S. Howe, G. Sierra, P.K. Townsend, *Supersymmetry in six dimensions*, Nucl. Phys. B **221** (1983) 331-348.

[26] O. Piguet, S.P. Sorella, *Algebraic renormalization: Perturbative renormalization, symmetries and anomalies*, Lect. Notes Phys. M **28** (1995) 1.

[27] T. Dennen, Yu-tin Huang, W. Siegel, *Supertwistor space for D6 maximal super Yang-Mills*, JHEP **1004** (2010) 127, arXiv:0910.2688 [hep-th].

[28] L.V. Bork, D.I. Kazakov, M.V. Kompaniets, D.M. Tokarev, V.E. Vlasenko, *Divergences in maximal supersymmetric Yang-Mills theories in diverse dimensions*, JHEP **1511** (2015) 059, arXiv:1508.05570 [hep-th].

[29] I.L. Buchbinder, A.Yu. Petrov, A.A. Tseytlin, *Two loop N=4 superYang-Mills effective action and interaction between D3-branes*, Nucl. Phys. B **621** (2002) 179-207, arXiv: hep-th/0110173.

[30] I.L. Buchbinder, E.A. Ivanov, *Complete N=4 structure of low-energy effective action in $\mathcal{N} = 4$ super Yang-Mills theories*, Phys. Lett. B **524** (2002) 208-216, arXiv:hep-th/0111062.

[31] I.L. Buchbinder, E.A. Ivanov, A.Yu. Petrov, *Complete low-energy effective action in $\mathcal{N} = 4$ SYM: A Direct $\mathcal{N} = 2$ supergraph calculation*, Nucl. Phys. B **653** (2003) 64-84, arXiv:hep-th/0210241.

[32] Z. Bern, S. Davies, T. Dennen, Yu-tin Huang, *Absence of three-loop four-point divergences in $\mathcal{N} = 4$ supergravity*, Phys. Rev. Lett. **108** (2012) 201301, arXiv:1202.3423 [hep-th].

[33] P. Tourkine, P. Vanhove, *An R^4 non-renormalization theorem in $\mathcal{N} = 4$ supergravity*, Class. Quant. Grav. **29** (2012) 115006, arXiv:1202.3692 [hep-th].

[34] Z. Bern, S. Davies, T. Dennen, Yu-tin Huang, *Ultraviolet cancellations in half-maximal supergravity as a consequence of the double-copy structure*, Phys. Rev. D **86** (2012) 105014, arXiv:1209.2472 [hep-th].

[35] Z. Bern, S. Davies, T. Dennen, *Enhanced ultraviolet cancellations in $\mathcal{N} = 5$ supergravity at four loops*, Phys. Rev. D **90** (2014)105011, arXiv:1409.3089 [hep-th].

[36] Andrei Smilga, *Ultraviolet divergences in non-renormalizale supersymmetric theories*, arXiv:1603.06811 [hep-th].