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Risk Measurement of Multivariate Credit Portfolio based on M-Copula Functions*

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In order to accurately connect the marginal distribution of portfolio credit risk, this paper constructs an M-Copula function by using the linear combination of Gumbel Copula and Clayton Copula. It employs GARCH (1, 1) model to fit the marginal distribution of the single asset logarithm yield sequence, uses the KMV model to calculate single asset default probability, then connects marginal default probability distribution of multiple credit portfolio risk by M-Copula functions and calculates the joint probability distribution and the corresponding value of default risk. Through the empirical study to the four healthy group companies and ST companies, it proves that the M-Copula functions can effectively fits the upper and lower tail correlation structures of credit risk marginal distributions, and that the model is able to accurately measure the credit risk for the two groups company's portfolio. The model provides an important reference for multiple credit portfolio risk measure.

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1. Introduction

Credit Risk, also called default risk, refers to that the counterparty cannot perform its obligations in accordance with the appointment which causes the risk of economic losses. Credit risk measurement has been a hot issue in the theory and practice research, which have developed a variety of risk measurement methods through the efforts of the scholars, such as Z-score method, the credit risk measurement model based on multivariate statistics, the credit risk measurement model based on the option theory model, the credit risk model based on artificial intelligence methods. These models provided important references for risk measurement, but these solutions were designed for single asset risk measurement. To the problem of diversified portfolio credit risk measurement, the related research is still very rare. And it's important to note, however, that when up to multiple assets portfolio credit risk measurement, the overall risk is not equal to the simple sum of the single risks because of the certain correlation among credit risks.

For the above conditions, early studies usually used the linear correlation coefficient to measure the correlation between assets, but the correlation in the financial markets usually had some characteristics, such as a nonlinear, asymmetry, thick tail distribution. Under this background, the Copula functions were introduced to related researches, and it could connect the marginal distribution of multiple variables to a joint distribution, and obtained default risk of the portfolio through calculating the default probability of joint distribution; what's more, Copula functions' asymmetric structure solved the problem of the return's thick tail on assets to a certain extent[1]. The foreign and domestic scholars studied portfolio risk measurement around Copula functions and achieved fruitful results [2-4]. Joshua and Dirk employ a generalization of the t-copula model to measure the risk of multivariate defaults with an asymmetric distribution, and show how the estimators proposed for the t-copula can be modified to estimate the portfolio risk under the skew t-copula model [2]. Choe and Jang construct a risk assessment model based on exchangeable Archimedean copulas and nested Gumbel copulas, and propose an appropriate density for importance sampling by analyzing multivariate Archimedean copulas[3]. Jonathan and Fernando use Copula theory to model the dependence across default rates in a credit card portfolio of a large UK bank, and prove that, when compared to traditional models, estimations based on asymmetric copulas usually yield results closer to the ratio of simultaneous extreme losses observed in the credit card portfolio[4]. These works extend the application of Copula theory in risk management area. However, all the risk measure models in the above studies use individual Copula function as the connection function, which is difficult to effectively connect the marginal distributions. Kole et al. show the importance of selecting an accurate copula for risk management[5] . In fact, there are many different kinds of Copula functions and categories can be divided into: ellipsoid Copula and Archimedes Copula. Among them, the commonly used ellipsoid Copula contains multivariate normal Copula and multivariate t-Copula. And the commonly used Archimedes Copula contains Gumbel Copula, Clayton Copula and Frank Copula. Ellipsoid Copula functions with elliptic contour line can construct different dependence degree's marginal distribution Copula functions.

But there is no closed form of expression for its distribution functions and its distribution functions is radial symmetry. Archimedean Copula functions are generated by a generating function, and it is convex, strictly decreasing continuous functions. Each Archimedean Copula functions have a unique generator. The form of single Copula functions is fixed and only suitable for fitting in the fixed tail distribution. And financial time series are changeful, a single Copula functions obviously is difficult to perfectly fitting its tail distribution. Recent studies show that M-Copula functions consisted of a linear combination of the multiple Copula functions which can depict the more flexible marginal distribution of financial time series, consequently improve portfolio risk measurement precision [6-11]. Inspired by this, This article will be the first to adopt Gumbel Copula and Clayton Copula, which can depict upper and lower tail correlation respectively, to build M-Copula, then use this functions to connect the portfolio's marginal distribution and measure credit risk combining with the classic KMV model, in order to provide meaningful reference for portfolio risk management.

2. Construction of the Risk Measurement Model

2.1 Construction of the M-Copula Function

The theory of Copula originated in 1959 when Sklar proposed the Sklar theorem in which the joint distribution and Copula function are combined, and it was noted that a joint distribution can be divided into a Copula function and n marginal distributions and the correlation of variables can be described by the Copula function. Therefore, the Copula function is essentially a function that connects a plurality of marginal distribution functions and their joint distribution function together. The N-dimension Copula function is considered to be a function $C(\bullet,\cdots,\bullet)$ having the following three properties:

- The domain of function $C(\bullet,\cdots,\bullet)$ is I^N ,that is $[0,1]^N$;
- The function $C(\bullet,\cdots,\bullet)$ has zero base and increases byN-dimension;
- The marginal distribution $C_n(\bullet)$ of function $C(\bullet,\cdots,\bullet)$, $n=1,2,\cdots,N$ meets $C_n(u_n)=C(1,\cdots,1,u_n,1,\cdots,1)=u_n$, where $u_n \in [0,1]$, $n=1,2,\cdots,N$.

In order to characterize the complex relationship in financial markets better, it can combine a variety of Copula functions to construct a more flexible mixed Copula: M-Copula function. I select a linear combination of Gumbel Copula and Clayton Copula to construct a N -dimension M-Copula function.

The formula of distribution function of Gumbel Copula is as follows:

$$C(u_1,u_2,\cdots,u_n)=\exp\{-[\sum_{i=1}^{n}(-\ln u_i)^{\alpha}]^{\frac{1}{\alpha}}\} \quad , \tag{2.1}$$

And Gumbel Copula processes the character that its upper tail is higher than other parts. The formula of distribution function of Clayton Copula is as follows:

$$C(u_1,u_2,\cdots,u_n)=[\sum_{i=1}^{n}u_i^{-\alpha}-n+1]^{-\frac{1}{\alpha}},\alpha>1, \tag{2.2}$$

Different from Gumbel Copula, Clayton Copula has the character that its lower tail is higher than other parts, which is shown in Fig. 2.

According to the formula of the above two Copula functions, it is easy to obtain the specific expression of M-Copula as follows:

$$C_M(u_1, u_2, \cdots, u_n; \theta) = \omega \exp\{-[\sum_{i=1}^{n}(-\ln u_i)^{\alpha}]^{\frac{1}{\alpha}}\} + (1-\omega)\exp\{-[\sum_{i=1}^{n}(-\ln u_i)^{\alpha}]^{\frac{1}{\alpha}}\}, \tag{2.3}$$

where $C_G(u_1, \cdots, u_N; a)$ and $C_C(u_1, \cdots, u_N; \theta)$ are N-dimension Gumbel Copula and Clayton Copula respectively; $\alpha \in (0,1)$, $\theta \in (0, \infty)$. M-Copula function has three parameters in which a and θ characterize the degree of correlation among variables; the weight parameters w and $1-w$ characterize the correlation form among variables and different combinations of weight parameters can characterize different correlation forms.

2.2 Fitting the Marginal Distribution

In the security market, return-loss distribution exist the severe phenomenon of excess kurtosis and heavy tail. Some models are created to fit finance time series, and a lot of empirical studies have shown GARCH family models can effectively describe the above behaviors of financial time series. So in this paper, I use GARCH (1, 1) model to fit the marginal distribution of financial time series. The GARCH (1, 1) proposed by Bollerslev [12] can be expressed as:

$$\begin{aligned} x_{i,t} &= \mu_i + \varepsilon_{i,t}, \\ \varepsilon_{i,t} &= \mathrm{e}_{i,t} h_{i,t}, \\ h_{i,t}^2 &= a_{i,0} + a_i \varepsilon^2{}_{i,t-1} + b_i h_{i,t-1}^2, \end{aligned} \tag{2.4}$$

where $x_{i,t}$ is the return series of financial asset i, $\mu_i = E(x_{i,t} \mid \Omega_{t-1})$, and Ω_{t-1} denotes the information set before $t-1$ moment. b_i is the coefficient of GARCH item and a_i is the coefficient of ARCH item, $\mathrm{e}_{i,t} \sim N(0, \sigma^2)$, $i = 1, 2, \cdots, n$.

2.3 Calculation of the Default Frequency of Credit Risk

This paper makes use of the KMV model to measure the default frequency of single asset's credit risk, and it can carry out the method following three steps: First, estimate the market value V and volatility σ_v; second, calculate the DD (Distance to Default); third, calculate the EDF (Expected Default Frequency).

In the KMV model, the volatility of market value of equity is calculated by using GARCH(1,1) model, and the risk-free interest rate r is seen as the one-year deposit interest rate announced by the central bank. If the risk-free interest rate has changes in the year, then the final risk-free interest rate is the weighted average of these rates. According to the existed research experience, I use the following formula to calculate the company's default point:

$$DP = LD + 0.75SD \tag{2.6}$$

where SD represents short-term debt and LD represents long-term debt.

Under the premise of having determined the default point, the distance to default can be given by the following equation:

$$DD = (V - DP) / V\sigma_v \tag{2.7}$$

Then, assuming the return on assets of the company obey normal distribution, it can calculate the expected default frequency for the company:

$$EDF = \Pr\left(E(\mathrm{V}) < DP\right) = N(DP - E(\mathrm{V})) \,/\, E(\mathrm{V})\sigma_v = N\left(-DD\right) \tag{2.8}$$

After calculating the default probability of a single asset, I adopt the M-Copula function to connect each marginal distribution of default probability, calculate the joint distribution of portfolio's default probability, and calculate the value at risk of combined credit risk in the final.

3. Empirical Analysis

3.1 Sample Selection and Statistical Description

I select 8 listed corporations as our study objects, among which four companies are in normal credit status: GNKJ, SHGF, HMQC, XALY, and others are under special treatment: ST-SD, ST-SW, ST-HH, ST-AG. Then I download stock closing prices of these public companies from the Resset Database (www.resset.cn) since January 4th 2011 to March 31st 2014， and obtain 614 valid samples. Then logarithmic treatment can be conducted with these stock yield sequences as follows:

$$r_t = \ln\left(p_{t+1}\right) - \ln\left(p_t\right) \ . \tag{3.1}$$

Then I obtain the statistical descriptions for these corporations' logarithmic yields as Table I shows. It's not hard to see from the table that GNKJ, SHGF, HMQC and ST-AG deviate to the right, and the others to the left. As we know, if a sample obeys normal distribution, then the sample kurtosis is supposed to be 3. However, I find it from the form that kurtosis coefficients of XALY, ST-SD and ST-AG are more than 3, especially ST-AG even reaches 10.55. Actually, further examinations for these statistics in table II and table III prove that they do not obey normal distribution but obey student-t distribution.

3.2 Estimation of M-Copula Function Parameters

Firstly, I perform Kendall rank test with portfolio samples and discover that all their correlation coefficients are not zero, which reveals their pertinence indeed. Secondly, I apply M-Copula function established in this paper to connect these companies' credit default distributions. As M-Copula function has parameters w , θ , α , it needs to use maximum likelihood method to estimate them. And default probabilities u_1, u_2, u_3, u_4 can be solved by KMV model at the same time. Then I divide 8 listed corporations into two groups as well-being listed companies and special-treated ones. While using maximum likelihood method to estimate parameters on the basis of those two groups' logarithmic yield time series, it needs to implement the following steps.

	GNKJ	SHGF	HMQC	XALY	ST-SD	ST-SW	ST-HH	ST-AG
mean	0.0001	0.0000	-0.0007	-0.0008	-0.0002	-0.0020	-0.0014	-0.0017
viation	0.0290	0.0350	0.0252	0.0276	0.0247	0.0294	0.0271	0.0179
Kurtosis	2.9653	2.7202	2.8467	3.2439	4.8666	2.2396	2.7238	10.5473
Skewness	0.0277	0.3274	0.0795	-0.5970	-0.8356	-0.1376	-0.2967	0.8223

Table 1: Statistical descriptions for 8 corporations' logarithmic yield

	Kolmogorov-Smirnov[a]			Shapiro-Wilk		
	Statistic	df	Sig.	Statistic	df	Sig.
GNKJ	.095	613	.000	.932	613	.000
SHGF	.069	613	.000	.959	613	.000
HMQC	.048	613	.002	.972	613	.000
XALY	.079	613	.000	.948	613	.000
ST-SD	.106	613	.000	.934	613	.000
ST-SW	.063	613	.000	.971	613	.000
ST-HH	.066	613	.000	.962	613	.000
ST-AG	.081	613	.000	.909	613	.000

Table 2: The test of Norm distribution

	Test Value = 0					
					95% Confidence Interval of the Difference	
	t	df	Sig. (2-tailed)	Mean Difference	Lower	Upper
ST-SW	-1.716	613	.087	-.002036095	-.00436661	.00029442
ST-SD	-.158	613	.874	-.00015608726	-.0020939363	.0017817617
ST-HH	-1.264	613	.207	-.001383074	-.00353119	.00076504
ST-AG	-2.262	613	.024	-.0016334922	-.003051556	-.000215428
GNKJ	.096	612	.923	.000113359	-.00219406	.00242078
SHGF	-.017	612	.986	-.000024245	-.00280284	.00275435
HMQC	-.616	613	.538	-.000627792	-.00262899	.00137340
XALY	-.750	613	.454	-.000835808	-.00302434	.00135273

Table 3: The Test of T-Student Distribution

Stock	GNKJ	SHGF	HMQC	XALY	ST-SD	ST-SW	ST-HH
The total value (Billion Yuan)	1.172	1.997	6.267	1.315	1.624	1,600	2.712
The volatility of total value	0.46	12.98	12.70	12.23	12.73	12.39	4.51
Default distance	1.9645	0.0351	0.0256	0.0731	0.0561	-0.1220	-0.0578
Default probability	0.0247	0.4859	0.4897	0.4708	0.4776	0.5485	0.5230

Table 4: Default Probability And Default Distance Of List Companies

Calculate the partial derivatives for u_1, u_2, u_3, u_4 in sequence, thus it can obtain the model's density function.

Invoke the historical default probabilities to acquire relevant maximum likelihood function:

$$L(\omega, \theta, \alpha) = \sum_{i=1}^{n} \ln c(u_{i1}, u_{i2}, u_{i3}, u_{i4}, \omega, \theta, \alpha), \tag{3.2}$$

The alphabet T means amounts of samples.

Solve parameters' values when the likelihood function reaches its maximum. And these values are their estimators.

Based on historical default probabilities of the healthy listed companies, I calculate the parameter values as follows: $w = 0.0001$, $\alpha = 2.4134$, $\theta = 0.5715$. As for special-treated group, I get different parameter values: $w = 0.0001$, $\alpha = 14.4411$, $\theta = 23.4666$. From the results of parameter estimation, it can conclude that the coefficient of Gumbel Copula function is nearly zero, representing that the upper tail correlation for these corporation default probabilities is weak.

3.3 Portfolio Default Probability Calculation

In this part, I first apply KMV model to calculate default probability for each company, and the results are shown in Table IV, and then measure the credit risk of the two groups' portfolio.

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At the beginning, the default probability values calculated by KMV model for healthy companies and special-treated group are as follows：

$$[u_1, u_2, u_3, u_4] = [0.0247, 0.4859, 0.4897, 0.4708] \quad (3.3)$$

$$[u_1, u_2, u_3, u_4]_{\mathrm{ST}} = [0.4776, 0.5485, 0.5230, 0.5459] \quad (3.4)$$

Then I substitute the two groups' default probability values and estimated parameter values into Formula (1), and I will acquire the M-Copula function for these two groups.

$$C(u_{1,}u_{2,}u_{3,}u_4)=0.0001\exp\{-[\sum_{i=1}^{4}(-\ln u_i)^{2.4134}]^{\frac{1}{2.4134}}\}+0.9999[\sum_{i=1}^{4}u_i^{-0.5715}-3]^{\frac{-1}{0.5715}} \quad (3.5)$$

$$C(u_{1,}u_{2,}u_{3,}u_4)=0.0001\exp\{-[\sum_{i=1}^{4}(-\ln u_i)^{14.4411}]^{\frac{1}{14.4411}}\}+0.9999[\sum_{i=1}^{4}u_i^{-23.4666}-3]^{\frac{-1}{23.4666}} . \quad (3.6)$$

From the above M-Copula functions, it is easy to find the weight of Clayton Copula is larger than that of Gumbel Copula. It implies that the four companies are more likely to crash together rather than boom together, because the shape of the cross-sectional plot of the Clayton Copula resembles the letter "L".

At last, through using Copula function expression above and combining the probability distribution of single asset default for each portfolio, I solve VaR value of the portfolio credit risk for well-being and special-treated group. And the value is 0.01148 and 0.4738, respectively.

The results disclose that value of default probability for well-being group is far less than special-treated ones, indicating that credit risk for well-being group is less than ST group's. In addition, comparing credit default probability of portfolios with single company, it is easy to find that value of the former is less than that of the latter, which also indicates portfolios' credit risk can be dispersed. Generally summarized, M-Copula function can be used to connect each default probability distribution of portfolio risk effectively, and fatherly lays a solid foundation of portfolio credit risk measurement.

4. Conclusion

In practice, Gumbel Copula and Clayton Copula can respectively connect the upper tail correlation structure and the lower correlation structure. To get more precise connection effect, this paper which aims at studying M-Copula function's feasibility applied to measure portfolio credit risk combines two types of Copula into M-Copula function linearly. By dividing object corporations into well-being and special-treated group, utilizing GARCH(1,1)-t model to fit yield sequence for each asset, applying KMV model to calculate default probability density of each company, and using M-Copula function to connect credit portfolios' marginal distribution, I work out the joint default probability density and relative VaR..

According to the study above, it can draw some conclusions from the empirical results. Firstly, portfolio credit risk's upper and lower tail correlation structure can be connected by using M-Copula. What's more, for each portfolio, single asset credit risk obviously exceeds portfolios', which reveals that portfolios can play a part of dispersing risk. Besides, portfolio credit risk values of well-being public companies are a great deal less than those of ST companies. So in general, the model proposed in this paper can measure VaR of multiple portfolio credit risk accurately and offer valuable reference for credit risk measurement in this area.

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