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Relation between the pole and $\overline{\mathrm{MS}}$ quark mass in QCD

Matthias Steinhauser*
Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), 76128 Karlsruhe, Germany
E-mail: matthias.steinhauser@kit.edu

We discuss the relation between heavy quark masses defined in the $\overline{\mathrm{MS}}$ and on-shell scheme with special emphasis on the four-loop corrections. The application to the heavy quark masses is particularly interesting for the top quark since the new results allow to estimate the residual irreducible uncertainty due to renormalon effects.

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1. Four-loop corrections to the $\overline{\text{MS}}$-on-shell relation

Quark masses enter the Standard Model of particle physics as fundamental parameters. Hence, on the one hand, it is important to determine their numerical values as precise as possible. On the other hand, it is necessary to have precise relations among the various renormalization schemes available, in particular between the $\overline{\text{MS}}$ and on-shell scheme.

The on-shell renormalization constant is introduced as a multiplicative factor

$$m_0 = Z_m^{\text{OS}} M \,, \tag{1.1}$$

where m^0 is the bare and M the on-shell (or pole) mass. Note that m_0 and M are μ-independent and Z_m^{OS} contains $\alpha_s(\mu)$ and $\log(\mu/M)$ terms. To determine Z_m^{OS} one requires that the inverse quark propagator has a zero for $q^2 = M^2$ which leads to the formula

$$Z_m^{\text{OS}} \;=\; 1 + \Sigma_S + \Sigma_V \,, \tag{1.2}$$

where Σ_S and Σ_V are the scalar and vector parts of the quark two-point function which have to be evaluated on-shell, i.e. for $q^2 = M^2$ (q is the external momentum). This requires the evaluation of n-loop on-shell integrals to obtain Z_m^{OS} to n-loop accuracy.

One-, two- and three-loop QCD results to Z_m^{OS} have been computed in Refs. [1], [2] and [3, 4, 5, 6], respectively, and four-loop corrections have been reported in Refs. [7, 8, 9]. Whereas in [8] an uncertainty of the four-loop coefficient of 3% was reported it could be reduced to about 0.2% in [9]. Furthermore, the explicit dependence on the number of (massless) fermions and the breakdown to SU(N_c) colour factors are given. Light-quark mass effects are known to two [2] and three loops [10].

In the following we present results for

$$z_m(\mu) \;=\; \frac{m(\mu)}{M} \;=\; \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}} \,, \tag{1.3}$$

which is finite. For $N_c = 3$ it reads in numerical form (the one-, two- and three-loop expressions are known analytically)

$$\begin{aligned} z_m(\mu = M) \;=\;& 1 - 1.333\frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2 (-15.374 + 1.041 n_f) \\ &+ \left(\frac{\alpha_s}{\pi}\right)^3 (-226.283 + 28.229 n_f - 0.653 n_f^2) \\ &+ \left(\frac{\alpha_s}{\pi}\right)^4 [-4455.25 \pm 1.64 + (845.941 \pm 0.040) n_f - 45.517 n_f^2 + 0.678 n_f^3] \,, \end{aligned} \tag{1.4}$$

where n_f is the total number of active quarks. Note that the four-loop corrections to z_m given in Section 3.1 of Ref. [9] are parametrized in terms of $n_l = n_f - 1$, i.e. the number of massless quarks. The strong coupling constant in Eq. (1.4) is defined in the n_f-flavour theory and evaluated at the scale $\mu = M$.

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For many applications the inverted relation, which can be used to compute the on-shell from the $\overline{\text{MS}}$ mass, is needed. It is given by $c_m(\mu) = M/m(\mu)$ which reads for $\mu = m(m)$

$$
\begin{aligned}
c_m(\mu = m(m)) \;=\; & 1 - 1.333\frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2 (14.485 - 1.041 n_f) \\
& + \left(\frac{\alpha_s}{\pi}\right)^3 (217.903 - 27.961 + 0.653 n_f^2) \\
& + \left(\frac{\alpha_s}{\pi}\right)^4 [4357.40 \pm 1.64 - (834.548 \pm 0.040) n_f + 45.431 n_f^2 - 0.678 n_f^3] \,,
\end{aligned} \tag{1.5}
$$

where $\alpha_s \equiv \alpha_s^{(n_f)}(m(m))$.

It is interesting to apply Eq. (1.5) to the case of the top and bottom quark which leads to

$$
\begin{aligned}
M_t &= m_t(m_t) \left(1 + 0.4244\,\alpha_s + 0.8345\,\alpha_s^2 + 2.375\,\alpha_s^3 + (8.615 \pm 0.017)\,\alpha_s^4\right) \\
&= 163.508 + 7.529 + 1.606 + 0.496 + (0.195 \pm 0.0004) \text{ GeV}\,, \qquad (1.6) \\
M_b &= m_b(m_b) \left(1 + 0.4244\,\alpha_s + 0.9401\,\alpha_s^2 + 3.045\,\alpha_s^3 + (12.685 \pm 0.025)\,\alpha_s^4\right) \\
&= 4.163 + 0.398 + 0.199 + 0.145 + (0.136 \pm 0.0003) \text{ GeV}\,, \qquad (1.7)
\end{aligned}
$$

where in the second lines of these equations $\alpha_s^{(6)}(m_t) = 0.1085$ and $\alpha_s^{(5)}(m_b) = 0.2253$ have been used. Equation (1.6) shows a nice convergence behaviour leading to a four-loop contribution of about 200 MeV which is about a factor 2.5 smaller than the three-loop term. On the other hand, no convergence is observed for the bottom quark where the two-, three- and four-loop contributions are of the same order of magnitude.

2. $\overline{\text{MS}}$-threshold mass relation

It is interesting to use the new four-loop result discussed in the previous section to construct precise relations between the $\overline{\text{MS}}$ and properly chosen threshold masses. The latter are used to parametrize quantum corrections to physical quantities related to the particle threshold like bound-state effects or threshold cross sections. Often such quantities are used to determine the numerical values of the quark masses by comparison with experimental measurements. In a first step the threshold mass value is obtained which in a second step is transformed to the $\overline{\text{MS}}$ mass. It has been shown in Refs. [8, 9] that the conversion is known to high accuracy after including the four-loop term of the $\overline{\text{MS}}$-on-shell relation.

In Refs. [8, 9] the potential subtracted (PS) [11], 1S [12, 13, 14] and renormalon subtracted (RS and RS′) [15] masses have been considered and detailed results for numerical effects are presented in Section 4 of Ref. [9]. In the following we want to provide compact formulae which allows for the computation of the $\overline{\text{MS}}$ quark mass for given threshold mass. We restrict ourselves to the top and bottom case and assume that the threshold masses are given by

$$
\begin{aligned}
m_t^{\rm PS} &= 168.049 \pm 0.100~\text{GeV}\,, \qquad \text{for} \qquad \mu_f = 80~\text{GeV}\,,\\
m_t^{\rm 1S} &= 172.060 \pm 0.100~\text{GeV}\,,\\
m_t^{\rm RS} &= 166.290 \pm 0.100~\text{GeV}\,, \qquad \text{for} \qquad \mu_f = 80~\text{GeV}\,, \qquad (2.1)\\
m_b^{\rm PS} &= 4.481 \pm 0.020~\text{GeV}\,, \qquad \text{for} \qquad \mu_f = 2~\text{GeV}\,,\\
m_b^{\rm 1S} &= 4.668 \pm 0.020~\text{GeV}\,,\\
m_b^{\rm RS} &= 4.364 \pm 0.020~\text{GeV}\,, \qquad \text{for} \qquad \mu_f = 2~\text{GeV}\,. \qquad (2.2)
\end{aligned}
$$

This leads to the formulae

$$
\begin{aligned}
&\text{PS} \qquad & m_t(m_t) &= \left[163.508 \pm 0.008 + 0.051\Delta_{\alpha_s} - 0.095\Delta_{m_t}^{\rm PS}\right]~\text{GeV}\,,\\
&\text{1S} \qquad & m_t(m_t) &= \left[163.508 \pm 0.005 + 0.090\Delta_{\alpha_s} - 0.096\Delta_{m_t}^{\rm 1S}\right]~\text{GeV}\,,\\
&\text{RS} \qquad & m_t(m_t) &= \left[163.508 \pm 0.009 + 0.029\Delta_{\alpha_s} - 0.095\Delta_{m_t}^{\rm RS}\right]~\text{GeV}\,,\\
&\text{PS} \qquad & m_b(m_b) &= \left[4.163 \pm 0.001 + 0.009\Delta_{\alpha_s} - 0.018\Delta_{m_b}^{\rm PS}\right]~\text{GeV}\,,\\
&\text{1S} \qquad & m_b(m_b) &= \left[4.163 \pm 0.004 + 0.010\Delta_{\alpha_s} - 0.019\Delta_{m_b}^{\rm 1S}\right]~\text{GeV}\,,\\
&\text{RS} \qquad & m_b(m_b) &= \left[4.163 \pm 0.002 + 0.005\Delta_{\alpha_s} - 0.018\Delta_{m_b}^{\rm RS}\right]~\text{GeV}\,, \qquad (2.3)
\end{aligned}
$$

where $(X \in \{\text{PS}, 1\text{S}, \text{RS}\})$

$$
\begin{aligned}
\Delta_{\alpha_s} &= \frac{0.1181 - \alpha_s(M_Z)}{0.0013}\,,\\
\Delta_{m_t}^{\rm X} &= \frac{m_t^X\Big|_{\text{from Eq. (2.1)}} - m_t^X}{0.1~\text{GeV}}\,,\\
\Delta_{m_b}^{\rm X} &= \frac{m_b^X\Big|_{\text{from Eq. (2.2)}} - m_b^X}{0.02~\text{GeV}}\,, \qquad (2.4)
\end{aligned}
$$

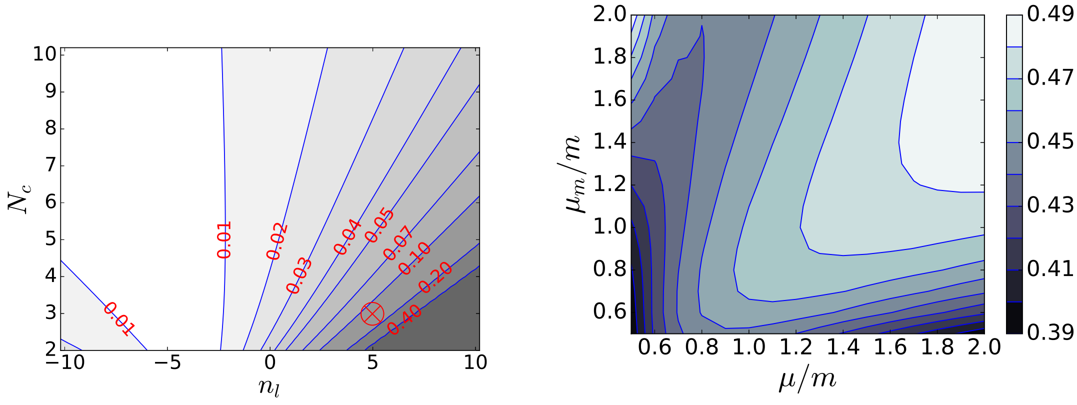
parametrize the deviation from the central values $\alpha_s(M_Z) = 0.1181$ and the threshold masses in Eqs. (2.1) and (2.2). The first uncertainty in Eq. (2.3) is obtained from the quadratic combination of 50% of the four-loop contribution[1] and the numerical uncertainty of the four-loop coefficient in the $\overline{\rm MS}$-on-shell relation. Note that the latter is basically negligible.

Formulae like (2.1) and (2.2) are easily derived with the help of `RunDec` [16] and `CRunDec` [17] where all mass relations obtained in Refs. [8, 9] are implemented.

3. Uncertainty of the top quark pole mass

As a further application of the four-loop corrections to the $\overline{\rm MS}$-on-shell relation we discuss in this section the renormalon contribution to the top quark pole mass. The basis for this analysis is the all-order calculation of the leading infrared renormalon contribution which has been computed in Refs. [18, 19] up to an overall normalization constant N. In Ref. [20] the N_c and n_l dependence

[1] which is a very conservative estimate of the higher order corrections considering the rapid convergence of the perturbative series, see Ref. [9]



(a) (b)

Figure 1: (a) Δ_{34}, which shows the consistency of N extracted with three- and four-loop accuracy, as a function of N_c and n_l, for $\mu = \mu_m = m(m)$. The cross corresponds to the case relevant for top, i.e. $N_c = 3$ and $n_l = 5$. (b) N as a function of μ and μ_m.

of c_m has been used to obtain N together with an estimate of the uncertainty which is based on the independent variation of the renormalization scale of the $\overline{\text{MS}}$ top quark mass (μ_m) and strong coupling constant (μ).

In Fig. 1(a) the N_c and n_l dependence of the quantity Δ_{34} (see Ref. [20] for a precise definition) is shown. Δ_{34} is an estimate of how close is the third order coefficient to the asymptotic value. It is likely to be an overestimate of the deviation of the fourth order coefficient from the asymptotic formula and should not be taken as an error on the normalization N. Still, from Fig. 1(a) one finds that generically $\Delta_{34} < 0.1$ and thus the four-loop coefficient indeed matches the asymptotic formula in the expected range of N_c and n_l values, including those of physical interest. An exception is the region where the one-loop coefficient of the beta function is small and the renormalon is not dominant, see lower-right corner of Fig. 1(a).

In Fig. 1(b) the normalization N is shown for $N_c = 3$ and $n_l = 5$ as a function of μ_m and μ. Both scales are varied by a factor two around the central value $\mu_m = \mu = m(m) \equiv m$; the maximal and minimal values are used to define the (asymmetric) uncertainty, see Ref. [20].

In this contribution we refrain from providing more details on intermediate quantities (which can be found in Ref. [20]) and immediately present the final result for the pole mass as computed by the $\overline{\text{MS}}$ mass. To obtain this relation we follow two approaches. The first one sums the asymptotic series up to the point where the expansion coefficients start to grow (see also Tab. 2 of [20]). This leads to the relation

$$\frac{M}{m(m)} = 1.06177^{+0.00010}_{-0.00025}\,(N) \pm 0.00001\,(c_4) \pm 0.00087\,(\alpha_s) \pm 0.00041\,(\text{ambiguity})\,, \quad (3.1)$$

where the uncertainties are due to the normalization constant, the numerical precision of the four-loop $\overline{\text{MS}}$-on-shell coefficient, the strong coupling constant and the irreducible ambiguity which is obtained from the first omitted term in the perturbative series.

The second method is based on the Borel transform of the asymptotic series coefficients and the subsequent computation of the Borel sum using principal value prescription. The ambiguity is estimated by the imaginary part of the corresponding integral when the contour is deformed into the upper complex plane, divided by Pi (see, e.g., [21], section 5.2). Our final result for the mass relation reads

$$\frac{M}{m(m)} = 1.06164^{+0.00009}_{-0.00023}\,(N) \pm 0.00001\,(c_4) \pm 0.00086\,(\alpha_s) \pm 0.00043\,(\text{ambiguity})\,, \quad (3.2)$$

in good agreement with Eq. (3.1).

Let us mention that the contribution in Eq. (3.2) from beyond four loops amounts to about 250 MeV. Furthermore, note that for a given $\overline{\text{MS}}$ mass, the top quark pole mass is determined from relations (3.1) and (3.2) with an accuracy of 0.92 per mil which is obtained by adding all uncertainties in quadrature. The irreducible uncertainty amounts to 0.41 per mil which corresponds to about 70 MeV, an uncertainty far below the accuracy that can be achieved at the Large Hadron Collider. Thus, the top quark pole mass is a useful concept in this context. This is different for the mass determination from a scan of the top pair production threshold at a high-energy e^+e^- collider [22, 23] where uncertainties below 100 MeV can be reached.

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