

Determination of V_{cb} and V_{ub} From Semileptonic Decays

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We review the methods employed in the determination of the CKM elements V_{cb} and V_{ub} . We focus on determinations that use semileptonic decays of B -mesons, both inclusively and through exclusive channels. We comment on anomalies seen in semileptonic decays with τ leptons, and the persistent discrepancies in inclusive vs. exclusive determinations.

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1. Introduction

A precise determination of the CKM elements V_{cb} and V_{ub} is of great interest and importance not only because they are 2 of the 18 parameters of the Standard Model (SM) but also because they are key in formulating precise tests of the model. For example, the unitarity triangle, that tests the extent to which the SM accounts for all observed CP violation, has its sides normalized to $|V_{cb}|$ and then $|V_{ub}|$ singles out a circle around the origin of the $\bar{\rho}-\bar{\eta}$ plane. This talk reviews the methods by which they are determined.

Since V_{cb} and V_{ub} appear in couplings of charged vector bosons to $V-A$ currents that transmute a b quark into a c or u quarks, the decay rates for hadrons with unit B -number are proportional to $|V_{cb}|^2$ or $|V_{ub}|^2$. Hence it is through measurements of these decay rates that these CKM elements are extracted from experiment. One can also consider single $|B|=1$ hadron production, but the theory and experiment of these processes is not at present competitive for a precise determination of these parameters. In fact, not all decays afford the same level of precision: semileptonic decays are much better understood theoretically than purely hadronic decays, and it is on these decays that we focus here. In fact, we will focus attention on decays of B -mesons; a separate contribution in these proceedings is dedicated to b -baryon decays.¹

We may classify broadly semileptonic B -decays by whether we concentrate on a specific decay mode, which we refer to as “Exclusive Decays”, versus measurements of semileptonic decays irrespective of the final hadronic state, or “Inclusive Decays.” While exclusive decay rates are smaller, they can be measured precisely in present day high luminosity colliders. The theory requires lattice input in the form of calculation of hadronic form factors. These can be computed over a limited range of invariant lepton pair mass, q^2 , and to compare with experimental data that spans the full range of available q^2 an extrapolation based on first principles, the “ z -expansion”, is used. Inclusive decays, on the other hand, can be computed analytically without recourse to lattice calculations. The method uses a double expansion in the inverse of the heavy b -quark mass, $1/m_b$, and in the QCD-coupling constant $\alpha_s(m_b)$, and relies on assuming the validity of quark-hadron duality over a very restricted region of phase space (and hence it is believed to be very accurate). We consider these two methods in turn.

2. Inclusive $B \rightarrow X_c \ell \nu$ decays

The theory of inclusive semileptonic b -hadron decays relies on a simultaneous application of the heavy quark expansion and an OPE [1]. Much like in the calculation of e^+e^- into hadrons, an integration over a kinematic variable is necessary to apply the OPE away from the physical region (by the magic of contour integration). Hence one obtains only $d\Gamma/dx$ for some suitably chosen kinematic variable x ; that is, one obtains only single differential decay rates. The result is an expansion in powers of Λ/m_b , with Λ being a hadronic scale that appears as a-priori unknown hadronic matrix elements. The lowest order of the expansion is free from unknown hadronic matrix elements and corresponds precisely to the perturbative decay rate of an unconfined b -quark, Γ_0 . Remarkably, there are no first order corrections, that is, the first correction to Γ_0 appears at order $(\Lambda/m_b)^2$. For B -meson decay this is a consequence of Luke’s theorem [2]; for b -baryons see [3, 4].

¹Talk by Svende Braun

m_b^{kin}	$\bar{m}_c(3\text{GeV})$	μ_π^2	ρ_D^3	μ_G^2	ρ_{LS}^3	$\text{BR}_{c\ell\nu}$	$10^3 V_{cb} $
4.553	0.987	0.465	0.170	0.332	-0.150	10.65	42.21
0.020	0.013	0.068	0.038	0.062	0.096	0.16	0.78

Table 1: Global fit to $B \rightarrow X_C \ell \nu$ from Ref. [17]. All parameters are expressed in (appropriate powers of) GeV. The first row gives central values and the second gives uncertainties.

The $1/m_b$ expansion introduces 2 unknown hadronic matrix elements at order $1/m_b^2$ and two more at order $1/m_b^3$ [5, 6, 7]. As pointed out in [1], moments of the single differential decay rate can be used to extract the hadronic matrix elements in the $1/m_b$ expansion; this is the basis of the moment analysis that is currently the preferred method for the determination of $|V_{cb}|$ [8, 9, 10, 11].

The calculation is in fact a double expansion in $1/m_b$ and in α_s [12, 13, 14, 15, 16]. The expansion of some differential rate Γ is of the form

$$\Gamma = \Gamma_0^{(0)} + \frac{\alpha_s}{\pi} \Gamma_0^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \Gamma_0^{(2)} + \left(\Gamma_{2,1}^{(0)} + \frac{\alpha_s}{\pi} \Gamma_{2,1}^{(1)}\right) \frac{\lambda_1}{m_b^2} + \left(\Gamma_{2,2}^{(0)} + \frac{\alpha_s}{\pi} \Gamma_{2,2}^{(1)}\right) \frac{\lambda_2}{m_b^2} + \Gamma_{3,1}^{(0)} \frac{\rho_1}{m_b^3} + \Gamma_{3,2}^{(0)} \frac{\rho_2}{m_b^3} + \dots \quad (2.1)$$

where we have only retained the terms that have been computed. Here $\lambda_{1,2} \sim \Lambda^2$ and $\rho_{1,2} \sim \Lambda^3$ are matrix elements of operators between B meson states. The precise definition of these varies among groups performing the analysis; for example, Ref. [17] uses $\mu_\pi^2 = -\langle B|\bar{b}(iD_\perp)^2 b|B\rangle$ and $\mu_G^2 = -\langle B|\bar{b}(iD_\perp^\mu)(iD_\perp^\nu)\sigma_{\mu\nu} b|B\rangle$ for $\lambda_{1,2}$. To determine $|V_{cb}|$ fits with six free parameters, $m_{b,c}$, $\lambda_{1,2}$, $\rho_{1,2}$, are performed. Fits to moments of the distributions are sensitive to the non-perturbative parameters and independent of $|V_{cb}|$, the latter being fixed by the total rate. Table 1 gives the result of a recent fit.

3. Exclusive Decays

$|V_{cb}|$ **from** $B \rightarrow D^* \ell \nu$ The decay rate is given by²

$$\frac{d\Gamma}{dw} = \frac{G_F^2 m_B^5}{48\pi^3} |V_{cb}|^2 \sqrt{w^2 - 1} P(w) (\eta_{ew} \mathcal{F}(w))^2.$$

The function $\mathcal{F}(w)$ is given in terms of form factors defined by

$$\langle D^*(p', \varepsilon) | V^\mu | B(p) \rangle = ig \varepsilon^{\mu\alpha\beta\gamma} \varepsilon_\alpha^* p'_\beta p_\gamma, \quad (3.1)$$

$$\langle D^*(p', \varepsilon) | A^\mu | B(p) \rangle = f_0 \varepsilon^{*\mu} + (\varepsilon^* \cdot p) [a_+(p+p')^\mu + a_-(p-p')^\mu], \quad (3.2)$$

and we used $w = p_B \cdot p_{D^*} / m_B m_{D^*}$. The function $P(w)$ is from phase space and $\eta_{ew} \approx 1.007$ is an electroweak short distance correction. Heavy quark symmetry gives $\mathcal{F}(1) = 1$ for $m_b = \infty$; remarkably the corrections to the infinite mass limit first appear at order $1/m_b^2$ and are given in terms of the same matrix elements as in the inclusive decay, $\lambda_{1,2}$. There are also computable $o(\alpha_s)$ corrections. The rate vanishes at the end-point, $w = 1$, so an extrapolation of data to this point is

²See <http://pdg.lbl.gov/2016/reviews/rpp2016-rev-vcb-vub.pdf> and references therein.

necessary to use this result; see below. Form factors are obtained in a restricted kinematic range from lattice QCD calculations.³

The lattice calculation of the form factors is restricted to a small region of q^2 , or equivalently of w , near the endpoint $w \approx 1$. To get the most out of the fit to data it is useful to have a reliable extrapolation of the lattice results to the entire physical region, $0 \leq q^2 \leq (m_B - m_{D^*})^2$. The z -expansion provides a parametrization that is fairly restricted once basic principles, such as Analyticity, crossing symmetry and unitarity, are incorporated. One writes the form factors in terms of a variable,

$$z = \frac{\sqrt{w+1} - \sqrt{2}}{\sqrt{w+1} + \sqrt{2}}.$$

This variable is very small over the whole physical region, $0 \leq z \leq 0.056$. The form factors exhibit a cut in the complex q^2 plane that is mapped onto the unit circle $|z| = 1$. The parametrization of a form factor $ff(w)$ reads,

$$ff(w) = \mathcal{P}(z) \sum_{n=0}^{\infty} a_n z^n, \quad \text{where} \quad \sum_{n=0}^{\infty} a_n^2 \leq 1.$$

The constraint on the coefficients a_n is what makes this work, and follows from first principles. In this expression, $\mathcal{P}(z)$ is a ‘‘Blaschke’’ factor, computable in terms of perturbative QCD and knowledge of the masses of physical resonances in cross channels. This is not just a ‘‘parametrization’’: since the physical region is so small, $n \geq 2$ terms give no more than 1% error, so a truncation is warranted at this level of precision. The HFAG collaboration obtains $|V_{cb}| \mathcal{F}(1) = 35.90(45) \times 10^{-3}$, so that using the lattice results from FNAL/MILC $\mathcal{F}(1) = 0.906(13)$ yields

$$|V_{cb}| = 39.04(49)_{\text{exp}}(53)_{\text{lat}}(19)_{\text{QED}} \times 10^{-3}.$$

$|V_{cb}|$ **from** $B \rightarrow D\ell\nu$ An entirely analogous analysis is made for $B \rightarrow D\ell\nu$. The determination of $|V_{cb}|$ is somewhat challenged by the faster vanishing of the rate at $w = 1$, $\Gamma \sim (w-1)^{3/2}$ vs $P(w) \sim \sqrt{w-1}$ for $B \rightarrow D^*$. The form factors are given by

$$\langle D(p') | V^\mu | B(p) \rangle = f_+(p+p')^\mu + (f_0 - f_+) \frac{m_B^2 - m_D^2}{q^2} q^\mu \quad q \equiv p - p',$$

and the rate is given by

$$\frac{d\Gamma}{dw} = \frac{G_F^2 m_B^5}{48\pi^3} |V_{cb}|^2 (m_B + m_D)^2 m_D^3 (\sqrt{w^2 - 1})^3 (\eta_{\text{ew}} \mathcal{G}(w))^2.$$

A determination of $|V_{cb}|$ from a global fit to $B \rightarrow D\ell\nu$ is obtained using the BGL z -expansion [20, 21, 22], giving the preliminary result⁴ $|V_{cb}| = 40.49(99) \times 10^{-3}$. It is interesting that the alternative CLN method [23], which truncates the BGL expansion non-systematically, does not do a good job of fitting the data. It gives $|V_{cb}| = 40.85(95) \times 10^{-3}$ but the fit has a p -value less than 10^{-5} when lattice results for f_0 are included. To quote the authors, ‘‘we are getting too precise for CLN!!’’

³See contribution of Ran Zhou to these proceedings.

⁴P. Gambino at Beauty 2016.

	$R(D)$	$R(D^*)$	Corr.
BaBar[24]	$0.440 \pm 0.058 \pm 0.042$	$0.332 \pm 0.024 \pm 0.018$	-0.27
Belle[25]	$0.375 \pm 0.064 \pm 0.026$	$0.293 \pm 0.038 \pm 0.015$	-0.49
LHCb[26]		$0.336 \pm 0.027 \pm 0.030$	
Average (HFAG)	$0.391 \pm 0.041 \pm 0.028$	$0.322 \pm 0.018 \pm 0.012$	-0.29
SM[31, 28]	0.300 ± 0.010	0.252 ± 0.005	

Table 2: Measurements of $R(D^{(*)})$, average and SM prediction. The experimental errors are statistical (first) and systematic (second).

$R_{D^{(*)}}$ **anomaly** A persistent problem in semileptonic B decays is the observed excess for $B \rightarrow D^{(*)} \tau \nu$. This is best characterized in terms of the ratio of these decays to the corresponding ones into light leptons,

$$R_{D^{(*)}} = \frac{\mathcal{B}(\bar{B} \rightarrow D^{(*)} \tau \bar{\nu})}{\mathcal{B}(\bar{B} \rightarrow D^{(*)} \ell \bar{\nu})}.$$

Table 2 compares the experimental results to the SM prediction. The SM shows a more than 4σ deficit relative to the average of data. The excess can be explained by assuming new physics at short distances [29]. In particular, models with leptoquarks with masses $\lesssim 1\text{TeV}$ that incorporate the principle of Minimal Flavor Violation (MFV) are viable explanations. Interestingly, models with vector leptoquarks can simultaneously account for these anomalies and the universality violation anomaly in $B \rightarrow K \mu \mu / B \rightarrow K e e$ [30].

Exclusive $|V_{ub}|$: $B \rightarrow \pi \ell \nu$ The analysis parallels that of $B \rightarrow D \ell \nu$. Form factors are defined via

$$\langle \pi(p') | V^\mu | B(p) \rangle = f_+(p+p')^\mu + (f_0 - f_+) \frac{m_B^2 - m_\pi^2}{q^2} q^\mu$$

in terms of which the decay rate is given by

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2 m_B^5}{48\pi^3} |V_{ub}|^2 |p_\pi|^3 |f_+(q^2)|^2$$

The form factors, $f_{+,0}(q^2)$, are determined from lattice QCD calculations.⁵ As in the case of $B \rightarrow D \ell \nu$ decays an interpolation is required and for this a z -fit is employed tailored to the heavy-to-light decay case [32, 33]. There is no heavy quark symmetry prediction for the form factor at zero recoil, so the z -fit is used for the shape of the lattice data, supplemented possibly by a model independent constraint at $q^2 = 0$ obtained from $B \rightarrow \pi\pi$ using the soft-collinear effective theory, and the overall normalization then gives $|V_{ub}|$ [34]. The UKQCD collaboration obtains $|V_{ub}| = 3.61(32) \times 10^{-3}$ [35] while FNAL/MILC finds $|V_{ub}| = 3.72(16) \times 10^{-3}$ [36].

4. Inclusive $B \rightarrow X_u \ell \nu$

While the theory of inclusive $B \rightarrow X_u \ell \nu$ decays is analogous to that of $B \rightarrow X_c \ell \nu$, there are practical complications that make interpretation of data challenging. Since $|V_{ub}/V_{cb}| \approx 10^{-1}$, the

⁵See contribution of Ran Zhou to these proceedings

rate for $B \rightarrow X_u \ell \nu$ decays is two orders of magnitude smaller than that of $B \rightarrow X_c \ell \nu$ decays, so the signal for X_u hides under that for X_c . The common approach to this problem is to focus on a region of kinematics for which the decay into charm is not allowed. However, to calculate the rate for a restricted region of kinematics requires knowledge of an infinite number of non-perturbative matrix elements that can be combined into a non-perturbative “shape function” (SF) [18, 19]. There are several possibilities: (i) $q^2 > (m_B - m_D)^2$ is marginally sensitive to SF, but has substantial $(\Lambda/m_b)^3$ corrections, has low rate, and requires measurement of the missing E to reconstruct q^2 ; (ii) $m_X < m_D$, is more sensitive to SF, but has higher rate; and (iii) $E_\ell > (m_b^2 - m_D^2)/2m_B$ is most sensitive to SF and has low rate, but is the simplest to access in data.

The need for a SF can be readily illustrated by considering $B \rightarrow X_s \gamma$, which shares the same SF as inclusive semileptonic decays. The OPE for inclusive decay leads us to consider

$$\bar{\Gamma} \frac{i}{m_b \not{v} - \not{q} + \not{k} + i\epsilon} \Gamma = \frac{i}{m_b^2 - 2m_b q_0 + i\epsilon} \bar{\Gamma} (m_b \not{v} - \not{q}) \Gamma + \dots$$

where $v^2 = 1$ and on the right hand side we have expanded in powers of $k/m_b \sim \Lambda_{\text{QCD}}/m_b$ but assumed $q/m_b \sim 1$ since we are interested in the endpoint region, $q_0 \approx m_b/2$. This gives a series in poles $q_0 - m_b/2$. For consistency we must retain all orders. With $x = 2q_0/m_b$, using $\text{Im} \frac{1}{1-x-i\epsilon} = \pi \delta(1-x)$ one obtains

$$\frac{d\Gamma}{dx} \propto \left(\delta(1-x) - \frac{\lambda_1 + 3\lambda_2}{2m_b^2} \delta'(1-x) + \dots \right) = f(1-x)$$

where the non-perturbative SF can be succinctly expressed as $2m_B f(\omega) = \langle B | \bar{h}_v \delta(\omega + in \cdot D) h_v | B \rangle$, ($n^2 = 0, n \cdot v = 1$).

The extraction of $|V_{ub}|$ requires an educated guess of the SF. Data analysis based on approaches proposed by several groups result in widely varying results. For example, a BaBar analysis using $m_X < 1.55$ GeV yields $|V_{ub}| \times 10^3$ varying between $3.81 \pm 0.14 \pm 0.11_{-0.20}^{+0.18}$ and $4.40 \pm 0.16 \pm 0.12_{-0.19}^{+0.24}$ [37]. The PDG gives a summary value of

$$|V_{ub}| = (4.41 \pm 0.15_{\text{exp}}^{+0.15} {}_{-0.17}^{\text{th}}) \times 10^{-3}$$

5. Summary

We have reviewed the methods for the determination of the CKM elements $|V_{cb}|$ and $|V_{ub}|$. While the determination is at the 1–2% precision level, the extraction from exclusive decays is $\sim 10\%$ below that from inclusive decays, both for $|V_{cb}|$ and $|V_{ub}|$. The discrepancy in $|V_{cb}|$ from inclusive *vs.* $B \rightarrow D^* \ell \nu$ is at about the 3σ level.

It is not easy to account for this discrepancy by invoking new physics. Consider accounting for the effects of new physics by supplementing the SM by four-fermion local operators. For example, if the new physics is in the form of right-handed currents, the effect on the extracted values of $|V_{cb}|$ is

$$\begin{aligned} |V_{cb}|_{\text{incl}} &= |V_{cb}| (1 + \frac{1}{2} \epsilon^2) \\ |V_{cb}|_{D^*} &= |V_{cb}| (1 + \epsilon) \\ |V_{cb}|_D &= |V_{cb}| (1 - \epsilon) \end{aligned}$$

where ε is a small parameter characterizing the strength of the four fermion $V + A$ operator. One can improve the agreement between inclusive *vs.* $B \rightarrow D^* \ell \nu$ determinations only at the cost of worsening the agreement with $B \rightarrow D \ell \nu$ determinations. A more general analysis of new physics modeled by dimension-6 operators does not improve the situation significantly for $|V_{cb}|$ [38], but it does decrease the tension in the $|V_{ub}|$ case [39].

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