Non-singlet coefficient functions for charged-current deep-inelastic scattering to the third order in QCD

J. Davies
Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, UK
E-mail: Joshua.Davies@liv.ac.uk

S. Moch
II. Institute for Theoretical Physics, University of Hamburg, D-22761 Hamburg, Germany
E-mail: Sven-Olaf.Moch@desy.de

J.A.M. Vermaseren
Nikhef Theory Group, Science Park 105, 1098 XG Amsterdam, The Netherlands
E-mail: t68@nikhef.nl

A. Vogt
Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, UK
E-mail: Andreas.Vogt@liv.ac.uk

We have calculated the coefficient functions for the structure functions $F_2$, $F_L$ and $F_3$ in $\nu - \bar{\nu}$ charged-current deep-inelastic scattering (DIS) at the third order in the strong coupling $\alpha_s$, thus completing the description of unpolarized inclusive $W^\pm$-exchange DIS to this order of massless perturbative QCD. In this brief note, our new results are presented in terms of compact approximate expressions that are sufficiently accurate for phenomenological analyses. For the benefit of such analyses we also collect, in a unified notation, the corresponding lower-order contributions and the flavour non-singlet coefficient functions for $\nu + \bar{\nu}$ charged-current DIS. The behaviour of all six third-order coefficient functions at small Bjorken-$x$ is briefly discussed.

XXIV International Workshop on Deep-Inelastic Scattering and Related Subjects
11-15 April 2016, DESY Hamburg, Germany

*Speaker.
1. Introduction

With six structure functions, $F_2$, $F_L$, and $F_3$ for $W^+$ and $W^-$ exchange [1, 2], inclusive charged-current DIS is an important source of information on the parton structure of nucleons and nuclei and on Standard Model parameters such as the strong coupling $\alpha_s$ and the weak mixing $\sin^2 \theta_W$ [3]. Future facilities, e.g., the LHeC [4], will be required to fully realize its phenomenological potential. Charged-current structure functions are, at very small values of the Bjorken variable $x$, also of interest for the scattering of high-energy cosmic neutrinos, see Ref. [5] and references therein.

Here we address the corresponding coefficient functions (mass-factorized partonic cross sections) in massless perturbative QCD. These functions are relevant also beyond DIS, e.g., for Higgs production via vector boson fusion [6, 7]. For recent progress on heavy-quark effects see Ref. [8].

2. First-order and $\nu + \bar{\nu}$ non-singlet coefficient functions

The first-order coefficient functions for unpolarized inclusive DIS were derived in the early days of QCD, see Refs. [1, 9]. The second- and third-order contributions for the $\nu + \bar{\nu}$ charged-current case have been calculated in Refs. [10–17]. Here we collect the corresponding flavour non-singlet results in the $\overline{\text{MS}}$ scheme for the standard choice $\mu_F = \mu_R = \mathcal{Q}$ of the renormalization and factorization scales. The additional contributions at other values of $\mu_F$ and/or $\mu_R$ are determined by these results and the corresponding splitting functions [18, 19]; see, e.g., Eqs. (2.16) – (2.18) in Ref. [20].

We denote the non-singlet quark coefficient functions for the charged-current structure functions $F_{2,3,L}^{\nu\bar{\nu} FF}(x, Q^2)$ in neutrino-proton DIS by $C_{a,\pm}$, and write their perturbative expansion as

$$C_{a,\pm}(x, Q^2) = \sum_{n=0} a_n x^{n} c_{a,\pm}^{(n)}(x) \quad \text{with} \quad a_n \equiv \frac{\alpha_s(Q^2)}{(4\pi)}.$$  

(2.1)

In this notation the zeroth- and first-order coefficient functions are given by

$$c_{2,\pm}^{(0)}(x) = x^{2H} \delta(x), \quad c_{3,\pm}^{(0)}(x) = 0, \quad c_{L,\pm}^{(0)}(x) = 4C_F x,$$

(2.2)

$$c_{2,\pm}^{(1)}(x) = C_F \{ 4 \mathcal{D}_1 - 3 \mathcal{D}_0 - (9 + 4 \xi_2) \delta(x_1) - 2(1 + x)(L_1 - L_0) - 4x_1^{-1}L_0 + 6 + 4x \},$$

(2.3)

$$c_{3,\pm}^{(1)}(x) = c_{2,\pm}^{(1)}(x) - 2C_F (1 + x),$$

(2.4)

with $C_F = \frac{3}{2} - 1)/(2n_c) = 4/3$ in QCD. Here and below we use the abbreviations

$$x_1 = 1 - x, \quad L_0 = \ln x, \quad L_1 = \ln x_1, \quad \mathcal{D}_k = \lfloor x_1^{-1}L_0^k \rfloor_+,$$

(2.5)

where $[a(x)]_+$ denotes $+-$distributions defined via $\int_0^1 dx [a(x)]_+ f(x) \equiv \int_0^1 dx a(x) \{ f(x) - f(1) \}$.

The coefficient functions $c_{a,\pm}^{(n)}$ and $c_{a,\pm}^{(n)}$ differ at $n > 1$. The 2nd- and 3rd-order contributions to the former quantities read, in an approximate but sufficiently accurate form given in Refs. [15–17],

$$c_{2,\pm}^{(2)}(x) \cong \frac{128}{9} \mathcal{D}_3 - \frac{184}{3} \mathcal{D}_2 - 31.1052 \mathcal{D}_1 + 188.641 \mathcal{D}_0 - 338.513 \delta(x_1) - 17.74L_1^3 + 72.24L_1^3 - 628.8L_1 - 181.0 - 806.7x + L_0L_1(37.75L_0 - 147.1L_1) + 0.719xL_0^3 - 28.384L_0 - 20.70L_0^3 - 80/27L_0^3 + n_f \left\{ 16/9 \mathcal{D}_2 - 232/27 \mathcal{D}_1 + 6.34888 \mathcal{D}_0 + 46.8531 \delta(x_1) - 1.500L_1^2 + 24.87L_1 - 7.8109 - 17.82x - 12.97x^2 + 8.113L_0L_1 - 0.185xL_0^3 + 16/3L_0 + 20/9L_0^3 \right\},$$

(2.6)

$$c_{L,\pm}^{(2)}(x) \cong - 37.338 + 89.53x + 33.82x^2 + 128/9xL_0^3 - 46.50L_0^3 + L_0(32.90 + 18.41L_0) - 84.094L_0L_1 - 128/9L_0 - 0.012 \delta(x_1) + 16/27n_f \left\{ 6L_1 - 12xL_0 - 25x + 6 \right\},$$

(2.7)
Non-singlet coefficient functions for charged-current DIS

\[ c_{3,3}^{(2)}(x) \cong 128/9 \delta_3 - 184/3 \delta_2 - 31.1052 \delta_1 + 188.641 \delta_0 - 338.572 \delta(x_1) - 16.40 L_1^3 \\
+ 78.46 L_1^3 - 470.6L_1 - 149.75 - 693.2x + 0.218xL_0^4 + L_0L_1(33.62L_0 - 117.8L_1) \\
- 49.30L_0 - 94/3L_0^2 - 104/27L_0^3 \]  

(2.8)

and

\[ c_{2,3}^{(3)}(x) \cong 512/27 \delta_5 - 5440/27 \delta_4 + 501.099 \delta_3 + 1171.54 \delta_2 - 7328.45 \delta_1 + 4442.76 \delta_0 \\
- 9170.38 \delta(x_1) - 512/27 L_1^3 + 704/3 L_1^3 - 3368 L_1^3 - 2978 L_1^3 + 18832 L_1 - 4926 \\
+ 7725 x + 57256x^2 + 12898 x^3 - 56000 x_1 L_1^3 - L_0 L_1 (6158 + 1836 L_0) + 4.719 xL_0^3 \\
- 775.8L_0 - 899.6 L_0^2 - 309.1 L_0^3 - 2932/81 L_0^4 - 32/27L_0^5 \\
+ n_f \left\{ 640/81 \delta_5 - 6592/81 \delta_4 + 220.573 \delta_3 + 294.906 \delta_2 - 729.359 \delta_1 \\
+ 2574.687 \delta(x_1) - 640/81 L_1^4 + 153.5 L_1^3 - 828.7 L_1^2 - 501.1 L_1 + 831.6 - 6752 x \\
- 2778 x^2 + 171.0 x L_1^4 + L_0 L_1 (4365 + 716.2 L_0 - 5983 L_1) + 4.102 xL_0^4 + 275.6 L_0 \\
+ 187.3 L_0^2 + 12224/243 L_0^3 + 728/243 L_0^4 \right\} \]  

(2.9)

\[ c_{L_3}^{(3)}(x) \cong 512/27 L_1^3 - 177.40 L_1^3 + 650.6 L_1^3 - 2729 L_1 - 2220.5 - 7884 x + 4168 x^2 \\
- (844.7 L_0 + 517.3 L_1) L_0 L_1 + (195.6 L_1 - 125.3) x_1 L_1^3 + 208.3 xL_0^3 - 1355.7 L_0 \\
- 7456/27 L_0^2 - 1280/81 L_0^3 + 0.113 \delta(x_1) \\
+ n_f \left\{ 1024/81 L_1^4 - 112.35 L_1^2 + 344.1 L_1 + 408.4 - 9.345 x - 919.3 x^2 \\
+ (239.7 + 20.63 L_1) x_1 L_1^2 + (887.3 + 294.5 L_0 - 59.14 L_1) L_0 L_1 - 1792/81 xL_0^3 \\
+ 200.73 L_0 + 64/3 L_0^2 + 0.006 \delta(x_1) \right\} \right\} \]  

(2.10)

\[ c_{3,3}^{(3)}(x) \cong 512/27 \delta_5 - 5440/27 \delta_4 + 501.099 \delta_3 + 1171.54 \delta_2 - 7328.45 \delta_1 + 4442.76 \delta_0 \\
- 9172.68 \delta(x_1) - 512/27 L_1^3 + 8896/27 L_1^3 - 1396 L_1^3 + 3990 L_1^3 + 14363 L_1 \\
- 1853 - 5709 x + x_1 x_1 (5600 - 1432 x) - L_0 L_1 (4007 + 1312 L_0) - 0.463 xL_0^3 \\
- 293.3 L_0 - 1488 L_0^2 - 496.95 L_0^3 - 4036/81 L_0^4 + 536/405 L_0^5 \\
+ n_f \left\{ 640/81 \delta_5 - 6592/81 \delta_4 + 220.573 \delta_3 + 294.906 \delta_2 - 729.359 \delta_1 \\
+ 2575.46 \delta(x_1) - 640/81 L_1^4 + 32576/243 L_1^3 - 660.7 L_1^2 - 959.1 L_1 + 31.95 x L_1^4 \\
+ 516.1 - 465.2 x + x_1 x_1 (635.3 + 310.4 x) - L_0 L_1 (1496 + 270.1 L_0 - 191 L_1) \\
- 1.200 xL_0^4 + 366.9 L_0 + 305.32 L_0^2 + 48512/729 L_0^3 + 304/81 L_0^4 \right\} \]  

(2.11)

\[ + n_f^2 \left\{ 64/81 \delta_5 - 64/81 \delta_4 + 7.67505 \delta_3 + 1.00830 \delta_2 - 103.2602 \delta(x_1) - 64/81 L_1^3 \\
+ 992/81 L_1^3 - 49.65 L_1 + 11.32 - x_1 x_1 (44.52 + 11.05 x_1) + 51.94 x + 0.0647 xL_0^2 \\
- L_0 L_1 (39.99 + 5.103 L_0 - 16.30 L_1) - 16.00 L_0 - 2848/243 L_0^3 - 368/243 L_0^3 \right\} \]  

+ f_{\mu_0} n_f \left\{ 2.147 L_1^3 - 24.57 L_1 + 48.79 - x_1 (242.4 - 150.7 x_1) - L_0 L_1 (81.70 + 9.412 L_1) \\
+ xL_0 (218.1 + 82.27 L_0^2) - 477.0 L_0 - 113.4 L_0^2 + 17.26 L_0^3 - 16/27L_0^5 \right\} x_1 .
Non-singlet coefficient functions for charged-current DIS

3. $\nu - \bar{\nu}$ non-singlet coefficient functions

The differences between corresponding $\nu$ and $\bar{\nu}$ coefficient functions are, as conjectured in Ref. [23], suppressed at large $x$ by two powers of $1-x$. Hence it is convenient to present the coefficient functions for $\nu - \bar{\nu}$ charged-current DIS in terms of differences which we define as

$$\delta C_{2L} \equiv C_{2L}^{\nu+\bar{\nu}} - C_{2L}^{\nu-\bar{\nu}}, \quad \delta C_{3} \equiv C_{3}^{\nu-\bar{\nu}} - C_{3}^{\nu+\bar{\nu}}.$$  (3.1)

The flavour class $f_{L02}$, see Fig. 1 of Ref. [16], does not contribute to the flavour asymmetries probed in the $\nu - \bar{\nu}$ combinations, hence it is understood that the corresponding part of Eq. (2.11) is removed before the difference for $F_3$ is formed. The $\delta C_a$ can be perturbatively expanded as

$$\delta C_a = \sum_{n=2}^\infty a_{n}^a \delta c_{a}^{(n)},$$  (3.2)

where $a_{n}$ is defined in Eq. (2.1) above. The second-order results were already given in Ref. [21] in exact and parametrized form. The later results, written in terms of the abbreviations (3.1), read

$$\delta c_{2}^{(2)}(x) \cong \{ -9.1587 - 57.70 x + 72.29 x^2 - 5.689 x^3 - xL_0(68.804 + 24.40 L_0) + 2.958 L_0^2 + 0.249 L_0 + 8/9 L_0^2 (2 + L_0) \} x_1,$$  (3.3)

$$\delta c_{L}^{(2)}(x) \cong \{ 10.663 - 5.248 x - 7.500 x^2 + 0.823 x^3 + xL_0(11.10 + 2.225 L_0) - 0.128 L_0^2 + 64/9 L_0 \} x_1^2,$$  (3.4)

$$\delta c_{3}^{(2)}(x) \cong \{ -29.65 + 116.05 x - 71.74 x^2 - 16.18 x^3 + xL_0(14.60 + 69.90 x - 0.378 L_0^2) - 8.560 L_0 + 8/9 L_0^2 (4 + L_0) \} x_1.$$  (3.5)

The corresponding third-order corrections are the new results of the present contribution. They supersede the previous approximate expressions in Eqs. (3.7) – (3.9) of Ref. [21], which were based on the lowest five even- integer and odd-integer Mellin moments of $C_{3-}$ and $C_{a-}$, $a = 2, L$, respectively, computed in Ref. [22]. Our new exact results can parametrized as

$$\delta c_{2}^{(3)} \cong \{ 273.59 - 44.95 x - 73.56 x^2 + 40.68 x^3 + 0.1356 L_0^3 + 8.483 L_0^4 + 55.90 L_0^6 + 120.67 L_0^5 + 388.0 L_0 - 329.8 L_0 L_1 - xL_0 (316.2 + 71.63 L_0 + 46.30 L_1) + 5.447 L_1 \} x_1 - 0.0008 \delta(x_1)$$  (3.6)

$$+ n_f \{ (-19.093 + 12.97 x + 36.44 x^2 - 29.256 x^3 - 0.76 L_0^3 - 5.317 L_0^4 - 19.82 L_0^5 - 38.958 L_0 - 13.395 L_0 L_1 + xL_0 (14.44 + 17.74 L_0) + 1.395 L_1 \} x_1 + 0.0001 \delta(x_1) \},$$

$$\delta c_{L}^{(3)} \cong \{ -620.53 - 394.5 x + 1609 x^2 - 596.2 x^3 + 0.217 L_0^3 + 62.18 L_0^2 + 208.47 L_0 - 482.5 L_0 L_1 - xL_0 (1751 - 197.5 L_0) + 105.5 L_1 + 0.442 L_1 \} x_1^2$$  (3.7)

$$+ n_f \{ -6.500 - 12.435 x + 23.66 x^2 + 0.914 x^3 + 0.015 L_0^3 - 6.627 L_0^2 - 31.91 L_0 - xL_0 (5.711 + 28.635 L_0) \} x_1^2,$$

$$\delta c_{3}^{(3)} \cong \{ -553.5 + 1412.5 x - 990.3 x^2 + 361.1 x^3 + 0.1458 L_0^3 + 9.688 L_0^4 + 90.62 L_0^6 + 83.684 L_0^5 - 602.32 L_0 - 382.5 L_0 L_1 - xL_0 (2.805 + 325.92 L_0) + 133.5 L_1 + 10.135 L_1 \} x_1 - 0.0029 \delta(x_1)$$  (3.8)

$$+ n_f \{ (-16.777 + 77.78 x - 24.81 x^2 - 28.89 x^3 - 0.7714 L_0^4 - 7.701 L_0^3 - 21.522 L_0^5 - 7.897 L_0 - 16.17 L_0 L_1 + xL_0 (43.21 + 67.04 L_0) + 1.519 L_1 \} x_1 + 0.0006 \delta(x_1) \}.$$
4. Discussion

With the exception of the $n_f$ part of Eq. (2.7) and the $n_f^2$ part of Eq. (2.10), which are exact, the second- and third-order expressions in sections 2 and 3 have been obtained by fitting the coefficients not written as fractions in the non-distribution parts to the exact coefficient functions at $x \geq 10^{-6}$. Where useful, the coefficients of $\delta(1-x)$ have been adjusted (even from zero) to fine-tune the accuracy of Mellin moments and convolutions. The resulting accuracy of Eqs. (2.6) – (2.11) and (3.3) – (3.8) and their convolutions with typical quark distributions of hadrons is 0.1% or better except where the functions are very small. Towards smaller $x$ the accuracy deteriorates, but the results are still accurate to about 1% and 3% at $x = 10^{-8}$ and $x = 10^{-10}$, respectively.

**Fortran** subroutines of these functions can be obtained from the preprint server arXiv.org by downloading the source of this note. They are also available from the authors upon request.

Analogous parametrizations for the pure-singlet quark and gluon coefficient functions for $F_2$ and $F_L$ have been given in Ref. [15] and section 4 of Ref. [16]. The partly very lengthy exact expression corresponding to Eqs. (2.6) – (2.11) and (3.3) – (3.5) can be found in Ref. [16] – where the $f_{l11}$ contribution has to be disregarded for the present charged-current case – and Refs. [17,21]; those for Eqs. (3.6) – (3.8) will be presented in Ref. [25]. Only the latter expressions allow for the analytical calculation of all integer Mellin moments of the coefficient functions.

The second moments of $\delta c_2^{(3)}$ and $\delta c_L^{(3)}$ are of particular relevance, since they enter the QCD corrections to the Paschos-Wolfenstein relation [24] for the determination of $\sin^2 \theta_W$ from charged-current DIS [3]. The truncated numerical values of these moments for $n_f$ light flavours are

$$\delta c_2^{(3)}(N=2) = -20.4001 + 0.72202 n_f, \quad \delta c_L^{(3)}(N=2) = -24.7755 + 0.80134 n_f. \quad (4.1)$$

Within their error estimates, the previous approximate results for these moments [21] agree with Eq. (4.1). The corresponding analytical expressions will also be presented in Ref. [25].

![Figure 1](image-url)

Figure 1: The third-order non-singlet coefficient functions for $F_2$, $F_L$, and $F_3$ in $\nu p + \bar{\nu} p \ (a = +)$ and $\nu p - \bar{\nu} p \ (a = -)$ charged-current DIS for four light flavours. The function $c_{3,a}^{(3)}(x)$ in Eq. (2.11) is shown with and without the $f_{l02}$ contribution. The factor $1/2000$ approximately converts the curves to an expansion in $\alpha_s$. 

---

**Additional Note:** The equations and figures should be correctly formatted according to the guidelines provided, ensuring proper alignment and clarity. The text should be checked for any errors or omissions that might affect its understandability.
Non-singlet coefficient functions for charged-current DIS

J. Davies

The \( a_s^3 \) coefficients \( c_{a,\pm}^{(3)} \) in Eq. (2.1) are illustrated in Fig. 1 over a wide range in \( x \). All six functions exhibit a sharp small-\( x \) rise, but only at \( x < 10^{-5} \) for \( F_2 \) and \( F_3 \) and at \( x \lesssim 10^{-4} \) for \( F_L \). With the exception of the flavour structure \( f_{102} \) that occurs at three loops for the first time but dominates \( c_{a,\pm}^{(3)} \) at small \( x \), at least four of five \( \ln k x \) small-\( x \) terms are required for a good approximation for \( c_{2,\pm}^{(3)} \) and \( c_{3,\pm}^{(3)} \), and all three such terms for \( c_{L,\pm}^{(3)} \), even below the \( x \)-range shown in the figure. Further discussions and illustrations of these coefficient functions will be presented in Ref. [25].

Acknowledgements

This work has been supported by the UK Science & Technology Facilities Council (STFC) grants ST/L000431/1 and ST/K502145/1, by the Deutsche Forschungsgemeinschaft (DFG) via contract MO 1801/1-1, and by the European Research Council (ERC) Advanced Grant 320651, HEPGAME.

References