Stability structure of the GLR equation

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We show that the GLR equation develops unstable modes when the parton distribution function is generalized to depend on azimuthal angle. This generally means the saturation scale Q_s acquires an azimuthal dependence. We argue that this process is a plausible alternative explanation for the origin of elliptic flow, one that naturally respects the scaling experimentally observed.

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Hydrodynamics is the most favored explanation for the anisotropic flow observed in nucleus-nucleus, and now proton-nucleus and even proton-proton collisions (for a review see [1]). However, several phenomenological puzzles (see [2] and references therein for a review) have accumulated challenging hydrodynamics as the origin of v_n. For instance, the near independence of v_2, and particularly, v_3, on system size, i.e. pA, dA, and even pp collisions give you, scaled for geometry, the same azimuthal coefficients as AA collisions [3, 4]. As a related item, the near independence of $v_2(p_T)$ on energy [6, 8] and rapidity [7, 9]. Also, the photon and dielectron v_2 is comparable to the hadron v_2. Usual explanations aim to provide an *enhancement* of this v_n but never to explain the *equality* [10].

All of these, summarized in [2], point to an origin of v_2 which is not hydrodynamic but rather based on initial conditions, since it resembles the scaling of structure functions $f(x, Q^2)$ and fragmentation functions $D_{q \rightarrow i}(z, Q^2)$ [11]. Hence, if we could just assume that either the structure function acquired an azimuthal dependence having the usual Bjorken scaling (strong and fundamentally non-perturbative dependence on x), logarithmically suppressed dependence on Q^2, and this azimuthal dependence generated most of v_2, all issues described in this introduction would be naturally resolved.

Such a "simple" suggestion of course is superficially extremely implausible: QCD has azimuthal symmetry, and the parton structure and fragmentation functions are based on factorization at high energy scales. Thus, they are "universal" and should not depend on relative angles, even if the target and/or projectile are spatially extended. Yet, ways compatible with *QCD* to incorporate azimuthal dependence of structure functions do exist. In fact, since the distributions functions depend on the probe used to measure them beyond tree level ("run"), a dipole colliding off-center with an extended object will measure azimuthally asymmetric parton distribution functions to precision to one-loop. Since the impact parameter $b \sim \mathcal{O}(1 - 10)\,fm$, these corrections are expected to be tiny$\sim \alpha_s^2/b$, hence, provided evolution is linear, their contribution to the observed v_n is undetectable.

In this work, however, we investigate whether azimuthal symmetry is *spontaneously broken* by the same processes that are thought to lie at the core of saturation physics. If it is, the above anisotropies are indeed tiny at high rapidities but could be amplified at mid-rapidity, in the same way as cylindrical boundary conditions in a flow of water from a faucet lead to an azimuthally asymmetric turbulent flow when density and flow velocity are high enough.

1. Stability of the GLR equation

The GLR equation is the leading-order non-linear correction to the BFKL equation, which is thought to be valid in the high density limit, when the transverse wavefunctions of the partons overlap.

We shall consider the GLR equation in its purely differential (Mueller-Qiu) limit. In terms of the structure function G, Bjorken x, strong coupling constant α_s, and transverse diameter of the nucleus $S_\perp$, it is

$$\frac{\partial^2 xG(x, Q^2)}{\partial \ln(1/x)\, \partial \ln(Q^2/\Lambda^2)} = \frac{\alpha_s N_c}{\pi} xG(x, Q^2) - \frac{\alpha_s^2 \pi}{S_\perp} \frac{1}{Q^2} [xG(x, Q^2)]^2. \tag{1.1}$$

Assuming azimuthal symmetry, everything depends on Q rather than Q_x and Q_y, or equivalently in Eq. 1.1 $G(x,k,\theta) = G(x,k,\theta+\phi)$ and the angle can be integrated out. Neglecting angular effects (which are assumed to be suppressed by factors of Q), the most trivial modification one can make is to relax this approximation. It is easy to see that this modifies Eq. 1.1 by a relatively simple substitution

$$\frac{1}{\Lambda^2}\left(\frac{\partial}{\partial Q} + \frac{1}{Q}\frac{\partial}{\partial \theta}\right)\frac{\partial xQ^2 G(x,Q^2)}{\partial \ln(1/x)} = \frac{\alpha_s N_c}{\pi}xG(x,Q^2) - \frac{\alpha_s^2 \pi}{S_\perp}\frac{1}{Q^2}[xG(x,Q^2)]^2, \tag{1.2}$$

We intend to perturb the solutions of Eq.1.1, i.e. $G_0(x,Q^2)$, the following way

$$G(x,Q^2,\theta) = G_0(x,Q^2)\left(1 + \sum_n u_n(x,Q^2)cos(n\theta+\beta_n)\right), \tag{1.3}$$

where the background $G_0(x,Q^2)$ is modeled as the (azimuthally symmetric) saturation scenario [12, 13], the solution of the azimuthally symmetric non-linear equation, which is a transcendental function approximately equal to

$$G_0(x,Q^2) = \frac{x^{2\lambda}}{2\alpha_s^4}\left[(1-tanh(\xi)) + \frac{Q_s^2(x)}{Q^2}(1+tanh(\xi))\right], \tag{1.4}$$

with $Q_s(x) = \alpha_s^2 \Lambda_{QCD}x^{-\lambda}$, $\xi = \frac{Q-Q_s(x)}{\zeta}$ and $\zeta \sim \frac{\alpha_s \Lambda_{QCD}}{N_c}$, so that $G_0(x,Q)$ has the approximate step function form determined by requiring the two sides of Eq. 1.1 to balance

$$Q_s^2(x) = \frac{\alpha_s \pi^2}{S_\perp N_c}xG_s(x,Q_s(x)) \rightarrow Q_s(x) \sim x^{2\lambda+1} \tag{1.5}$$

It is of special interest for us to study parton distribution functions when the probe energy is small compared to the saturation scale; i.e. $Q/Q_s(x) \ll 1$. In this limit we obtain

$$(2\lambda+1)\frac{Q}{2}\frac{\partial u_n(x,Q^2)}{\partial Q} + \frac{Q}{2}x\frac{\partial^2 u_n(x,Q^2)}{\partial Q\partial x} = \left[\frac{\alpha_s N_c}{\pi} + \frac{N_c\pi}{C_F S_\perp \alpha_s^2}\frac{x^{2\lambda+1}}{Q^2}\right]u_n(x,Q^2) + \delta u_n(x,Q) \tag{1.6}$$

$$\delta u_n(x,Q) = \frac{N_c\pi}{2C_F S_\perp \alpha_s^2}\frac{x^{2\lambda+1}}{Q^2} - \left[\frac{1}{2}\sum_k^{n-1}u_k(x,Q^2)u_{n-k}(x,Q^2)cos(\beta_n-\beta_k-\beta_{n-k})\right.$$

$$\left. + \sum_k u_k(x,Q^2)u_{n+k}(x,Q^2)cos(\beta_n+\beta_k-\beta_{n+k})\right]. \tag{1.7}$$

Assuming δu_n is small (a Taylor expansion can go beyond this approximation), one can study the behavior of linearized instabilities (where the instability interacts with the azimuthally symmetric "saturating" component) relatively easily.

A thought needs to go into boundary conditions. The running scenario we described earlier would assume that the perturbation starts at a non-zero x and grows with $x \rightarrow 0$. An ansatz appropriate for these boundary conditions is $u(x,Q^2) = x^p f(t)$ with $t = Bx^C/Q^2 = z^2/2D$, $B = \pi^2/C_F S_\perp \alpha_s^3$,

$C = 2\lambda + 1$, and $D = 2\alpha_s N_c/(2\lambda + 1)\pi$. So, we get a Bessel's equation with pure imaginary index (order) $i\nu$ for f

$$f'' + \frac{1}{z}f' + (1 + \frac{\nu^2}{z^2})f = 0, \tag{1.8}$$

when $p = -C$ and $\nu = \sqrt{2D}$. The solution of this equation is inspired by Bessel functions of the first kind [14]

$$f_\nu(z) = A(z)cos(\nu lnz) + B(z)sin(\nu lnz) \quad , \quad \begin{matrix} A(z) \\ B(z) \end{matrix} = \sum_{n=0}^{\infty} \begin{matrix} A_{2n} \\ B_{2n} \end{matrix} \left(\frac{z}{2}\right)^{2n} \tag{1.9}$$

with a recursive definition

$$A_{2n} = -\frac{nA_{2n-2} - \nu B_{2n-2}}{n(n^2 + \nu^2)} \quad , \quad B_{2n} = -\frac{\nu A_{2n-2} + nB_{2n-2}}{n(n^2 + \nu^2)}$$

for $n \geq 1$ and A_0, B_0 known. Finally, our solution reads

$$u(x, Q^2) = x^{-(2\lambda+1)}f_\nu(z) \quad , \quad z = \sqrt{\left(\frac{4\pi N_c}{(2\lambda+1)C_F\alpha_s^2}\right)\frac{1}{S_\perp}\frac{x^{2\lambda+1}}{Q^2}} \quad , \quad \nu = \left(\frac{4\alpha_s N_c}{\pi(2\lambda+1)}\right)^{1/2}. \tag{1.10}$$

We start evolution at a critical moderately high x to fix the parameters characterizing the theory (A_0, B_0 for the Bessel function) to reproduce $u_2/\varepsilon \sim 10^{-2}$, in line with expectation from semi-perturbative higher twist processes. We then go lower in x to analyze the behavior of u_2, v_2 as one gets closer to mid-rapidity. As can be seen in Fig. 1 (left panel) the instability indeed grows extremely rapidly from a broad value of Bjorken x. In fact, very quickly it becomes so large that the linearization ansatz we used becomes inapplicable.

As a phenomenological fix to deal with this issue, we shall assume corrections beyond the leading order "saturate" u_2. An ad-hoc but consistent way to do this is replacing $u_2(x, Q) \to u_2^{max} \tanh\left(\frac{u_2(x,Q)}{u_2^{max}}\right)$, so that at low u_2 this recovers the linearized equations but never goes above the parameter u_2^{max}. Physically, this will be the scale where higher order corrections, assumed here to essentially cutoff u_2 growth, take over. The price for this adjustment is that we lose predictivity of u_2 at a given Q, x.
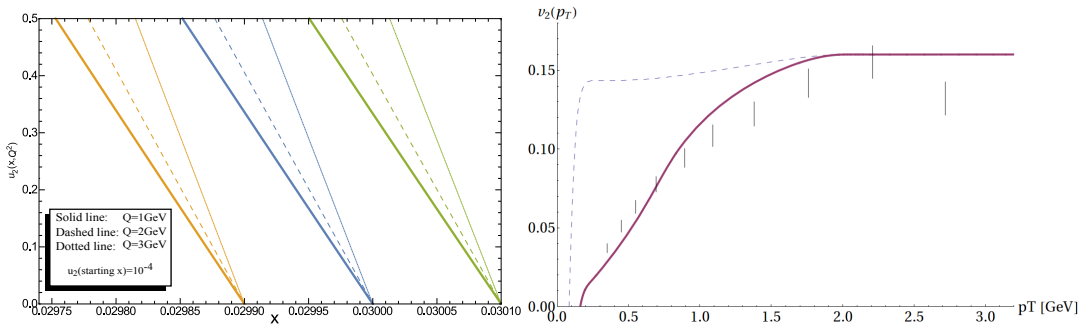

Figure 1: Left panel: u_2 as a function of x, Q when started as a small value at a "moderate" x (where processes such as higher twist can occur) and evolved to $x \to 0$. Right panel: a comparison of the data with a stabilized perturbation, converted into hadrons via k_T factorization and limiting fragmentation

To convert these asymmetric distribution functions to something that can be related to experimental measurement, we would need to perform microscopic quark-and gluon scattering as well as use fragmentation functions. For a rough estimate we use the k_T factorization ansatz [11, 13]

$$\frac{dN}{dp_T dy} \sim \frac{1}{p_T^2} \int \delta\left(x_A + x_B - e^{-y}\right) k\, dk\, d\theta\, f_A(x_A, k, \theta) f_B(x_B, p_T - k, \pi + \phi - \theta)\Big|_{x_{A,B} = \frac{p_T e^{\pm y}}{\sqrt{s}}}. \quad (1.11)$$

Motivated by Eq. 1.11 we set u_2^{max} at a value of 0.5, where its contribution to hadron production becomes comparable to the unprerturbed, saturation value. We note that the experimental v_2 is curiously quantitatively similar to this limit.

Together with quark-hadron duality, k_T factorization can be used to provide a calculation for v_2. The shifting of momentum due to fragmentation from parton p_T to hadron p_{Th}, necessary to provide the right limit for dN/dp_T, can be accomplished by updating Eq. 1.11 using

$$\frac{dN_h}{dp_{Th} dy_h} = \int_0^1 dz \int_{\Lambda_{QCD}}^{Q_s} dp_T\, \delta\left(z - \frac{p_{Th}}{p_T}\right) z D(z, pT) \frac{dN}{dp_T dy}\Big|_{p_T} = \int_{p_{Th}/Q_s}^1 dz\, z \int_{\Lambda_{QCD}/z}^{Q_s} D(z, pT) \frac{dN}{dp_T dy}\Big|_{p_T}. \quad (1.12)$$

We do this with a Schwinger function-inspired Gaussian fragmentation function, $D(z, pT) \sim e^{-z^2}$. Note that the integration limits are for the *parton* to have momentum larger than Λ_{QCD} and smaller than Q_s. The results, to be interpreted as a rough estimate, are shown in the right panel of Fig. 1.

In conclusion, we have discussed the phenomenological scaling of azimuthal anisotropy coefficients in hadronic collisions, arguing that this suggests an initial state origin compatible with "Bjorken" type scaling phenomenology. We have also noted that saturation dynamics corresponds also to a regime where azimuthal instabilities, in the full (2+1) evolution equations, could acquire growing modes. We have used the azimuthally asymmetric GLR equation as a laboratory to test these growing modes, and found intriguing hints that they are indeed possible and have some qualitative features required for modeling v_n. This suggests that this model should be developed further in order to connect it with data at a quantitative level, including the full integro-differential equation, realistic particle production and fragmentation, etc. This is something that we plan to do in a subsequent work.

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