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# Determinations of $V_{us}$ from Hadronic $\tau$ Decay Data

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The current status of determinations of $V_{us}$ from exclusive and inclusive hadronic $\tau$ decay data is reviewed. In the case of the inclusive determinations, we describe (i) recent progress on the implementation of the conventional flavor-breaking sum rule approach and (ii) a new dispersive approach in which lattice data, rather than the OPE, is used as theory input and only the inclusive strange experimental distribution is required. The former provides a resolution of the long-standing puzzle of $> 3\sigma$ low values of $V_{us}$ obtained from previous implementations of the flavor-breaking sum rule approach, while the latter is shown to make possible a significant reduction in the impact of the region of larger experimental errors above $s \sim 2\ GeV^2$. Advantages of, and near-term prospects for, this new approach are also discussed.

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## 1. Introduction

Improvements in the precision of the determination of $|V_{us}|$ are of interest in the context of further refining 3-family-unitarity tests. The latest determination, $|V_{ud}| = 0.97417(21)$, from super-allowed $0^+ \to 0^+$ nuclear $\beta$ decays [1], yields the 3-family-unitarity expectation

$$|V_{us}| = 0.2258(9)\,. \tag{1.1}$$

For comparison, the direct determination from $K_{\ell 3}$, using the updated 2014 FlaviaNet result $f_+(0)|V_{us}| = 0.2165(4)$ [2], and 2016 FLAG $n_f = 2+1+1$ assessment, $f_+(0) = 0.9704(33)$ [3], yields

$$|V_{us}| = 0.2231(9)\,. \tag{1.2}$$

Similarly, the direct determination from $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$, using the updated 2014 FlaviaNet result $|f_K V_{us}|/|f_\pi V_{ud}| = 0.2760(4)$ [2], $|V_{ud}|$ from Ref. [1], and the 2016 FLAG $n_f = 2+1+1$ average $f_K/f_\pi = 1.193(3)$ [3], yields

$$|V_{us}| = 0.2253(7)\,. \tag{1.3}$$

Given the (albeit mild) tension between the $K_{\ell 3}$ and $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$ results, additional independent determinations, such as those provided by hadronic $\tau$ decays, are of interest. In what follows, we discuss exclusive determinations based on the measured $\tau \to K\nu_\tau$, $\tau \to \pi\nu_\tau$ and $\tau \to K\pi\nu_\tau$ branching fractions, the inclusive determination based on the conventional flavor-breaking (FB) sum rule analysis of the non-strange and strange decay distributions [4], and a new alternate determination employing lattice data and the inclusive strange decay distribution. The conventional FB sum rule analysis is of particular interest because of the long-standing puzzle of the rather low $|V_{us}|$ values obtained using its conventional implementation [4, 5, 6], the most recent of which [6] yields

$$|V_{us}| = 0.2176(21)\,, \tag{1.4}$$

$3.6\sigma$ below the 3-family-unitarity expectation of Eq. (1.1).

## 2. Hadronic $\tau$ decays in the Standard Model

In the Standard Model (SM), with $R_{V/A;ij} \equiv \Gamma[\tau^- \to \nu_\tau \,\text{hadrons}_{V/A;ij}(\gamma)]/\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma)]$, the differential distribution, $dR_{V/A;ij}/ds$, for decays mediated by the flavor $ij = ud, us$ vector (V) or axial vector (A) currents, is related to the spectral function, $\rho^{(J)}_{V/A;ij}$, of the spin $J = 0, 1$, flavor $ij$, V or A current-current two-point function scalar polarizations, $\Pi^{(J)}_{V/A;ij}$, by [7]

$$\begin{aligned}\frac{dR_{V/A;ij}}{ds} &= \frac{12\pi^2|V_{ij}|^2 S_{EW}}{m_\tau^2}\left[w_\tau(y_\tau)\rho^{(0+1)}_{V/A;ij}(s) - w_L(y_\tau)\rho^{(0)}_{V/A;ij}(s)\right] \\ &\equiv \frac{12\pi^2|V_{ij}|^2 S_{EW}}{m_\tau^2}\,(1-y_\tau)^2\,\tilde{\rho}_{V/A;ij}(s)\,,\end{aligned} \tag{2.1}$$

where $y_\tau = s/m_\tau^2$, $w_\tau(y) = (1-y)^2(1+2y)$, $w_L(y) = 2y(1-y)^2$, $S_{EW}$ is a known short-distance electroweak correction, and $V_{ij}$ is the flavor $ij$ CKM matrix element. The dominant, non-chirally-suppressed $J = 0$ contributions are determined by $f_\pi$ and $f_K$, and hence accurately known. The

remaining continuum, doubly-chirally-suppressed $J = 0$ contributions are negligible for $ij = ud$, and for $ij = us$, both small and highly constrained by the known value of $m_s$, through the associated $ij = us$ scalar and pseudoscalar sum rules. This makes possible mildly model-dependent determinations of continuum $\rho^{(0)}_{V/A;us}(s)$ contributions in the range $s \leq m_\tau^2$ relevant to hadronic $\tau$ decays [8, 9]. With this input, $dR_{V/A;ud,us}/ds$ provides a direct determination of $\rho^{(0+1)}_{V/A;ud,us}(s)$.

## 3. Exclusive mode determinations of $|V_{us}|$

With $B_\pi$ and $B_K$ the single-prong $\pi$ and $K$ branching fractions, $B_e$ the electronic branching fraction, and $R_P = B_P/B_e$, $P = \pi, K$, one has, from Eq. (2.1),

$$\begin{aligned} R_\pi &= \frac{24\pi^2 S_{EW}}{m_\tau^2}\left(1-\frac{m_\pi^2}{m_\tau^2}\right)^2 |f_\pi V_{ud}|^2 \\ R_K &= \frac{24\pi^2 S_{EW}}{m_\tau^2}\left(1-\frac{m_K^2}{m_\tau^2}\right)^2 |f_K V_{us}|^2, \end{aligned} \tag{3.1}$$

with $f_{\pi,K}$ the $\pi$, $K$ decay constants in the ChPT convention ($f_\pi \simeq 92\ MeV$). $|V_{us}|$ can thus be determined from $R_K$ with external input for $f_K$, or from $R_K/R_\pi$ with external input for $|V_{ud}|$ and $f_K/f_\pi$. With branching fractions $B_K$, $B_\pi$ and $B_e$ from the HFAG Summer 2014 fit [10], $|V_{ud}|$ from Ref. [1], and 2016 FLAG $n_f = 2+1+1$ input for $f_K/f_\pi$ and $f_K$, one obtains

$$|V_{us}| = 0.2222(17) \qquad (\text{from } R_K) \tag{3.2}$$

$$|V_{us}| = 0.2230(18) \qquad (\text{from } R_K/R_\pi). \tag{3.3}$$

The latter is the $\tau$ analogue of the $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$ determination. Comparing these results to those of Eqs. (1.1), (1.2) and (1.3), we see the $\tau$ determinations are compatible, within errors, with those from $K_{\ell 3}$ and $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$, as well as (at the 1.4 and 1.9 $\sigma$ levels) with 3-family unitarity expectations. The errors on the $\tau$ determinations, however, are a factor of $\sim 2$ larger.

One can also, in principle, determine $|V_{us}|$ from the normalization of any exclusive strange decay mode, given reliable theoretical input for the $s$-dependence of the corresponding decay distribution. This is, in principle, feasible for the $K\pi$ modes, where dispersive representations can be employed for the timelike $K\pi$ form factors $f_{+,0}(s)$. Ref. [11] has investigated expectations for the $K\pi$ branching fractions, employing $K_{\ell 3}$ and Belle [12] $\tau^- \to \bar{K}^0\pi^-\nu_\tau$ decay distribution data. Triply subtracted versions of the dispersion relations are employed to reduce sensitivity to the high-$s$ region where the form factors phases are not known. The predicted branching fractions, including estimated long-distance electromagnetic and strong isospin-breaking effects, are

$$\begin{aligned} B[\tau^- \to \bar{K}^0\pi^-\nu_\tau] &= 0.00851(30) \\ B[\tau^- \to K^-\pi^0\nu_\tau] &= 0.00471(18), \end{aligned} \tag{3.4}$$

where the errors are 100% correlated, and totally dominated by the uncertainties in the phase space integrals, reflecting errors in the dispersive form factor results induced by the current data errors. The $B[\tau^- \to \bar{K}^0\pi^-\nu_\tau]$ result is compatible within errors with the HFAG 2012 input, $0.00821(18)$, for the normalization of the corresponding $\tau$ decay distribution, but some tension exists between

the $B[\tau^- \to K^-\pi^0\nu_\tau]$ result and the corresponding 0.00432(15) HFAG 2012 normalization input. One should also bear in mind that the 2007 BaBar result, $B[\tau^- \to K^-\pi^0\nu_\tau] = 0.00416(19)$ [13], is more precise by almost a factor of 2 than all other determinations entering the 2012 HFAG average and hence dominates that average. This average, however, differs significantly from the more recent, but still preliminary, BaBar thesis result, $B[\tau^- \to K^-\pi^0\nu_\tau] = 0.00500(15)$, obtained from an analysis focussed specifically on improving $\pi^0$ identification [14]. The experimental situation for the normalization of the $K^-\pi^0$ mode is thus somewhat unsettled. In addition, with current data as input, the branching fraction expectations of Eqs. (3.4) have errors of 3.5% and 3.8%, respectively, which would produce $\sim 0.0030$ uncertainties in $|V_{us}|$. This approach is thus not currently competitive, but might become so with errors on the $\tau \to K\pi\nu_\tau$ distributions reduced to the level expected once the full Belle II data set is available [11].

## 4. Inclusive flavor-breaking sum rules for $|V_{us}|$

The conventional inclusive FB $\tau$ decay determination of $|V_{us}|$ [4] employs finite energy sum rules (FESRs) involving the FB polarization difference, $\Delta\Pi_\tau \equiv \Pi^{(0+1)}_{V+A;ud} - \Pi^{(0+1)}_{V+A;us}$, and associated spectral function, $\Delta\rho_\tau \equiv \rho^{(0+1)}_{V+A;ud} - \rho^{(0+1)}_{V+A;us}$. These FESRs have the form

$$\int_0^{s_0} w(s)\Delta\rho_\tau(s)\,ds = -\frac{1}{2\pi i}\oint_{|s|=s_0} w(s)\Delta\Pi_\tau(s)\,ds\,, \tag{4.1}$$

and are valid for any analytic $w(s)$ and any $s_0 > 0$. Experimental data is used on the LHS, the OPE (for sufficiently large $s_0$) on the RHS.

The $J = 0+1$ FESR form Eq. (4.1) requires $\Delta\rho_\tau(s)$, obtained by applying the mildly model-dependent continuum *us* $J = 0$ subtraction to the experimental $dR_{V+A;us}/ds$ distribution. This subtraction could be avoided for FESRs involving the alternate FB spectral difference $\tilde\rho_{V+A;ud-us}(s)$ and associated FB polarization difference $\tilde\Pi_{V+A;ud-us}(Q^2)$ (where $Q^2 = -s$), defined by

$$\tilde\Pi_{V+A;ud-us}(Q^2) \equiv \left(1 - \frac{2Q^2}{m_\tau^2}\right)\Pi^{(1)}_{V+A;ud-us}(Q^2) + \Pi^{(0)}_{V+A;ud-us}(Q^2)\,. \tag{4.2}$$

The $J = 0$ $D = 2$ OPE series entering these alternate FESRs, however, displays very bad convergence and has fixed order truncations badly violating spectral positivity constraints [15]. Use of the $J = 0+1$ form and associated continuum *us* $J = 0$ subtraction are thus unavoidable.

Applying the *us* $J = 0$ subtraction to $dR_{V/A;ij}/ds$ yields a $J = 0+1$ analogue, $dR^{(0+1)}_{V/A;ij}/ds$. Re-weighted versions

$$R^w_{V+A;ij}(s_0) \equiv \int_0^{s_0} ds\,\frac{w(s)}{w_\tau(s)}\,\frac{dR^{(0+1)}_{V+A;ij}(s)}{ds} \tag{4.3}$$

may then be constructed for any $w$ and any $s_0 \le m_\tau^2$. Defining the FB difference

$$\delta R^w_{V+A}(s_0) \equiv \frac{R^w_{V+A;ud}(s_0)}{|V_{ud}|^2} - \frac{R^w_{V+A;us}(s_0)}{|V_{us}|^2}\,, \tag{4.4}$$

and replacing the LHS by its OPE representation via Eq. (4.1), one finds, solving for $|V_{us}|$ [4],

$$|V_{us}| = \sqrt{R^w_{V+A;us}(s_0)\Big/\left[\frac{R^w_{V+A;ud}(s_0)}{|V_{ud}|^2} - \delta R^{w,OPE}_{V+A}(s_0)\right]}\,. \tag{4.5}$$

Provided all theoretical and experimental inputs to the RHS of Eq. (4.5) are reliable, results for $|V_{us}|$ will be independent of $s_0$ and $w$. Employing a range of $s_0$ and $w$ thus allows any assumptions entering the analysis to be tested for self-consistency. The $s_0$-independence test is particularly useful if assumptions about unknown higher dimension effective OPE condensates are employed, since integrated $D = 2k+2$ OPE contributions scale as $1/s_0^k$, and problems with these assumptions will thus manifest themselves as an unphysical $s_0$-dependence in the results for $|V_{us}|$[1].

The conventional implementation of Eq. (4.5) [4] (which leads to the low values of $|V_{us}|$ noted above) employs the single weight $w = w_\tau$ and single $s_0$ choice $s_0 = m_\tau^2$. Since $w_\tau$ has degree 3, OPE contributions up to dimension $D = 8$, unsuppressed by additional powers of $\alpha_s$, appear in $\delta R_{V+A}^{w,OPE}(s_0)$. The leading $D = 2$ and sub-leading $D = 4$ OPE contributions are fixed by $\alpha_s$, the light and strange quark masses and the light and strange quark condensates [16], and hence can be taken as external input [3, 17, 18]. The $D = 6$ and 8 condensates, however, are not known experimentally. Typically [4, 5], $D = 6$ contributions have been estimated using the vacuum saturation approximation (VSA) and $D = 8$ contributions neglected. A very strong double cancellation is, however, present in the $D = 6$ VSA estimate, a factor of $\sim 3$ reduction coming from the cancellation in the individual *ud* and *us* V+A sums, and a further factor of $\sim 6$ from the cancellation in the subsequent FB *ud*-*us* V+A difference. With studies in the *ud* sector showing the VSA to be very crude, and VSA violations to be significantly channel-dependent [19], this high degree of cancellation makes reliance on the $D = 6$ VSA estimate potentially dangerous. Since $D = 8$ contributions are assumed negligible largely on the grounds that the $D = 6$ VSA estimate is already small, a similar comment applies to this assumption as well.

The restriction to $w = w_\tau$ and $s_0 = m_\tau^2$ in the conventional implementation has the experimental advantage that the associated spectral integrals $R_{V+A;ud,us}^{w_\tau}(m_\tau^2)$ are fixed by the inclusive non-strange and strange hadronic $\tau$ branching fractions, making knowledge of the details of the differential distributions unnecessary. This advantage, however, comes at the cost of forgoing the $s_0$- and weight-independence checks which would serve to test the assumptions regarding $D = 6$ and 8 OPE contributions for self-consistency.

$|V_{us}|$ results from analyses with variable $s_0$ and weights other than $w_\tau$ [5], in fact, show significant $s_0$- and weight-dependence. A particularly illuminating comparison is provided by the results of the $w_\tau(y) = 1 - 3y^2 + 2y^2$ and $\hat{w}(y) = 1 - 3y + 3y^2 - y^3$ FESRs, where $y = s/s_0$. Since the coefficients of $y^2$ in $w_\tau$ and $\hat{w}$ differ only by a sign, the integrated $D = 6$ OPE contributions for the two cases will be identical in magnitude but opposite in sign. If, as sugested by the VSA estimate, the $D = 6$ contributions are small for $w_\tau$, they will also necessarily be small for $\hat{w}$ as well. Similarly, the integrated $D = 8$ contributions for $w_\tau$ are $-2$ times those for $\hat{w}$ and, if negligible for the former, will also be negligible for the latter. If the assumptions employed for the $D = 6$ and 8 contributions in the conventional implementation are reliable, the $|V_{us}|$ results obtained using $w_\tau$ and $\hat{w}$ should display good individual $s_0$ stability and be in good agreement. In contrast, if the effective $D = 6$ and/or 8 condensates are not negligible, this should show up as $s_0$-instabilities of opposite signs in the two cases, with the $1/s_0^2$ and $1/s_0^3$ scalings of the integrated $D = 6$ and 8 contributions leading to differences between the two sets of $s_0$-dependent results which decrease with increasing $s_0$.

[1] Explicitly, ignoring $\alpha_s$-suppressed logarithmic variations, and writing the $D > 4$ contribution to $\Delta\Pi_\tau(Q^2)$ as $C_D/Q^D$, with $C_D$ the effective dimension $D$ condensate, the integrated $D = 2k+2$ OPE contribution for a polynomial weight $w(y) = \sum_{k=0} x_k y^k$, with $y = s/s_0$, is $(-1)^k C_{2k+2}\, w_k/s_0^k$.

The left panel of Figure 1 shows the results of this comparison. The results clearly follow the pattern expected if $D > 4$ contributions are not negligible, and are incompatible with the $D = 6, 8$ assumptions underlying the conventional implementation. The right panel shows the results of additional $s_0$- and weight-independence tests involving the weights $w_N(y)$, $N = 2, 3, 4$, with[2]

$$w_N(y) = 1 - \frac{N}{N-1} y + \frac{1}{N-1} y^N \,. \tag{4.6}$$

The solid lines show the results produced using the conventional implementation treatment of $D > 4$ OPE contributions, the dashed lines those produced by the alternate implementation discussed below, in which $D > 4$ effective condensates are fitted to experimental data. For all five weights, the $s_0$-dependent $|V_{us}|$ results of the conventional implementation show evidence of converging toward a common value at $s_0 > m_\tau^2$, precisely as expected if the $s_0$-instabilities result from $D > 4$ OPE contributions larger than those employed in the conventional implementation.

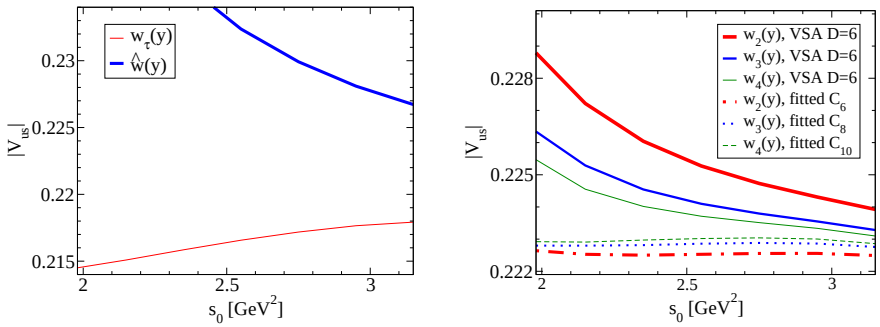


**Figure 1:** Left panel: $|V_{us}|$ from the $w_\tau$ and $\hat{w}$ FESRs with conventional OPE input (including contour improved perturbation theory for the $D = 2$ series). Right panel: Comparison of conventional implementation results (solid lines) with those obtained using central fitted $C_{6,8,10}$ and the fixed order perturbation theory $D = 2$ prescription favored by lattice results, for the weights $w_{2,3,4}$ (dashed lines).

A potential additional problem for the FB FESR approach is the slow convergence of the $D = 2$ OPE series which, to four loops, neglecting $O(m_{u,d}^2/m_s^2)$ corrections, is given by [16]

$$\left[\Delta\Pi_\tau(Q^2)\right]_{D=2}^{OPE} = \frac{3}{2\pi^2} \frac{m_s(Q^2)}{Q^2} \left[1 + \frac{7}{3}\bar{a} + 19.93\bar{a}^2 + 208.75\bar{a}^3 + \cdots\right], \tag{4.7}$$

where $\bar{a} = \alpha_s(Q^2)/\pi$, and $m_s(Q^2)$ and $\alpha_s(Q^2)$ are the running $\overline{MS}$ strange quark mass and coupling. With $\bar{a}(m_\tau^2) \simeq 0.1$, the 4-loop, $O(\bar{a}^3)$ term is, in fact, larger than the 3-loop, $O(\bar{a}^2)$ term at the spacelike point on the contour $|s| = s_0$ for all $s_0$ accessible in $\tau$ decays. This raises the questions of appropriate $D = 2$ truncation order and the reliability of conventional OPE error estimates.

It is possible to investigate these issues by comparing OPE expectations to lattice data for $\Delta\Pi_\tau(Q^2)$. This comparison has been carried out using $n_f = 2+1$ RBC/UKQCD lattice data [21]

[2]The $w_N(y)$, like $w_\tau(y)$, have a double zero at $s = s_0$ ($y = 1$), a property which keeps duality violating (DV) contributions safely small above $s \simeq 2\ GeV^2$ [20].

in Ref. [22]. An excellent match between lattice data and the $D = 2+4$ OPE sum is obtained over a broad high-$Q^2$ interval stretching from $Q^2 \sim 10\ GeV^2$ down to $\sim 4\ GeV^2$, provided one employs the 3-loop truncated version of the $D = 2$ series with fixed (rather than local) scale treatment of logarithmic contributions [22]. The fixed-scale and local-scale treatments are the analogues of the "fixed-order" (FOPT) and "contour-improved" (CIPT) FESR $D = 2$ series treatments. The high-$Q^2$ comparison also shows that conventional $D = 2+4$ OPE error estimates are extremely conservative, in spite of the slow convergence of the $D = 2$ OPE series [22]. Below $Q^2 \sim 4\ GeV^2$ one sees clear deviations of the $D = 2+4$ OPE sum from the lattice data [22], confirming the presence, already indicated by the $w_\tau$-$\hat{w}$ FESR comparison discussed above, of $D > 4$ OPE contributions much larger than those employed in the conventional implementation.

While these results make clear that the conventional implementation approach must thus be abandoned, they also suggest an obvious alternate implementation in which the 3-loop-truncated, FOPT treatment of the $D = 2$ OPE series favored by the lattice is used, and, rather than making assumptions about their values, the $D > 4$ effective OPE condensates $C_D$ are obtained from fits to the experimental data. Results of such an analysis, employing the weights $w_N(y)$, were reported in Ref. [22]. The analysis uses the following experimental input: $\pi_{\mu 2}$, $K_{\mu 2}$ and SM expectations for the $\pi$ and $K$ pole contributions, recent ALEPH data for the continuum *ud* V+A distribution [23], Belle [12] and BaBar [13, 14] results for the $\bar{K}^0\pi^-$ and $K^-\pi^0$ distributions, BaBar [24] and Belle [25] results for the $K^-\pi^+\pi^-$ and $\bar{K}^0\pi^-\pi^0$ distributions, a combination of BaBar [26] and Belle [27] results for the very small $\bar{K}K\bar{K}K$ distribution and normalization, and 1999 ALEPH results [28] for the distribution of the sum of the remaining exclusive strange modes not remeasured by the B-factory experiments. Two versions were employed for the $K^-\pi^0$ branching fraction, which normalizes the corresponding exclusive mode distribution: $0.00433(15)$ from the 2014 HFAG summer fit [10], and the preliminary BaBar thesis result $0.00500(14)$ [14]. The latter is preferred by the BaBar collaboration, whose earlier result dominates the HFAG average; the central results reported below thus correspond to this choice.

The $w_N$ employed in Ref. [22] have the useful property that the $w_N$ FESR involves only two free parameters, $|V_{us}|$ and $C_{2N+2}$. These were fitted using the range $2.15\ GeV^2 \le s_0 \le 3.15\ GeV^2$, within which integrated DV contributions should be small for "doubly pinched" weights like the $w_N$ [20]. Excellent consistency is observed between the $|V_{us}|$ obtained from the $w_2$, $w_3$ and $w_4$ FESRs [22]. The $s_0$ dependence of $|V_{us}|$ obtained from these FESRs, when central fitted values for the effective condensates $C_{6,8,10}$ are employed in place of the assumed values used in the conventional implementation, are shown by the dashed lines in the right panel of Figure 1. Clearly the $s_0$- and weight-dependence problems displayed by the conventional implementation results are completely cured once a range of $s_0$ is employed, and this range is used to fit the $D > 4$ effective condensates to experimental data.

The excellent consistency of $w_2$, $w_3$ and $w_4$ FESR results makes possible a combined 3-weight fit. Using the updated preliminary BaBar $K^-\pi^0$ branching fraction as input, one finds [22]

$$|V_{us}| = 0.2228(5)_{th}(23)_{exp}\,, \tag{4.8}$$

where the theory error is dominated by the uncertainty in $\langle m_s\bar{s}s\rangle$ and the experimental error by the uncertainties in the strange exclusive distributions [22]. The result is in very good agreement with that from $K_{\ell 3}$, and compatible within errors with 3-family unitarity expectations. Using the

alternate HFAG 2014 $K^-\pi^0$ branching fraction as input yields instead $|V_{us}| = 0.2200(5)_{th}(23)_{exp}$, 0.0024 higher than the conventional implementation result employing the same $K^-\pi^0$ branching fraction. Further clarification of the experimental situation for this mode is of obvious interest.

## 5. An alternate, non-FB, inclusive $|V_{us}|$ determination

The OPE-lattice comparison demonstrates that the alternate implementation of the FB FESR approach enjoys very favorable theoretical errors. With current experimental errors, however, the resulting $|V_{us}|$ uncertainty is a factor of $\sim 2$ larger than those from $K_{\ell 3}$ and $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$. The FB FESR error is currently strongly dominated by the uncertainties on the weighted strange spectral integrals. To understand what near-term improvements might be possible, it is useful to look at the relative contributions of the different exclusive modes to the weighted *us* V+A spectral integrals entering the analysis of Ref. [22]. Table 5 shows these results for $w_{2,3,4}$ at the two $s_0$ fit-window endpoints (the results vary monotonically between these two endpoints). The target

**Table 1:** Relative $w_N$-weighted *us* spectral integral contributions. "$K\pi$" column entries represent the $K^-\pi^0$ and $\bar{K}^0\pi^-$ sum, "$K\pi\pi$ (B factory)" column entries the $K^-\pi^+\pi^-$ and $\bar{K}^0\pi^-\pi^0$ sum and "Other" column entries the sum of the ALEPH 1999 residual mode and very small $K\bar{K}\bar{K}$ contributions.

| Weight | $s_0$ $[GeV^2]$ | $K$ | $K\pi$ | $K\pi\pi$ (B factory) | Other |
|---|---|---|---|---|---|
| $w_2$ | 2.15 | 0.496 | 0.426 | 0.062 | 0.017 |
| | 3.15 | 0.360 | 0.414 | 0.162 | 0.065 |
| $w_3$ | 2.15 | 0.461 | 0.446 | 0.073 | 0.019 |
| | 3.15 | 0.331 | 0.415 | 0.182 | 0.074 |
| $w_4$ | 2.15 | 0.441 | 0.456 | 0.082 | 0.021 |
| | 3.15 | 0.314 | 0.411 | 0.194 | 0.081 |

for a competitive $|V_{us}|$ determination is sub-0.5%, corresponding to a sub-% level determination of the inclusive weighted *us* V+A spectral integrals. At present, the errors on the lower-multiplicity BaBar- and Belle-based $K^-\pi^0$, $\bar{K}^0\pi^-$, $K^-\pi^+\pi^-$ and $\bar{K}^0\pi^-\pi^0$ contributions are dominated by the uncertainties on the corresponding branching fractions, which normalize the unit-normalized experimental distributions. These are currently 1.5%, 3.0%, 2.8% and 2.3%, respectively, and are a natural target for near-term improvement. More difficult to improve are the errors on the residual mode contributions, currently based on the old 1999 ALEPH data, which not only have low statistics but involve Monte Carlo input for a number of the higher-multiplicity mode distributions. These data produce $\sim 25\%$ uncertainties on the weighted residual mode spectral integral contributions. With the relative inclusive $w_{2,3,4}$-weighted spectral integral contributions shown in the table, these $\sim 25\%$ uncertainties generate $\sim 1.6\%$, 1.8% and 2.0% errors on the corresponding weighted $s_0 = 3.15\ GeV^2$ inclusive *us* spectral integrals from residual mode contributions alone. A factor of $> 2$ improvement in the residual mode contribution errors, and hence significant improvements to the experimental analyses of the higher multiplicity strange $\tau$ decay mode *differential distributions* would thus be required before the FB FESR approach would become fully competitive. This is unlikely to be feasible in the near future.

Further progress is, however, possible using an alternate analysis which employs lattice data in place of the OPE and has the flexibility to reduce the relative contribution of higher-error, higher-multiplicity modes to the relevant weighted strange spectral integrals. The analysis is based on generalized dispersion relations involving the product of the spectral function combination $\tilde{\rho}_{V+A;us}(s)$ with weights consisting of products of factors having poles at Euclidean $Q^2$ [29]. The dispersive integral over this product is then given by a sum of residues involving the corresponding polarization $\tilde{\Pi}_{V+A;us}(Q^2)$ evaluated at those pole locations. These polarization values can be evaluated very accurately on the lattice, provided the pole locations are at moderate $Q^2$ [29].

This works as follows [29]. From Eq. (2.1), the combination $|V_{us}|^2\,\tilde{\rho}_{V+A;us}(s)$ is directly determinable from the experimental $dR_{us;V+A}/ds$ distribution, *without the need of any (albeit mildly model-dependent) continuum* $J=0$ *subtraction.* For weights

$$W_N(s) \equiv \frac{1}{\prod_{k=1}^{N}(s+Q_k^2)}\,, \tag{5.1}$$

with $N \geq 3$ and all $Q_k^2 > 0$, one has the convergent, unsubtracted dispersion relation

$$\int_{th}^{\infty} ds\, W_N(s)\,\tilde{\rho}_{us;V+A}(s) \;=\; \sum_{k=1}^{N} \frac{\tilde{\Pi}_{us;V+A}(Q_k^2)}{\prod_{j\neq k}\left(Q_j^2-Q_k^2\right)}\,. \tag{5.2}$$

The $\tilde{\Pi}_{V+A;us}(Q_k^2)$ on the RHS of Eq. (5.2) are to be evaluated using lattice data. The LHS, up to the unknown factor $|V_{us}|^2$, is to be evaluated from the experimental $dR_{us;V+A}/ds$ distribution. Since the spectral integral on the LHS extends up to infinity, while $dR_{us;V+A}/ds$ data is available only up to $s=m_\tau^2$ (with, moreover, significant errors in the upper part of the kinematically allowed region), one aims to use pQCD to approximate the contribution from $s > m_\tau^2$, and choose the number, $N$, and locations, $Q_1^2,\cdots,Q_N^2$, of the poles, in such a way that contributions to the LHS from both $s > m_\tau^2$ and the region where *us* data errors are large are strongly suppressed.

These goals can be accomplished by choosing $N$ large enough and keeping all $Q_k^2$ below $\sim 1\ GeV^2$ [29]. Increasing $N$ typically lowers the error on the LHS of Eq. (5.2) by more strongly suppressing contributions from the high-$s$ region, but increases the errors on the RHS (the level of cancellation in the sum of residues can be shown to grow with increasing $N$). The error on $|V_{us}|$ is to be minimized by optimizing the choice of $N$ and the pole locations, $\{Q_k^2\}$, subject to these competing constraints. Checking that the results for $|V_{us}|$ are independent of the choice of $N$ and the $\{Q_k^2\}$ also provides useful self-consistency tests for the method.

Preliminary results obtained using this framework were presented by RBC/UKQCD and its external collaborators at Lattice 2016 [29]. Two RBC/UKQCD $n_f = 2+1$ ensembles with near-physical light quark masses [30] were employed, one of size $48^3\times 96$ with $1/a = 1.73\ GeV$, $m_\pi = 139\ MeV$, $m_K = 499\ MeV$, the other of size $64^2\times 128$ with $1/a = 2.36\ GeV$, $m_\pi = 139\ MeV$, $m_K = 508\ MeV$. Good agreement with 3-family-unitarity expectations and a preliminary total error of $\sim 0.6\%$ for $|V_{us}|$ using 88 configurations of the former and 80 of the latter were reported [29]. The lattice component of this error is improvable with improved statistics. Final results for $|V_{us}|$ are yet to be released, but expected soon, once studies of systematics uncertainties associated with isospin-breaking corrections, continuum extrapolation, corrections for the small mis-tunings of the light quark masses, and finite volume effects are completed [29].

## 6. Summary

Exclusive mode $|V_{us}|$ determinations employing $\tau$ branching fraction results for $B_K$ and $B_K/B_\pi$ have current errors 0.0017 and 0.0018. A factor of $\sim 2$ improvement would make these competitive with those from $K_{\ell 3}$ and $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$. Belle II data should help in this regard, and may also make feasible a precision dispersive determination using $K\pi$ distribution results.

Recent results show that the $> 3\sigma$ low $|V_{us}|$ values seen in the conventional implementation of the FB FESR approach result from problems with assumptions about $D = 6,8$ OPE condensates. This problem cannot be cured using the conventional implementation. An alternate approach which allows $D > 4$ condensates to be fit to data cures the $s_0$- and weight-dependence problems of the conventional implementation and yields results in much better agreement with other $|V_{us}|$ determinations. This alternate approach requires the full $dR/ds$ distributions and can not be carried out using branching fraction information alone. Comparisons with lattice data show that this alternate FB FESR approach has very favorable theoretical errors. Significant improvements in the high-multiplicity *us* experimental distributions are, however, required to make the approach fully competitive.

The new inclusive approach using lattice data and the inclusive *us* V+A distribution only [29] has a number of advantages over the FB FESR approach. It requires no continuum $J = 0$ *us* subtraction, uses precision lattice data rather than the OPE, and allows the use of weights which can be chosen to much more strongly suppress spectral integral contributions from the higher-multiplicity, high-error part of the spectrum without blowing up the errors on the sum of residues which are to be evaluated using lattice data [29]. The much stronger dominance of the resulting spectral integrals by $K$ and $K\pi$ exclusive mode contributions ensures smaller experimental errors for the new approach and makes significant near-term improvements in those errors possible through improvements to the $K\pi$ branching fractions alone.



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