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Entanglement Entropy in Field Theory and Gravity

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This is an introductory set of lectures given by the author at the Modave Summer School 2015 on the various incarnations of entanglement entropy. Topics covered include an overview of the basic properties of entanglement entropy, its structure in quantum field theory, its computation in two-dimensional conformal field theory, a simple application to c-theorems in two dimensions, and the Ryu-Takayanagi formula and its extensions in holography.

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1. Introduction

In recent years, there has been an explosion of interest around the understanding of a very particular numerical quantity associated with dividing a quantum system into two pieces. This set of lecture notes will discuss this particular quantity from various points of view, and provider the reader with the tools to compute it in the systems where it is most well-understood.

However, before plunging into details it is worth thinking briefly about *why* entanglement entropy has recently become such a hot topic. There are probably many answers to this question. One first thing to note is that quantum mechanical systems are *complicated*: they really have many moving parts, far more than a typical classical system. Suppose you have N *classical* spins, each of which can be either up or down: to store the full information of the system all you need to do is store N numbers (where each number is actually either just 0 or 1). On the other hand, if you have N quantum-mechanical spins, the dimension of the Hilbert space is 2^N. The fact that N appears in the exponent means that the difficulty of a quantum problem scales with the number of degrees of freedom in a drastically different way than in a classical problem. The fact that we typically care about systems where N is large means that a direct assault is almost always guaranteed to fail, since there is simply too much information to keep track of. It is a fun exercise (first suggested, I think, by Brian Swingle) to imagine filling the whole universe (i.e. the observable Hubble volume) with standard issue hard drives and asking how many spins you can simulate in this way. The answer is around 213.

What this really means is that Hilbert space is large and complicated. This is all the fun of quantum field theory – it means that very interesting things can happen, new infrared states of matter, phase transitions, emergence, etc. etc. etc. – but it means that to access all of this fun physics we require a way to *slice* the Hilbert space and break it into pieces that are somehow more manageable. Sometimes there are natural ways to do this slicing: for example, if we look at quantum field theories that are close to free, we can think about states with only a few weakly interacting particles, and the dynamics will not take us very far from this subspace. On the other hand, if we care about strongly interacting systems, it is not obvious how to slice up Hilbert space into bite-size pieces, and we should look instead for tools that will let us organize the large amount of information that is present in Hilbert space. One such tool is the entanglement entropy.

In these notes we discuss entanglement entropy in its various incarnations. We begin with its definition and applications in very general quantum-mechanical systems in Section 2. In Section 3 we then specialize to quantum field theory, where the tensor factorization of field-theoretical Hilbert spaces in *physical* space imbues entanglement entropy with extra geometrical significance. Finally in Section 4 we specialize further to those quantum field theories that possess holographic duals.

Familiarity with basic quantum field theory and general relativity is assumed throughout these notes. Some familiarity with the basic mechanics of holography will be useful for understanding Section 4. No knowledge of string theory or advanced conformal field theory is assumed.

2. Entanglement entropy in quantum mechanics

Consider the canonical example of an entangled state, an Einstein-Podolsky-Rosen pair of entangled spins:

$$|\psi\rangle_{EPR} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\uparrow\rangle|\downarrow\rangle\right) \tag{2.1}$$

The two spins are clearly correlated in some manner, but as we are dealing with a single pure state that completely specifies the system, the information stored here is not a simple classical correlation. Rather we say that the two spins are quantum mechanically entangled. The number that we use to quantify how tightly the two spins are correlated is the *entanglement entropy*, which we now define.

2.1 Definition

Consider a state $|\psi\rangle$ in a Hilbert space $\mathscr{H}$ that can be broken into a tensor product of two subspaces: $\mathscr{H} = A \otimes \overline{A}$. We can now associate a *reduced density matrix* ρ_A with the factor A by tracing out the subfactor $\overline{A}$:

$$\rho_A = \operatorname*{Tr}_{\overline{A}} |\psi\rangle\langle\psi| \tag{2.2}$$

The entanglement entropy S_A associated with the tensor factor A of the Hilbert space is then the ordinary von Neumann entropy of the reduced density matrix ρ_A.

$$\boxed{S_A \equiv -\operatorname{Tr}_A \rho_A \log \rho_A} \tag{2.3}$$

This is the object that we will study for the remainder of these lectures.

To understand what it measures, let us compute it in boringly excruciating detail for two states: first we consider the simple product state

$$|\psi\rangle_{prod} = |\uparrow\rangle|\downarrow\rangle \tag{2.4}$$

and take subfactor A to be the first spin, making subfactor $\overline{A}$ the second. Tracing out the second spin we find $\rho_{A,prod} = |\downarrow\rangle\langle\downarrow|$, which is still an outer product of a single state, i.e. it is still *pure*. We have not lost any information in the tracing out. The von Neumann entropy of this density matrix is zero, and thus for the product state we find that $S_{A,prod} = 0$.

Now we repeat the exercise for the EPR state (2.1): computing the reduced density matrix we find instead

$$\rho_{A,EPR} = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) . \tag{2.5}$$

In this case we have actually lost some information when tracing out the second spin: the reduced density matrix is no longer that of a pure state. Its resulting mixed-ness is a probe of how entangled $\overline{A}$ was with A. This mixed-ness is measured by the von Neumann entropy of ρ_A, which gives us $S_{A,EPR} = \log 2$.

In the above example, the entanglement entropy was defined starting with a pure state $|\psi\rangle$. One can also start with a density matrix ρ, defining a smaller reduced density matrix the same way as before:

$$\rho_A = \operatorname{Tr}_{\overline{A}} \rho , \tag{2.6}$$

which clearly reduces to (2.2) in the special case where ρ is pure, i.e. when it can be written $\rho = |\psi\rangle\langle\psi|$. Note that if ρ was mixed to begin with, then S_A is no longer a direct probe of the entanglement between A and $\overline{A}$, as ρ already had some von Neumann entropy to start with, and S_A will now be "contaminated" with this. In a (standard) abuse of notion, we will still call the number S_A the entanglement entropy.

In the rest of these lectures, we will discuss the entanglement entropy in vastly more complicated situations, when we will not be able to trace out degrees of freedom so easily. It is important to note that the entanglement entropy is completely determined by the *state*, which can be chosen freely to be any state at all. However if we take the state in question to be the ground state of a particular Hamiltonian, then the entanglement entropy can lead to considerable information about the dynamics associated with the Hamiltonian, as we discuss below.

Before moving on, we define a related quantity called the *Renyi entropy*:

$$S_n \equiv -\frac{1}{n-1} \log \operatorname{Tr}_A \rho_A^n . \tag{2.7}$$

These are also measures of entanglement, but do not have as many nice properties as the entanglement entropy. However, as we will see below, they are often easier to compute, and if known for all n in a suitably nice way they can be analytically continued to obtain the entanglement entropy:

$$S_A = \lim_{n\to 1} S_n \tag{2.8}$$

2.2 General properties

The entanglement entropy so defined has a number of properties. We list some of the most commonly used properties below. For further discussion of the physical significance of the inequalities below see [1].

1. If we start with a pure state, then the entanglement entropy of a sub-factor is equal to that of its complement:
$$S_A - S_{\overline{A}} = 0 \tag{2.9}$$
In fact, all non-zero eigenvalues of ρ_A and $\rho_{\overline{A}}$ are the same. (2.9) states that the entanglement entropy of a pure state should not be thought of as a property of A or of $\overline{A}$, but rather as a property of the *division* of the system into two halves.

2. If we start with a *mixed* state ρ with entropy $S(\rho)$, then (2.9) is modified to read instead
$$|S_A - S_{\overline{A}}| \leq S_\rho \leq S_A + S_{\overline{A}} \tag{2.10}$$
The first inequality is called the Araki-Lieb inequality [2], and states that in a mixed state the entropy of A differs from that of $\overline{A}$, but the amount by which it can differ is bounded by how impure the original state was. The second inequality is called sub-additivity, to contrast it with:

3. **Strong sub-additivity**, which is the strongest set of constraints on the entanglement entropy. Suppose the Hilbert space is a product not of two but of three or more tensor factors, requiring a change of notation: $\mathscr{H} = \otimes_i \mathscr{H}_i$. Denote by S_1 the entanglement entropy of sub-factor $\mathscr{H}_1$, S_{12} the entanglement entropy of sub-factor $\mathscr{H}_1 \otimes \mathscr{H}_2$, etc. Then it can be shown that
$$S_{12} + S_{23} \geq S_2 + S_{123} \qquad S_{12} + S_{23} \geq S_1 + S_3 \tag{2.11}$$
These are very powerful inequalities that have a variety of physical applications, as we will see. In fact (2.10) can be derived from (2.11). The general proof of (2.11) is fairly non-trivial [3].

3. Entanglement entropy in field theory

The above discussion of entanglement entropy was very abstract, applying to any quantum mechanical system. However, when applied to quantum *field* theory entanglement entropy takes on a deeper geometric significance. This happens because states in a quantum field theory are defined with reference to a spatial manifold $\mathscr{M}$, and the Hilbert space of a quantum field theory is (typically) a tensor product of degrees of freedom localized at different points in space. This fact is most obvious if one considers regularizing the quantum field theory with a lattice in the UV. Thus for any region of space $A \subset M$ there is a tensor factor of the Hilbert space $\mathscr{H}_A$ associated with it, and it makes sense to trace out things that are inside (or outside) A. Thus we can ask how entangled a *spatial* region A is with the rest of the universe, etc. etc.

In a general QFT in d space-time dimensions A is a $d-1$ dimensional region and its boundary ∂A is a $d-2$ dimensional closed submanifold that is usually called the *entangling surface*.

3.1 General structure

How do we compute the entanglement entropy? In general, given an arbitrary state in an arbitrary quantum field theory, this is a somewhat difficult problem. However in any reasonable state, the answer is always dominated by very short-range correlations across the entangling surface [4, 5]:

$$S_A = \frac{\text{Area}(\partial A)}{\varepsilon^{d-2}} + \cdots, \tag{3.1}$$

where ε is the UV cutoff and this formula applies only for $d > 2$. The form of this *area law* for the entanglement entropy is independent of the state, as sufficiently short-range correlations in any field theory are sensitive only to the structure of the vacuum. The result above is telling us that the vacuum in a quantum field theory is highly entangled. See [6] for a review of area laws.

This is always the most UV-divergent term. There can be other terms that are still UV-divergent, but less so: their precise forms depends on the dimension and are known if we assume that we are starting with a conformal field theory in the UV. Here we will simply present results without proof, referring the reader to the references for further justification.

The structure of the answer in $1+1$ dimensions will be discussed in detail in the next subsection. In $2+1$ dimensions ∂A is a closed curve and we have the intuitively obvious

$$S_{A,(2+1)} = \frac{\text{Length}(\partial A)}{\varepsilon} + \text{finite}, \tag{3.2}$$

where the finite piece depends on the state (and, for example, the shape of the entangling region).

In $3+1$ dimensions we have a subleading logarithmic divergence [7, 8]:

$$S_{A,(3+1)} = \frac{\text{Area}(\partial A)}{\varepsilon^2} - 8\pi \log(\varepsilon) \int_{\partial A} d^2x \sqrt{h} \left(-aE_2 + cI_2\right) + \text{finite} \tag{3.3}$$

where a and c are the central charges of a 4d CFT, E_2 is the Euler density of ∂A and I_2 a different conformally invariant density that we do not describe here. This logarithmic term changes additively under a rescaling of the cutoff, which in a CFT has an effect that is very well-understood and captured completely by the Weyl anomaly. Thus the structure of this logarithmic term can be understood as being determined by the Weyl anomaly of the 4d field theory. This explains also the absence of a logarithmic term in $2+1$ dimensions, where there is no anomaly.

3.2 Entanglement in CFT$_2$

There are very few general formulas for entanglement entropy. A notable exception is two-dimensional conformal field theory, where there is a universal and celebrated formula for the entanglement entropy of an interval in the vacuum [9–11].

$$S(L) = \frac{c}{3} \log\left(\frac{L}{\varepsilon}\right) \tag{3.4}$$

where L is the length of the interval, c the central charge of the CFT, and ε a UV cutoff as before. Note that the logarithmic dependence on the cutoff can be viewed as the degenerate limit of the area law exhibited in (3.1).

There are many ways to derive this formula: the essential physics behind all of them is the fact that in a 2d CFT there are no scales, and thus the dependence on IR quantities (such as the length of the interval) is essentially determined by the dependence on the UV cutoff, which is in turn determined by the Weyl anomaly, which in two dimensions is very constraining. In these notes we will take a slightly nonstandard route and derive this formula by adapting a technique from [12] to map the entanglement entropy to a thermal entropy.

Consider the CFT_2 on the Lorentzian plane with coordinates $x^{\pm} = t \pm x$. We are interested in computing the entanglement entropy of an interval A of length L, which we place at $t = 0$ and take to stretch from $x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$.

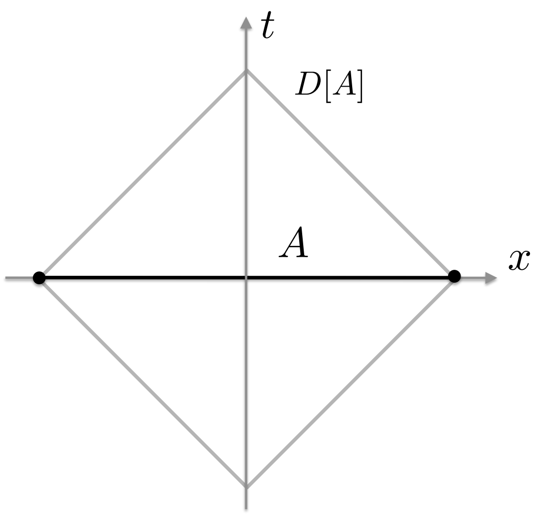

Figure 1: An interval A in a 2d CFT, together with its causal development $D[A]$.

Now in a relativistic theory there is a trick to implement the procedure of tracing out the degrees of freedom outside A. Tracing out all such degrees of freedom means that we are interested only in the physics that is in the causal development $D[A]$ of the interval A, as shown in Figure 1. Thus consider performing the following coordinate transformation

$$x^{\pm} = \frac{L}{2}\tanh\left(\frac{y^{\pm}}{2}\right) \qquad y^{\pm} = \tau \pm y \tag{3.5}$$

It is easy to see that the $y^{\pm}$ coordinates only cover $D[A]$; thus, in the $y^{\pm}$ coordinates the trace has already been implemented.

Next, note from (3.5) that the y "time" coordinate τ has acquired a Euclidean periodicity:

$$\tau \sim \tau + 2i\pi \tag{3.6}$$

A Euclidean periodicity in time implies that the physics in the y coordinate system is actually at a finite temperature $T = \frac{1}{2\pi}$. Thus the state of the field theory in the y coordinates is actually that of a thermal density matrix:

$$\rho_{(y^\pm)} = \exp(-\pi H_\tau) \tag{3.7}$$

where H_τ is the Hamiltonian that generates translations in the τ coordinate.

Finally, we note that this coordinate transformation is a separate mapping of the left and right-moving coordinates, and so is part of the symmetry group of a 2d CFT. Thus there is a unitary mapping U that takes all operators in the $x^\pm$ coordinates to those in the $y^\pm$ coordinates: in other words, the reduced density matrix $\rho_{A,(x^\pm)}$ in the $x^\pm$ coordinates is unitarily related to the thermal density matrix (3.7)

$$\rho_{(y^\pm)} = U\rho_{A,(x^\pm)}U^\dagger \tag{3.8}$$

But the von Neumann entropy is invariant under unitary transformations: thus to determine the von Neumann entropy of $\rho_{A,(x^\pm)}$, which is interpreted as the entanglement entropy in the $x^\pm$ coordinates we need only find the von Neumann entropy of $\rho_{(y^\pm)}$, which is interpreted a *thermal* entropy in the $y^\pm$ coordinates.

It turns out that in any 2d CFT the thermal entropy of a system at sufficiently high temperature compared to its spatial extent is completely determined by the *Cardy formula* [13].[1] The entropy of any 2d CFT at a finite temperature T on a (very long) line of length $R \gg T^{-1}$ is:

$$S = \frac{\pi c}{3} RT \tag{3.9}$$

where c is the central charge of the CFT. T is $\frac{1}{2\pi}$: but what is R? It is the length of the interval in the y coordinate at $\tau = 0$: but we see from (3.5) that even if we take $y \to \infty$ we never actually reach the end of the interval, and thus naively R appears infinite.

This divergence in R is actually a consequence of the fact that the entanglement entropy is UV divergent: we are attempting to include correlations all the way up to the endpoint of the interval at $x = \pm\frac{L}{2}$. We may regulate the divergence by introducing UV cutoff ε, which in this context means that we study instead an interval that extends only from $x \in [-\frac{L}{2} + \varepsilon, \frac{L}{2} - \varepsilon]$. In that case we find the length in the y coordinate to be finite:

$$R = 2\log\left(\frac{L}{\varepsilon}\right), \tag{3.10}$$

and thus the final answer for the entanglement entropy is

$$S = \frac{c}{3}\log\left(\frac{L}{\varepsilon}\right) \tag{3.11}$$

as claimed above.

This answer may have seemed very slick. In reality we used conformal symmetry to transfer all of the difficulty to a different problem, that of determining the thermal entropy of a system in (3.9). A universal formula for the thermal entropy is very nontrivial and exists only in two dimensions, which means that such methods can only result in a general formula for the entanglement entropy in 2d CFT.

[1] This formula is derived in many places, in particular in the notes by Blagoje Oblak in this volume. See also Appendix A of [14] for a particularly streamlined derivation in notation similar to mine.

We have derived a formula for the entanglement entropy of an interval on the plane. However in 2d CFT an infinite cylinder can be conformally mapped to the plane via the exponential map $z = e^w$: thus a cylinder is essentially equivalent to the plane, and we can also derive a formula for the entanglement entropy on a cylinder. The details of the derivation are left to the exercises. Here we discuss only the physical interpretation of the resulting expressions.

If the compact cylinder direction is taken to be time and have periodicity β, then we are studying a system with a compact Euclidean time direction, which is thus at finite temperature. We find for the entanglement entropy of an interval of length L at a finite temperature β^{-1}:

$$S(L)_\beta = \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\sinh\left(\frac{\pi L}{\beta}\right)\right) \tag{3.12}$$

Note that at short distances $L \ll \beta$ we are probing scales much smaller than the thermal wavelength, and the entanglement entropy reduces to the vacuum result (3.11), as it cannot tell that we are at finite temperature. On the other hand, at large distances we find

$$S(L \gg \beta)_\beta \sim \frac{\pi c}{3}\frac{L}{\beta}, \tag{3.13}$$

i.e. the entanglement entropy is becoming extensive in system size: it now obeys a *volume law*. At this point the entanglement entropy is dominated by the ordinary thermal entropy arising from the fact that the system is at a finite temperature and is thus in a mixed state. Indeed the entropy density extracted from (3.13) agrees exactly with that arising from the Cardy formula (3.9), as it must.

If we now take the compact cylinder direction to be *space* and have length R, then we are studying an interval in the CFT vacuum on a *circle*. We then find the entanglement entropy to be

$$S(L)_R = \frac{c}{3}\log\left(\frac{R}{\pi\epsilon}\sin\left(\frac{\pi L}{R}\right)\right) \tag{3.14}$$

Again, the short-distance result agrees with that of the vacuum, but at longer distances we see that $S(L) = S(R-L)$, as expected from (2.9).

3.3 An application: c-theorems

We would now like to turn to an application of entanglement entropy techniques: we will prove a *c-theorem* using the techniques above. This will also require us to understand some interesting properties of the entanglement entropy in a relativistic field theory.

A 2d CFT is characterized by a number, the central charge c, which can be heuristically understood as a measure of the number of degrees of freedom. Now consider deforming a CFT with central charge c_{UV} by some relevant operator with some characteristic energy scale m – what happens in the infrared? We might expect that in general we will gap out some degrees of freedom, and we will always have *less* degrees of freedom in the infrared – i.e. if we flow to a new infrared CFT with a central charge c_{IR}, we will have $c_{UV} \geq c_{IR}$. A simple example is a free scalar with mass m in two dimensions, where in the UV we may ignore the mass, leading to $c_{UV} = 1$ and in the IR we do not have enough energy to excite the scalar at all, so $c_{IR} = 0$.

The fact that $c_{UV} \geq c_{IR}$ in full generality was proven by Zamolodchikov in 1986 using properties of the correlation functions of the stress tensor [15]. In this section we will follow instead the beautiful papers [16, 17] and present an alternative proof that uses properties of the entanglement entropy.

First, consider an RG flow from one CFT to another, deformed by a relevant operator with associated energy scale m. Consider the entanglement entropy of an interval of length L. Depending on the relative size of L versus m^{-1}, we expect the entanglement entropy to be given by (3.4) with the appropriate central charge, i.e.

$$S(L \ll m^{-1}) \sim \frac{c_{UV}}{3}\log\left(\frac{L}{a}\right) \qquad S(L \gg m^{-1}) \sim \frac{c_{IR}}{3}\log\left(\frac{L}{a'}\right), \tag{3.15}$$

where the behavior of $S(L\sim m)$ in the intermediate region depend on the details of the RG flow. In other words, we may define a c-function as a function of L by:

$$c(L) = 3\frac{d}{d\log L}S(L), \tag{3.16}$$

which effectively measures, via the entanglement entropy, the number of degrees of freedom[2] that are active at the scale L. It is clear that $c(L)$ smoothly interpolates from c_{UV} to c_{IR}. Our task now is to prove that

$$\frac{dc(L)}{dL} \leq 0 \tag{3.17}$$

This can be established via the strong subadditivity of the entanglement entropy (2.11). First we note that for three geometric regions A, B, and C, the SSA relation can be written:

$$S(A) + S(B) \geq S(A\cap B) + S(A\cup B) \tag{3.18}$$

Let us first see how far we can get with this alone. Consider arranging three intervals as shown in

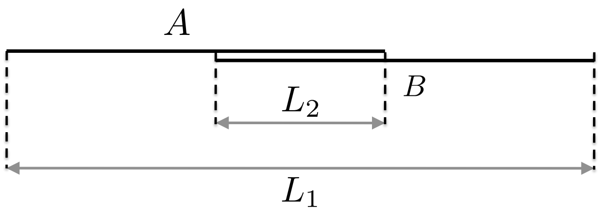

Figure 2: Two intervals A and B arranged to overlap symmetrically. Note that the length of $A\cap B$ is L_2 and that of $A\cup B$ is L_1, and the lengths of A and B individually are $\frac{1}{2}(L_1+L_2)$.

Figure 2: the lengths of A and B are both $\frac{L_1+L_2}{2}$, the length of $A\cup B$ is L_1 and the length of $A\cap B$ is L_2. Thus we immediately find the geometric relation:

$$S\left(\frac{L_1+L_2}{2}\right) \geq \frac{1}{2}\left(S(L_1)+S(L_2)\right) \tag{3.19}$$

for all possible choices of L_1, L_2: in other words, we have just established that $S(L)$ is a *concave down* function as a function of L:

$$\frac{d^2}{dL^2}S \leq 0 \tag{3.20}$$

[2] This idea has been generalized to higher dimensions in [18].

This is a useful relation that holds for any translationally invariant system: however, it is not yet strong enough to imply (3.17): as $c(L)$ is a logarithmic derivative of $S(L)$, we require it to be concave down as a function not of L but of $\log L$.

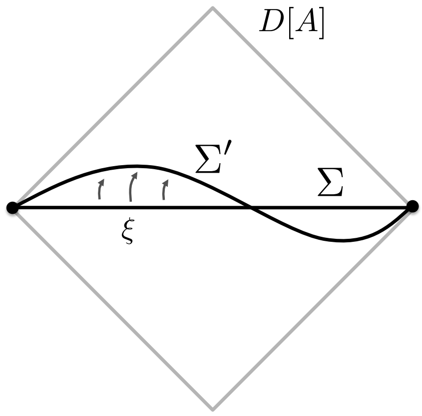

Figure 3: An interval and its causal development: note that the two Cauchy slices Σ and Σ' have are related by a diffeomorphism ξ and thus have the same information (and entanglement entropy) in a Lorentz-invariant theory.

Luckily, we have another symmetry at our disposal: Lorentz invariance. In a Lorentz-invariant theory the entanglement entropy of an interval A should not really be thought of as a function of the interval A, but rather of its causal diamond $D[A]$, as is familiar from Figure 3, as any Cauchy slice on the same interval can be related by a unitary transformation implementing a diffeomorphism, and so has the same information. This actually has practical consequences. Let us consider arranging two intervals A and B as before, but now on two different time slices as in Figure 4. The geoometry is arranged so that the proper lengths of interval A and B are both $\sqrt{L_1 L_2}$.

Now consider the entanglement entropy of $S(A \cup B)$. The causal diamond of $A \cup B$ is equivalent to the causal diamond of the interval extending from a_1 to b_1, which has proper length L_1. Thus we have $S(A \cup B) = S(L_1)$, and similarly, $S(A \cap B) = S(L_2)$. The SSA relation (3.18) now becomes

$$S(\sqrt{L_1 L_2}) \geq \frac{1}{2}\left(S(L_1) + S(L_2)\right) \tag{3.21}$$

This is a stronger relation than (3.19), as it implies that $S(L)$ is a concave down function as a function of $\log L$:

$$\frac{d^2}{d(\log L)^2} S \leq 0, \tag{3.22}$$

which is equivalent to the monotonicity relation (3.17) that we set out to prove. In other words, we have used entanglement entropy to prove the c-theorem from very fundamental principles: the structure of quantum mechanics (used in proving SSA) and Lorentz invariance.

What of higher dimensions? In $2+1$ dimensions we have the so-called F-theorem [19–21], where F is a number defined to be the partition function of an S^3:

$$F = -\log Z(S^3) \tag{3.23}$$

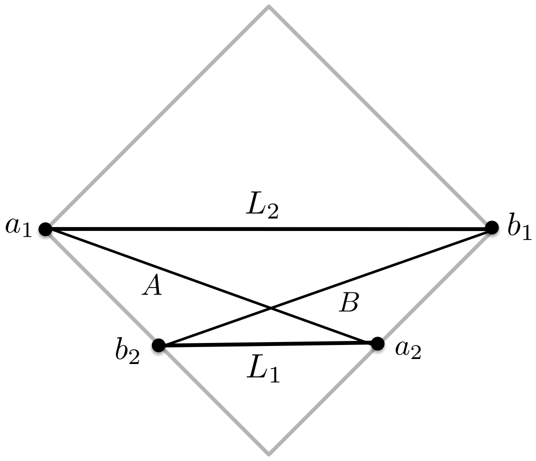

Figure 4: Two intervals A and B, now boosted relative to one another. The proper lengths of A and B both are $\sqrt{L_1 L_2}$.

Considerable evidence led to the conjecture that F should decrease under *RG*, but an analog of Zamolodchikov's proof remained elusive.

Interestingly, however, F can be related to the constant term in the entanglement entropy of a circular disc on the vacuum on $\mathbb{R}^3$ (3.2). A higher-dimensional generalization of the arguments used above – resting on SSA and Lorentz-invariance – can now be used to show that F decreases as we move to the infrared, establishing the F-theorem [22, 23]. At the moment this is the only proof of the F-theorem, demonstrating the power of entanglement-based techniques.

Going up one dimension higher, there are two central charges (called a and c) that characterize a (3+1)-d CFT. It was recently shown by Komargodski and Schwimmer that there is an a-theorem, in that the central charge conventionally called a also decreases monotonically under RG [24]. The proof uses general properties of conformal invariance and effective field theory. a also appears in the entanglement entropy in $3+1$ dimensions (see (3.3)), but at the moment there is no entanglement-based proof of the a-theorem.

3.4 Replica trick

Before moving on to the manifestation of entanglement entropy in gravity, we present a a framework to compute entanglement entropies from a path integral. This is done by the *replica trick*. We will first illustrate it in the simplest possible example.

Consider for concreteness the quantum field theory of a free scalar field in $1+1$ dimensions, defined through a Euclidean action $S[\phi(x)]$. The details of this Lagrangian will not be important at all. We will first develop a framework to understand the reduced density matrix of a spatial region (in this case an interval) A from path integrals involving the Euclidean action.

We first consider a simpler problem. In quantum mechanics where the dynamical degree of freedom is typically a particle position x, we are used to the idea that for every state $|\psi\rangle$ there is a wave-function defined as $\psi(x) = \langle x|\psi\rangle$. In quantum field theory the dynamical degrees of freedom are instead space-dependent fields $\phi(x)$: thus for every state $|\psi\rangle$ in the quantum field theory we have a wave-function*al*

$$\Psi[\phi(\vec{x})] = \langle \phi(\vec{x}) | \psi \rangle \tag{3.24}$$

This wave-functional is a map from the space of fields $\phi(\vec{x})$ to the complex numbers. In quantum field theory this is a somewhat more cumbersome and consequently less familiar object than in quantum mechanics, but it will be useful for our purposes.

Let us now consider the vacuum of our quantum field theory. How do we construct the ground-state wave-functional $\Psi_0[\phi(\vec{x})]$? This can be done by performing the following path integral:

$$\Psi_0[\bar{\phi}(\vec{x})] = \int [\mathscr{D}\phi]_{\phi(\tau=0,\vec{x})=\bar{\phi}(\vec{x})} \exp(-S[\phi]_{\tau<0}) \tag{3.25}$$

In other words, to determine the value wave-functional evaluated at a particular field configuration $\bar{\phi}(\vec{x})$, we should perform the Euclidean path integral over half of the Euclidean plane (i.e. that with $\tau < 0$), with the boundary condition that at $\tau = 0$ the dynamical field $\phi(\tau = 0, \vec{x}) = \bar{\phi}(\vec{x})$. The integration over all negative Euclidean times has the effect of suppressing all states that are not the vacuum, and thus the only state that ultimately contributes is the Euclidean vacuum in the quantum field theory.

We turn now to the reduced density matrix ρ_A for a spatial region A in the vacuum. In the wave-functional picture this is a map from the space of *two* fields (one for each index of the matrix) to the complex numbers. From its formal definition in terms of the trace, it is:

$$\rho_A(\phi_1(\vec{x}_A);\phi_2(\vec{x}_A)) = \int [\mathscr{D}\phi(\vec{x}_{\bar{A}})] \Psi_0[\phi_1(\vec{x}_A);\phi(\vec{x}_{\bar{A}})] \Psi_0^*[\phi_2(\vec{x}_A);\phi(\vec{x}_{\bar{A}})] \tag{3.26}$$

where we have introduced some notation: $\vec{x}_A$ means that $\vec{x} \in A$, the two arguments in $\Psi_0[\phi_1(\vec{x}_A);\phi(\vec{x}_{\bar{A}})]$ simply indicate that the wave-functional depends on fields both inside and outside A, and the path integral above is over all fields localized in $\bar{A}$, thus implementing the trace over all degrees of freedom outside A.

Now by using (3.25) to evaluate each wavefunctional factor in (3.26), we see that there is actually an elegant functional integral representation for the reduced density matrix. The path integration over all points outside A sews together the two path integrals (one from each wave-functional factor) to result in

$$\rho(\phi_1(\vec{x}_A);\phi_2(\vec{x}_A)) = \int [\mathscr{D}\phi]_{(\phi_1,\phi_2)} \exp(-S[\phi]) \tag{3.27}$$

where now we integrate over the entire Euclidean plane but with a cut made along the interval A, with the following boundary conditions along the cut:

$$\phi(\tau = 0^+, \vec{x}_A) = \phi_1(\vec{x}_A) \qquad \phi(\tau = 0^-, \vec{x}_A) = \phi_2(\vec{x}_A) \tag{3.28}$$

Finally, from here we may evaluate the Renyi entropy (2.7):

$$S_n = -\frac{1}{n-1} \log \mathrm{Tr}(\rho^n) \ . \tag{3.29}$$

Now the n-fold product of ρ^n takes the form in terms of functional integrals:

$$\mathrm{Tr}(\rho^n) \sim \int [\mathscr{D}\phi_{1,2\cdots n}(\vec{x}_A)] \rho(\phi_1(x_A);\phi_2(x_A)) \rho(\phi_2(x_A);\phi_3(x_A) \cdots \rho(\phi_n(x_A);\phi_1(x_A)) \tag{3.30}$$

This has the effect of gluing together n different copies of the original manifold (i.e. the "replicas") along the cut A; thus all we need to do to compute the Renyi entropy is evaluate the partition function on a complicated manifold that we will call $\mathcal{M}_n$, as shown in Figure 5. Now, *if* we can do this analytically as a function of n, then we can analytically continue $n \to 1$ to obtain the entanglement entropy:

$$S_A = \lim_{n\to 1} S_n = -\lim_{n\to 1} \frac{1}{n-1} \log Z[\mathcal{M}_n] \tag{3.31}$$

This may seem like a somewhat suspicious set of manipulations: the difficulty in justifying the analytic continuation earns this procedure the disreputable name replica *trick* rather than (for example) replica *method*. Nevertheless, as far as this author is aware, whenever this procedure can be implemented in an entanglement context it gives trustworthy results. It should also be clear that the restriction to a single scalar field was only for notational convenience, and the final result (3.31) applies in full generality.

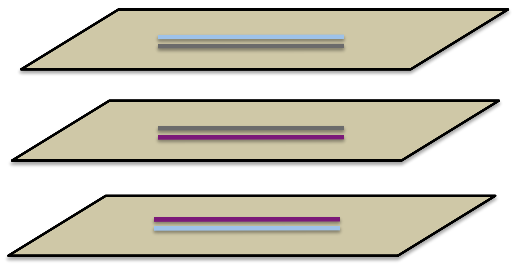

Figure 5: Schematic view of manifold $\mathcal{M}_n$ for $n = 3$. Color coding indicates how the edges of consecutive sheets of the manifold are glued together.

Exercises

1. Find the analog of (3.5) for the case of (1): an interval in a 2d CFT at finite temperature β^{-1} but on an infinite spatial line, and (2): an interval in the vacuum of a 2d CFT compactified on a circle of circumference 2π. Use it to derive (3.12) and (3.14).
2. Find the analog of (3.5) for a sphere in the vacuum of a a CFT$_d$ with $d > 2$. Why can we not use the formula we find to derive a general formula for the entanglement entropy of a sphere in CFT$_d$?

4. Entanglement entropy in gravity

Having given a whirlwind tour of some results in entanglement entropy in field theory that this particular author feels are particularly interesting, we now move to a new topic: entanglement entropy in holography.

Holographic duality is the surprising fact that certain quantum field theories are precisely equivalent to certain theories of quantum gravity that live in one higher dimension [25]. This is both conceptually profound and practically useful: it is conceptually profound because it promises to allow the somewhat amorphous set of ideas that is quantum gravity to be precisely defined in terms of much more well-understood objects, i.e. quantum field theories. It is practically useful because it is a weak/strong duality: when the quantum field theory is strongly coupled, the dual gravitational theory is weakly coupled and classical, effectively mapping strongly correlated quantum field theory problems to exercises in classical geometry.

There are many excellent reviews on AdS/CFT (see e.g. [26–29]). In what follows we will assume some basic familiarity with the subject and do not review how to do basic AdS/CFT computations.

We do briefly re-emphasize the philosophy. Essentially one considers a gravitational theory defined on an asymptotically AdS space: by performing measurements or operations at the boundary of AdS, one can compute field-theory observables. The details of course depend on what one wants to compute. For example, if we are interested in studying a field theory partition function Z evaluated on a d-dimensional Euclidean manifold $\mathscr{M}$, then the central formula of AdS/CFT might be summarized as

$$Z_{CFT}[\mathscr{M}_d] = Z_{\text{string}}[g_{d+1}] \to \exp\left(-S_{\text{gravity}}[g_{d+1}]\right) \tag{4.1}$$

where g_{d+1} is a $d+1$-dimensional manifold whose boundary (defined in the sense of AdS/CFT) is $\mathscr{M}_d$. Here Z_{string} is a somewhat metaphysical object that is the "string theory partition function", which reduces under favorable circumstances to a gravity partition function that can sometimes be approximated by an on-shell action. Manipulations of this and similar formulas allow one to compute boundary theory correlation functions, thermodynamics, and entanglement entropy.

4.1 Ryu-Takayanagi: minimal areas and entanglement

In fact, it turns out that there is a beautifully simple formula due to Ryu and Takayanagi [30,31] for the entanglement entropy $S(A)$ associated with a spatial region A in the field theory. We will prove it in the next subsection: for now, we simply state that the entanglement entropy (in a theory of Einstein gravity) is simply given by the *area of the bulk minimal surface m_A that hangs down into the bulk and ends on the boundary of A*, as shown in Figure 6.

$$S_A = \frac{1}{4G_N}\text{Area}(m_A), \tag{4.2}$$

where G_N is the effective bulk Newton's constant. In this formula it should be understood that we should always pick m_A so that it can be continuously deformed to the boundary region A: this constraint is called the *homology condition* and will be important in what follows.

Note that this prescription relates two very primitive objects on the two sides of the duality: on one side, we have entanglement entropy of a spatial region, which requires only the structure of quantum mechanics and spatial locality to define; on the other, we have minimal areas, or geometry, one of the most basic concepts in any theory of gravity. The existence of this relation suggests that the entanglement structure of the dual field theory is the correct way to organize our understanding of holography. This idea has been at the heart of much recent work into "constructing spacetime

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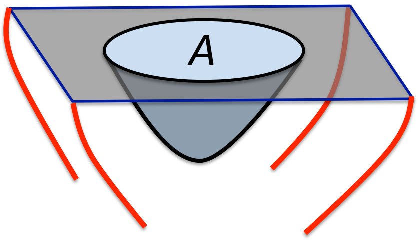

Figure 6: The Ryu-Takayanagi prescription: a minimal surface hangs down into the bulk and measures entanglement entropy.

from entanglement". I will not attempt to give a complete list of references here, but foundational papers on this subject include [32, 33].

This formula also makes it somewhat easier to *visualize* entanglement, by relating a somewhat abstract quantum mechanical object to something as intuitive as a minimal area. We now discuss some simple examples, to illustrate how the 2d CFT results discussed in Section 3.2 are geometrized in AdS_3/CFT_2 holography.

In AdS_3 holography we have in the bulk a solution to Einstein gravity with a negative cosmological constant

$$S = -\frac{1}{4\pi G_N}\int d^3x\sqrt{g}\left(R - \frac{2}{\ell^2}\right) \tag{4.3}$$

where the AdS_3 radius ℓ is related to the central charge of the dual field theory by [34]

$$c = \frac{3\ell}{2G_N} \ . \tag{4.4}$$

We begin by studying the dual field theory defined on the plane $\mathbb{R}^2$. In this case the bulk spacetime is given by AdS_3 in Poincaré coordinates:

$$ds^2 = \ell^2\left(r^2\left(-dt^2 + dx^2\right) + \frac{dr^2}{r^2}\right), \tag{4.5}$$

where the boundary is at $r \to \infty$. In $2+1$ bulk dimensions, minimal "surfaces" are actually bulk geodesics: thus consider a geodesic with its endpoints separated by a distance L in x at the AdS boundary, which we take to be at some large value of $r = r_\Lambda$. The computation of its proper length is a simple exercise in geometry, after which we find the entanglement entropy to be

$$S(L) = \frac{\ell}{2G_N}\log\left(r_\Lambda L\right) = \frac{c}{3}\log\left(\frac{L}{\varepsilon}\right), \tag{4.6}$$

where in the second equality we have used (4.4) to express the bulk AdS radius in terms of the central charge, as well as identifying the maximum value of r_Λ with the UV cutoff ε^{-1}. We see that the UV divergence of the entanglement entropy in field theory arises geometrically from the fact that the AdS boundary is infinitely far away from all points in the bulk. In fact, it is easy to see that even in higher dimensions, the divergence from arising from the fact that the boundary is infinitely far away will always result in an area law of the form (3.1).

The answer grows logarithmically with the separation L at the boundary because as we make the interval distance bigger, the minimal geodesic hangs deeper and deeper into the bulk, and AdS is *smaller* in the deep bulk, as is clear from (4.5). Thus the answer grows more slowly than linear in L.

Consider now heating the field theory up to a finite temperature T. The finite temperature state is dual to the planar BTZ black hole, which has metric

$$ds^2 = \ell^2 \left(-dt^2(r^2 - r_+^2) + r^2 dx^2 + \frac{dr^2}{(r^2 - r_+^2)} \right) . \tag{4.7}$$

Here r_+ is the location of the black hole horizon, and the field theory temperature is

$$T = \frac{r_+}{2\pi} \tag{4.8}$$

From a straightforward computation we can verify that the minimal geodesic distance is indeed given by the expected expression (3.12). Note that as we make the interval length longer and longer, eventually the geodesic dips deeper into the bulk, approaching the horizon at $r = r_+$. However, it never penetrates the horizon; instead it hangs just outside it, essentially measuring the proper distance along the horizon. This is simply the thermal entropy density of the black hole multiplied by L , via the usual Bekenstein-Hawking formula. Thus we see that the volume law of the entanglement entropy (3.13) appears in a very natural way.

Finally, consider studying the dual field theory on a cylinder of circumference 2π. The appropriate bulk spacetime is now AdS in global coordinates:

$$ds^2 = \ell^2 \left(-dt^2(r^2+1) + r^2 dx^2 + \frac{dr^2}{r^2+1} \right), \tag{4.9}$$

and the geodesic distance is again given by (3.14). Note from Figure 7 that the assertion that $S_A = S_{\bar{A}}$ takes on a simple geometric form: if there is no *obstruction* in the interior of the spacetime, then in computing the entanglement entropy of A or of $\bar{A}$, we are computing the length of the same minimal area (i.e. $m_A = m_{\bar{A}}$).

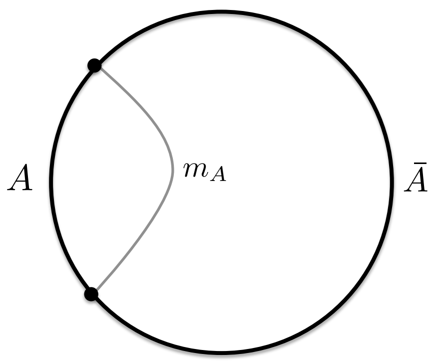

Figure 7: The minimal geodesic used to compute the entanglement entropy of an interval A, *and* its complement $\bar{A}$. Note that $m_A = m_{\bar{A}}$.

What if there is an obstruction, e.g. a black hole in the interior? Now the homology condition plays a role: we see from Figure 8 that the bulk minimal surfaces m_A and $m_{\overline{A}}$ corresponding to A and to $\overline{A}$ are different, as we cannot deform the minimal surface through the black hole horizon. Thus now generically $S_A \neq S_{\overline{A}}$. This makes sense, as the state dual to the black hole is the thermal density matrix, which is a *mixed* state.

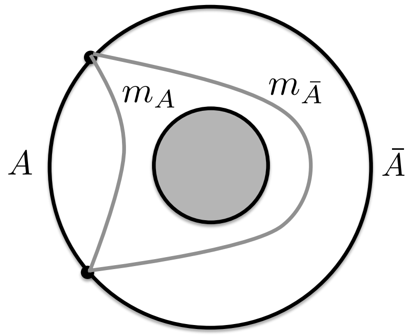

Figure 8: Minimal geodesics used to compute the entanglement entropy of an interval A, and its complement $\overline{A}$ in the presence of a black hole. Note $m_A \neq m_{\overline{A}}$.

Let us turn now to the strong sub-additivity condition, written in the geometric form (3.18):

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B) \tag{4.10}$$

As mentioned earlier, the proof of this inequality for a general quantum mechanical system is fairly intricate. However, in a holographic theory it is almost an embarrassingly simple argument [35]. We present the pictorial proof in Figure 9.

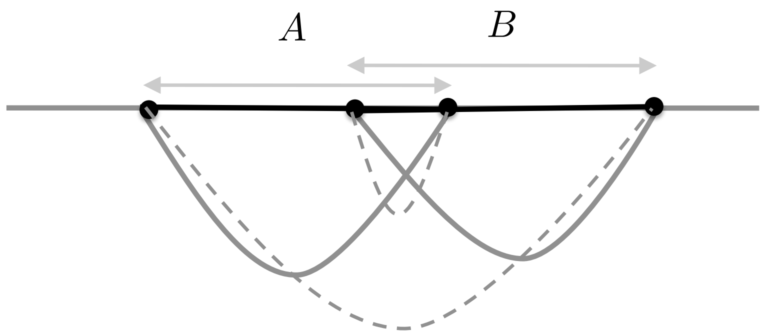

Figure 9: Pictorial proof of strong sub-additivity. Solid grey lines represent $S(A) + S(B)$. Dashed grey lines represent $S(A \cup B) + S(A \cap B)$. Now continuously deform the dashed lines into the solid ones, thus making them touch at a single point and creating a kink there without breaking the lines. The length of the deformed dashed lines is now equal to that of the solid lines; but as the *original* dashed grey lines were minimal, this procedure will necessarily increase their length. We conclude that $S(A) + S(B) \geq S(A \cup B) + S(A \cap B)$.

4.2 Sketch of proof using replica trick

In this section we present a proof of the Ryu-Takayanagi prescription, following [36]. At the moment the prescription can be proven only for a certain set of states – those that can be obtained from a Euclidean path integral. Importantly, this excludes all time-dependent states, which are different in important ways that will be discussed later on.

We first discuss the basic idea. Recall from Section 3.4 that the replica trick allows us to express the entanglement entropy of a region in field theory in terms of a partition function evaluated on an n-sheeted d-manifold $\mathscr{M}_n$, (3.31)

$$S_A = \lim_{n\to 1} S_n = -\lim_{n\to 1}\frac{1}{n-1}\left(\log Z[\mathscr{M}_n] - n\log Z[\mathscr{M}_1]\right) \tag{4.11}$$

where we have included an extra term to account for the possibility that $Z[\mathscr{M}_1]$ (i.e. the original partition function[3]) has not been normalized to 1. Now to compute $Z[\mathscr{M}_n]$ from AdS/CFT, we need to find a $d+1$-dimensional bulk geometry $g_{(n)}$ that is a solution to Einstein's equations and asymptotes at the AdS boundary to the d-manifold $\mathscr{M}_n$. Via the AdS/CFT correspondence we then have

$$Z[\mathscr{M}_n] = \exp\left(-S[g_{(n)}]\right), \tag{4.12}$$

with $S[g_{(n)}]$ the Einstein-Hilbert action, evaluated here on the bulk geometry. If we can find this action for all n we can then analytically continue to $n=1$ to determine the entanglement entropy.

Given that $\mathscr{M}_n$ is very complicated, implementing this procedure for all n may seem impossibly complicated – remarkably, however, it can be explicitly performed in some cases, including AdS_3/CFT_2 [37,38] and spheres in the vacuum of a CFT_d for any d [39]. We will take a different route here: we will instead analytically continue $n\to 1$ directly in the bulk and attempting to physically interpret the resulting geometry. It was argued in [36] that in the $n\to 1$ limit the essential data needed to characterize this bulk geometry is really simply that of a minimal surface m_A that probes the original $n=1$ geometry.

To understand this, we follow [40] in a slight reformulation of the original argument. Consider first the bulk geometry $g_{(n)}$ for integer n. This geometry is regular in the interior. It also admits a natural $\mathbb{Z}_n$ action: this $\mathbb{Z}_n$ cyclically permutes the n sheets of $\mathscr{M}_n$ at the boundary, and can be extended naturally into the bulk. Importantly, the $\mathbb{Z}_n$ has fixed points: at the boundary the fixed points of this action are the entangling surface ∂A, and this locus of fixed points will be extended into the bulk to constitute a codimension 2 surface that we call m_A.

Consider now the *orbifold* geometry $\hat{g}_{(n)} \equiv g_n/\mathbb{Z}_n$, formed by identifying points that are related by the $\mathbb{Z}_n$ action. As $g_{(n)}$ *was* regular in the interior, its orbifold will be singular at the fixed points of the $\mathbb{Z}_n$ action, i.e. on the surface m_A, where it will have a conical deficit of angle $2\pi\left(1-\frac{1}{n}\right)$.

Assuming the replica symmetry is unbroken, the action of $g_{(n)}$ will be

$$S[g_{(n)}] = nS[\hat{g}_{(n)}]\ . \tag{4.13}$$

Thus we can reformulate (4.11) in the holographic context as

$$S_A = \lim_{n\to 1}\frac{n}{n-1}\left(S[\hat{g}_{(n)}] - S[g_{(1)}]\right) \tag{4.14}$$

[3]In other words, we are allowing for the possibility that the original density matrix ρ_A is not correctly normalized

So far everything is well-defined, as n is still integer.

We now take the bold step of analytically continuing $n \to 1$ in the above formula and keeping track only of the information to first order in $n-1$. We assume that in this treacherous domain we may still *define* $g_{(n)}$ as a geometry containing a codimension-2 surface m_A, along which we have a conical deficit of angle $2\pi\left(1-\frac{1}{n}\right) \approx 2\pi\varepsilon$ with $\varepsilon = n-1$. We then need only evaluate:

$$S_A = \partial_\varepsilon(S[\hat{g}_{(n)}]) \tag{4.15}$$

at $\varepsilon = 0$, i.e. the linear part of the variation of the action with the opening angle ε.

We now note that it may seem dangerous to evaluate the action of a singular geometry $\hat{g}_{(n)}$: how do we treat the singularity? Should we supplement the action with boundary terms? If we want (4.13) to hold we should not add any further boundary terms to the action, as $g_{(n)}$ was smooth and had no boundary terms there. Thus we should simply evaluate the original Einstein-Hilbert action on a conical geometry, integrating up to the site of the defect. For concreteness, we put down coordinates near the site of the defect as

$$ds^2 = \rho^{-2\varepsilon}\left(d\rho^2 + \rho^2 d\tau^2\right) + (g_{ij} + K_{aij}x^a)\,dy^i dy^j + \cdots \tag{4.16}$$

where the defect is at $\rho = 0$, (ρ, τ) and (x^1, x^2) parametrize the same two-dimensional space orthogonal to m_A, and y^i parametrize the directions along m_A, and we have ignored higher order terms in x^a. Note that the K_{aij} are precisely the extrinsic curvatures of m_A. The geometry is singular at $\rho = 0$ unless $\varepsilon = 0$.

There are multiple ways to evaluate the action: their equivalence is spelled out in [36, 40]. An aesthetically pleasing method is the following: consider introducing a small artificial cutoff in ρ at $\rho = a$, and for $\rho < a$ we close off the inside of the geometry with a *smooth* geometry that smoothens out the conical deficit. The full action is

$$S_{tot} = S(\rho < a) + S(\rho > a) \tag{4.17}$$

We are interested in computing $\lim_{a\to 0}\partial_\varepsilon S(\rho > a)\big|_{\varepsilon=0}$. Now at $\varepsilon = 0$, S_{tot} is the action of a smooth solution to the Einstein equations with no boundaries: thus its variation with respect to any bulk fields (and in particular with respect to the 1-parameter variation of bulk fields induced by a chance of ε) vanishes. We then conclude that

$$\partial_\varepsilon S(\rho > a)\big|_{\varepsilon=0} = -\partial_\varepsilon S(\rho < a)\big|_{\varepsilon=0} \tag{4.18}$$

However the action $S_(\rho < a)$ is now just the action of a regulated cone in Einstein gravity. This is easy to compute (see e.g. [41]) and the formula is well-known. If the Einstein-Hilbert action is

$$S = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{g}R \tag{4.19}$$

then the action of a cone with opening angle ε along a surface m_A is precisely proportional to this opening angle:

$$S_{cone} = -\frac{\varepsilon}{4G_N}\text{Area}(m_A)\ . \tag{4.20}$$

Using this to evaluate $\partial_\varepsilon S(\rho < a)$ and then inserting the answer into (4.15), we conclude that the holographic entanglement entropy is precisely:

$$S_A = \frac{1}{4G_N}\text{Area}(m_A) \tag{4.21}$$

We are not yet entirely done: while we have shown that the entanglement entropy is given by the area of m_A, we have not yet provided an algorithm to determine m_A itself. This can be done by studying the Einstein equations near $\rho = 0$ in the coordinates (4.16): one finds that they possess intractable singularities near $\rho \to 0$ unless the traces of the extrinsic curvatures K_{aij} vanish:

$$K_{ai}{}^{i} = 0 \tag{4.22}$$

However, this is precisely the condition for the surface m_A to be minimal. This completes the proof.

It should be clear from the derivation that this proof relies heavily on the idea of the Euclidean path integral, and thus only works for states that can be prepared via such a path integral. Nevertheless, this is sufficient for many of the states that we are interested in for holographic applications.

4.3 Extensions

Here we discuss very briefly some extensions of the simplest situations discussed above.

4.3.1 Time dependence

Everything discussed above essentially applies only to static situations with no time dependence. Nevertheless, it is clearly of great interest to understand entanglement entropy in time-dependent situations (e.g. the formation of a black hole, which is dual to thermalization in the boundary field theory). A natural covariant extension of the Ryu-Takayanagi formula to the time-dependent case has been proposed [42]

$$S_{EE} = \frac{1}{4G_N}\text{Area}(e_A), \tag{4.23}$$

where e_A is now the covariant bulk *extremal* surface ending on the boundary at ∂A. At the moment there is no proof for this formula, but it has been applied in many settings and there is a widespread expectation that it is correct.

While this seems very natural, it is important to emphasize that it is conceptually rather different: for example, in a Lorentzian setting, there is actually no notion of a *minimal* surface, as wigglings of the surface in the time direction can arbitrarily reduce its covariant area. Thus the object appearing above is *extremal* rather than *minimal*. This has consequences: for example, the proof of strong subadditivity for the covariant entanglement entropy [43] is quite intricate and not at all a kinematic triviality like the one for the static case discussed above. Importantly, one finds SSA only when the bulk geometry satisfies a null curvature condition (meaning that on-shell the bulk matter supporting the geometry should satisfy a null energy condition).

4.3.2 Higher derivative corrections to gravity

The discussion so far has applied only to the simplest limit of holography, that where we consider only classical gravity in the bulk, and furthermore only the leading term in an expansion of the bulk action in powers of derivatives (i.e. only the Ricci scalar term in the gravitational action). It is helpful to remind ourselves what this corresponds to on the field theory side. For the simplest example of the duality between Type IIB string theory on $\mathrm{AdS}_5 \times S^5$, and $\mathcal{N} = 4$ Super Yang-Mills, we have the following relations between bulk and boundary quantities:

$$\lambda^{\frac{1}{4}} = \frac{\ell}{\ell_s} \qquad g_s = \frac{\lambda}{4\pi N} \tag{4.24}$$

where λ and N are respectively the field theory t'Hooft coupling and rank of the gauge group, ℓ is the bulk AdS radius, ℓ_s is the string length, and g_s is the bulk string coupling. Thus the $N \to \infty$ limit corresponds to turning off the bulk string coupling and thus discussing a classical theory in the bulk. Further taking $\lambda \to \infty$ corresponds to making the classical theory one of gravity and not of strings.

Thus moving away from the strict $\lambda \to \infty$ limit corresponds to allowing higher-derivative corrections in the bulk, that are suppressed by powers of the string length, e.g. schematically

$$S = \frac{1}{4\pi G_N} \int d^5x\sqrt{-g}\left(R + \ell_s^6 R^4 + \cdots\right) \tag{4.25}$$

where R^4 refers to a term involving four powers of the Riemann tensor, etc. Their precise form can be determined from string theory if desired: thus it is of intrinsic interest to understand how entanglement entropy behaves in the presence of such corrections to the bulk effective action

This problem has been studied by [40, 44] by implementing the algorithm of [36] to higher-derivative actions, resulting in a corrected formula for entanglement entropy that includes the contributions of higher order terms. Their result applies to any diffeomorphism-invariant theory of gravity and can be written schematically as

$$S_{EE} = \int_{m_A} d^{d-2}x\sqrt{h}\left(\frac{\delta\mathcal{L}}{\delta R} + \frac{\delta^2\mathcal{L}}{\delta R\delta R} K\cdot K\right) \tag{4.26}$$

Here the answer is expressed in terms of derivatives of the bulk Lagrangian $\mathcal{L}$ with respect to the bulk Riemann tensor, and K refers to the extrinsic curvature of the bulk "minimal" surface, which we continue to call m_A (and where the condition to be satisfied by this surface is now modified, and it is thus no longer precisely minimal). For Einstein gravity the bulk action is linear in the Riemann tensor, and only the first term contributes, resulting in the usual area formula.

It is interesting to note that the functional to be evaluated on m_A is *not* equivalent to the functional appearing in Wald's formula for the *thermal* entropy of a black hole in a higher derivative theory of gravity [45, 46]. The difference is the second term involving the extrinsic curvatures, which vanish when the formula is evaluated on the event horizon of a black hole.

Finally, we should point out that there is an important class of higher-derivative corrections that are not captured by the above formulas: those that involve bulk gravitational Chern-Simons terms, which are only diffeomorphism-invariant up to a total derivative and so cannot be written as functions of a bulk Riemann tensor. The presence of such terms in a bulk gravitational action indicate that the dual field theory suffers from *gravitational anomalies*, i.e. a breakdown of stress-energy conservation [47, 48]. The entanglement entropy in such theories possesses interesting additional structure, and has been studied holographically in [49,50] and field-theoretically in [51–54].

4.3.3 Quantum bulk corrections

We turn now to including *quantum* effects in the bulk gravitational theory: from (4.24) we see that this corresponds to studying the dual field theory at finite N. Thus one now needs to include effects arising from the quantum fluctuations associated to the dynamical fields in the bulk, viewed as a low-energy effective theory.

There are multiple ways to study this quantity: for example, if the form of the bulk manifold $g_{(n)}$ that asymptotes to an n-sheeted replica geometry $\mathscr{M}_n$ is known, then it is possible to directly compute functional determinants in the bulk and obtain explicit answers for the Renyi entropies, which can then (hopefully) be analytically continued to $n = 1$ to obtain the entanglement entropy. This program has been initiated by [55] in the case of AdS_3/CFT_2.

However, there is also an expectation for the general form of the one-loop correction to the entanglement entropy itself. [56] argue that the answer takes a very elegant form. Note first that the bulk minimal surface m_A divides the bulk into two pieces; thus one can consider the bulk region B_A bounded by m_A and the boundary region A. One can thus consider the *bulk* entanglement entropy of dynamical fields in the bulk $S_{\mathrm{EE,bulk}(B_A)}$. The key correction to the boundary entanglement entropy is argued to be this bulk entanglement itself, i.e.

$$S_A = \frac{1}{4G_N}\mathrm{Area}(m_A) + S_{\mathrm{EE,bulk}}(B_A) + K_{\mathrm{local}}(m_A) \tag{4.27}$$

Here $K_{\mathrm{local}}(m_A)$ is associated with the renormalization of *bulk* dynamical fields and is a set of local terms evaluated on m_A whose form is discussed in [56].

It is intriguing to note that bulk geometric areas and bulk entanglement together conspire to give boundary entanglement, and this author feels that this formula is likely to have a deep significance.

Exercises

1. Re-derive in a holographic context the universal CFT findings (3.12) and (3.14) by studying appropriate geodesics on the BTZ black hole (4.7) and global AdS_3 (4.9).

2. Consider the BTZ black hole but now with a compactified spatial coordinate $x \sim x + 2\pi$. Consider the entanglement entropy of an interval of length $L < 2\pi$. What happens as the length L is varied from very small lengths up to 2π? Discuss the relationship between your findings and the Araki-Lieb inequality (2.10). The physics that you have just discovered is discussed in [57].

3. Study the Einstein equations near the tip of a singular cone given by (4.16) and verify that regularity of the resulting equations requires the vanishing of the extrinsic curvatures (4.22).

5. Conclusion

We have reached the end of our whirlwind tour of the theory and applications of entanglement entropy.

It is worth emphasizing at this point that entanglement entropy is – while undoubtedly great fun in various ways – merely a novel way to understand and visualize the structure of quantum correlations. While it is likely to give us a new way to organize physics, this author sometimes feels that it should perhaps be thought of as being something like "energy". The energy of a state is something that is often useful to calculate, and everyone should know how to do it. Separating physics into low-energy and high-energy is a very fruitful way to think about a wide variety of physical systems, and all physicists should develop intuition as to how to do this. Yet nevertheless very few physicists would say that they study "energy" itself: rather the virtue of "energy" is that it tells us how to organize our thinking.

It seems likely that "entanglement" will play a similar role in the physics of the 21st century, in telling us how to organize and slice the incredible complexity that is present in quantum mechanical systems.

It remains, of course, up to the readers of these notes to figure out how to use it to do so.

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References

[1] E. H. Lieb and M. B. Ruskai, *A fundamental property of quantum-mechanical entropy*, *Phys. Rev. Lett.* **30** (Mar, 1973) 434–436.

[2] H. Araki and E. H. Lieb, *Entropy inequalities*, in *Inequalities*, pp. 47–57. Springer, 2002.

[3] E. H. Lieb and M. B. Ruskai, *Proof of the strong subadditivity of quantum-mechanical entropy*, in *Inequalities*, pp. 63–66. Springer, 2002.

[4] M. Srednicki, *Entropy and area*, *Phys. Rev. Lett.* **71** (1993) 666–669, [hep-th/9303048].

[5] R. D. Sorkin, *1983 paper on entanglement entropy: "On the Entropy of the Vacuum outside a Horizon"*, arXiv:1402.3589.

[6] J. Eisert, M. Cramer, and M. B. Plenio, *Area laws for the entanglement entropy - a review*, *Rev. Mod. Phys.* **82** (2010) 277–306, [arXiv:0808.3773].

[7] D. V. Fursaev, A. Patrushev, and S. N. Solodukhin, *Distributional Geometry of Squashed Cones*, *Phys. Rev.* **D88** (2013), no. 4 044054, [arXiv:1306.4000].

[8] S. N. Solodukhin, *Entanglement entropy, conformal invariance and extrinsic geometry*, *Phys. Lett.* **B665** (2008) 305–309, [arXiv:0802.3117].

[9] C. Holzhey, F. Larsen, and F. Wilczek, *Geometric and renormalized entropy in conformal field theory*, *Nucl. Phys.* **B424** (1994) 443–467, [hep-th/9403108].

[10] P. Calabrese and J. L. Cardy, *Entanglement entropy and quantum field theory*, *J.Stat.Mech.* **0406** (2004) P06002, [hep-th/0405152].

[11] P. Calabrese and J. L. Cardy, *Entanglement entropy and quantum field theory: A Non-technical introduction*, *Int.J.Quant.Inf.* **4** (2006) 429, [quant-ph/0505193].

[12] H. Casini, M. Huerta, and R. C. Myers, *Towards a derivation of holographic entanglement entropy*, *JHEP* **05** (2011) 036, [arXiv:1102.0440].

[13] H. W. J. Bloete, J. L. Cardy, and M. P. Nightingale, *Conformal Invariance, the Central Charge, and Universal Finite Size Amplitudes at Criticality*, *Phys. Rev. Lett.* **56** (1986) 742–745.

[14] S. Detournay, T. Hartman, and D. M. Hofman, *Warped Conformal Field Theory*, *Phys. Rev.* **D86** (2012) 124018, [arXiv:1210.0539].

[15] A. B. Zamolodchikov, *Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory*, *JETP Lett.* **43** (1986) 730–732. [Pisma Zh. Eksp. Teor. Fiz.43,565(1986)].

[16] H. Casini and M. Huerta, *A Finite entanglement entropy and the c-theorem*, *Phys. Lett.* **B600** (2004) 142–150, [hep-th/0405111].

[17] H. Casini and M. Huerta, *A c-theorem for the entanglement entropy*, *J. Phys.* **A40** (2007) 7031–7036, [cond-mat/0610375].

[18] H. Liu and M. Mezei, *A Refinement of entanglement entropy and the number of degrees of freedom*, *JHEP* **04** (2013) 162, [arXiv:1202.2070].

[19] D. L. Jafferis, *The Exact Superconformal R-Symmetry Extremizes Z*, *JHEP* **05** (2012) 159, [arXiv:1012.3210].

[20] I. R. Klebanov, S. S. Pufu, and B. R. Safdi, *F-Theorem without Supersymmetry*, *JHEP* **10** (2011) 038, [arXiv:1105.4598].

[21] D. L. Jafferis, I. R. Klebanov, S. S. Pufu, and B. R. Safdi, *Towards the F-Theorem: N=2 Field Theories on the Three-Sphere*, *JHEP* **06** (2011) 102, [arXiv:1103.1181].

[22] H. Casini and M. Huerta, *On the RG running of the entanglement entropy of a circle*, *Phys. Rev.* **D85** (2012) 125016, [arXiv:1202.5650].

[23] H. Casini, M. Huerta, R. C. Myers, and A. Yale, *Mutual information and the F-theorem*, *JHEP* **10** (2015) 003, [arXiv:1506.0619].

[24] Z. Komargodski and A. Schwimmer, *On Renormalization Group Flows in Four Dimensions*, *JHEP* **12** (2011) 099, [arXiv:1107.3987].

[25] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113–1133, [hep-th/9711200]. [Adv. Theor. Math. Phys.2,231(1998)].

[26] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183–386, [hep-th/9905111].

[27] J. Polchinski, *Introduction to Gauge/Gravity Duality*, in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2010). String Theory and Its Applications: From meV to the Planck Scale: Boulder, Colorado, USA, June 1-25, 2010*, pp. 3–46, 2010. arXiv:1010.6134.

[28] J. McGreevy, *Holographic duality with a view toward many-body physics*, *Adv. High Energy Phys.* **2010** (2010) 723105, [arXiv:0909.0518].

[29] S. A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, *Class. Quant. Grav.* **26** (2009) 224002, [arXiv:0903.3246].

[30] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602, [hep-th/0603001].

[31] S. Ryu and T. Takayanagi, *Aspects of Holographic Entanglement Entropy*, *JHEP* **08** (2006) 045, [hep-th/0605073].

[32] B. Swingle, *Entanglement Renormalization and Holography*, *Phys. Rev.* **D86** (2012) 065007, [arXiv:0905.1317].

[33] M. Van Raamsdonk, *Building up spacetime with quantum entanglement*, *Gen. Rel. Grav.* **42** (2010) 2323–2329, [arXiv:1005.3035]. [Int. J. Mod. Phys.D19,2429(2010)].

[34] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, *Commun. Math. Phys.* **104** (1986) 207–226.

[35] M. Headrick and T. Takayanagi, *A Holographic proof of the strong subadditivity of entanglement entropy*, *Phys. Rev.* **D76** (2007) 106013, [arXiv:0704.3719].

[36] A. Lewkowycz and J. Maldacena, *Generalized gravitational entropy*, *JHEP* **08** (2013) 090, [arXiv:1304.4926].

[37] T. Hartman, *Entanglement Entropy at Large Central Charge*, arXiv:1303.6955.

[38] T. Faulkner, *The Entanglement Renyi Entropies of Disjoint Intervals in AdS/CFT*, arXiv:1303.7221.

[39] L.-Y. Hung, R. C. Myers, M. Smolkin, and A. Yale, *Holographic Calculations of Renyi Entropy*, *JHEP* **12** (2011) 047, [arXiv:1110.1084].

[40] X. Dong, *Holographic Entanglement Entropy for General Higher Derivative Gravity*, *JHEP* **01** (2014) 044, [arXiv:1310.5713].

[41] D. V. Fursaev and S. N. Solodukhin, *On the description of the Riemannian geometry in the presence of conical defects*, *Phys. Rev.* **D52** (1995) 2133–2143, [hep-th/9501127].

[42] V. E. Hubeny, M. Rangamani, and T. Takayanagi, *A Covariant holographic entanglement entropy proposal*, *JHEP* **07** (2007) 062, [arXiv:0705.0016].

[43] A. C. Wall, *Maximin Surfaces, and the Strong Subadditivity of the Covariant Holographic Entanglement Entropy*, *Class. Quant. Grav.* **31** (2014), no. 22 225007, [arXiv:1211.3494].

[44] J. Camps, *Generalized entropy and higher derivative Gravity*, *JHEP* **03** (2014) 070, [arXiv:1310.6659].

[45] R. M. Wald, *Black hole entropy is the Noether charge*, *Phys. Rev.* **D48** (1993) 3427–3431, [gr-qc/9307038].

[46] V. Iyer and R. M. Wald, *A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes*, *Phys. Rev.* **D52** (1995) 4430–4439, [gr-qc/9503052].

[47] P. Kraus and F. Larsen, *Holographic gravitational anomalies*, *JHEP* **01** (2006) 022, [hep-th/0508218].

[48] S. N. Solodukhin, *Holography with gravitational Chern-Simons*, *Phys. Rev.* **D74** (2006) 024015, [hep-th/0509148].

[49] A. Castro, S. Detournay, N. Iqbal, and E. Perlmutter, *Holographic entanglement entropy and gravitational anomalies*, *JHEP* **07** (2014) 114, [arXiv:1405.2792].

[50] T. Azeyanagi, R. Loganayagam, and G. S. Ng, *Holographic Entanglement for Chern-Simons Terms*, arXiv:1507.0229.

[51] A. C. Wall, *Testing the Generalized Second Law in 1+1 dimensional Conformal Vacua: An Argument for the Causal Horizon*, *Phys.Rev.* **D85** (2012) 024015, [arXiv:1105.3520].

[52] N. Iqbal and A. C. Wall, *Anomalies of the Entanglement Entropy in Chiral Theories*, arXiv:1509.0432.

[53] T. L. Hughes, R. G. Leigh, O. Parrikar, and S. T. Ramamurthy, *Entanglement Entropy & Anomaly Inflow*, arXiv:1509.0496.

[54] T. Nishioka and A. Yarom, *Anomalies and Entanglement Entropy*, arXiv:1509.0428.

[55] T. Barrella, X. Dong, S. A. Hartnoll, and V. L. Martin, *Holographic entanglement beyond classical gravity*, *JHEP* **09** (2013) 109, [arXiv:1306.4682].

[56] T. Faulkner, A. Lewkowycz, and J. Maldacena, *Quantum corrections to holographic entanglement entropy*, *JHEP* **11** (2013) 074, [arXiv:1307.2892].

[57] V. E. Hubeny, H. Maxfield, M. Rangamani, and E. Tonni, *Holographic entanglement plateaux*, *JHEP* **08** (2013) 092, [arXiv:1306.4004].