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A short review about the Bethe-Salpeter equation and the solution of the $\chi^2\phi$ model in Minkowski space

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We present a brief review of the Bethe-Salpeter equation including a list of references with which the reader can undertake deeper studies of the subject. We also present some recent results for the $\chi^2\phi$ model solving the Bethe-Salpeter equation with the light-front projection and the Nakanishi integral representation of the Bethe-Salpeter amplitude.

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1. Review

Since the advent of the relativistic description of the electron in 1925 the developments and applications of formalisms for the treatment of bound states based on quantum field theory faced a few drawbacks. Beyond the problems with divergences, present also in quantum electrodynamics which were eventually helped by renormalization theory, a fully covariant description for bound states has also to deal with the difficulty of giving a suitable physical meaning for the two-particle wave function where each elementary particle has its proper time defined together with its own spatial coordinates.

The Bethe-Salpeter equation (BSE), proposed in the early 50s [1, 2], is an attempt to put bound states composed of two or more elementary particles on a fully covariant relativistic formalism. The construction of the Bethe-Salpeter amplitude (BSA) relies on the so-called Feynman two-body interaction kernel and on the proof that its usual power series expansion can be re-expressed as an integral equation. Studying the analyticity in the complex-energy plane, Mandelstam [3] noted that the homogeneous BSE for bound states is a pole of the four-point Green's function for on-shell total momentum.

The integral equation that defines the BSE is intrinsically related to singularities and branch points (cuts) of the amplitude along the real axis of the relative energy in Minkowski space, which make the procedure to solve the BSE in Minkowski space a hard task. The first successful attempt to obtain a solution of the BSE was achieved within the Wick-Cutkosky model [4], where two scalar fields interact with each other by a scalar field with null mass, in the first order of the ladder approximation for the interaction kernel. Wick formulated the BSE in Euclidean space by rotating the relative energy in the complex plane, which leads to a well defined non-singular integral equation which can be solved by standard numerical methods. This method simplifies the analytical structure of the BSE considerably and remained for several decades as one of the main recipes for solving the BSE. Using Wick rotation prescription a wide range of bound-state systems were proposed and investigated in Euclidean space. A quite complete bibliography of the works developed in the first years can be found in [5]. Furthermore, a formalism for the two-fermion bound states within the Bethe-Salpeter approach in Euclidean space was developed in [6] and references quoted therein.

Still in Euclidean space, the BSE was solved for the $\chi^2\phi$ model (scalar-scalar bound states) by using the Feynman-Schwinger representation. The set of all ladder and crossed-ladder irreducible diagrams was considered for the two-body interaction kernel and it was found that the crossed ladders contribute significantly to the binding energy [7].

The intrinsic nonperturbative nature of the BSE turns out for its utility in hadron physics, (see, e.g., [8] and references quoted therein). However, for the moment, the price one has to pay to deal with more realistic systems is to construct a kind of hybrid models which include a mapping between the Euclidean and Minkowski solutions and thus a general study of the analytic continuation of the BSE is required. After the Wick rotation has been performed, the backward analytic continuation is quite difficult even for the ladder approximation, and for more complicated cases its proper implementation is unclear or, at least, highly non-trivial. In this sense, solutions of the BSE directly in Minkowski space could be very helpful in order to better understand the analytic continuation in the complex plane.

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Using the Nakanishi integral representation (NIR) for the three-point vertex function and the nonperturbative conjecture of uniqueness, one of the first off-shell solutions of the BSE for bound states directly in Minkowski space was developed by Kusaka et al [9]. However, the method to deal with the singularities was quite cumbersome and still only applicable for the $\chi^2\phi$ model for the first order in the ladder approximation. The solution of the $\chi^2\phi$ model with dressed propagators was studied in [10].

Another proposal to solve the BSE in Minkowski space takes advantage of the NIR for the Bethe-Salpeter amplitude (BSA) together with the light-front (LF) projection of the BSA [11, 12, 13]. This approach improves the method proposed by Kusaka in order to easily treat the singularities associated with the interaction kernel, allowing the solution of the BSE in Minkowski space taking into account higher orders of the ladder approximation [14]. Furthermore, it is well known that the BSA projected in LF is the valence component of the Fock space [15, 16]. Therefore, projecting the BSA in the LF one can somehow recover the probabilistic interpretation of the amplitude. Since then, scrutinies studies have been developed in order to achieve solutions of the BSE in Minkowski space. Based on the NIR and also the nonperturbative conjecture of uniqueness, the half-off-shell solution for scattering states composed by two massive scalars interacting through the exchange of a massive scalar particle was computed in [12, 17].

Recently a new method to access the off-mass-shell scattering amplitude by solving the BSE in Minkowski space was developed based on the direct integration of the singularities presented in the propagators and in the interaction kernel [18], without resorting to the NIR. Although the method seems to be quite cumbersome even for spinless particles interacting by a one-boson exchange, it can be a useful technique when other components beyond the valence are relevant.

In possession of the off-shell BSA for bound and scattering states, observables like electromagnetic elastic form factor [19] as well as the transitions one [20], can be easily computed even considering cross-ladder kernels [21]. At this point it is interesting to note that the availability of the BSA in Minkowski space is necessary once the Wick rotation is not well-defined in the integrals of the form factor and then, calculations in Euclidean and Minkowski space can show considerable differences [19]. The method for the solution of the BSE in Minkowski space based on the NIR was also used to achieve the structure for the excited states [22] and was successfully extended to solve the two-fermion bound state [23].

Finally, the fact that the NIR is metric independent means that it might be plausible to extract the Nakanishi weight function from Euclidean space [24]. If so, we could propose a useful method for the mapping of the Euclidean amplitudes computed, e.g. in Lattice QCD, to Minkowski space. All issues concerning the differences between observables calculated in Euclidean/Minkowski spaces are necessary to clarify the analytical continuation in the complex plane and make the solutions of the BSE in Minkowski space an essential field to be further explored.

2. Solution of the $\chi^2\phi$ model in Minkowski space

The BSE for the bound state, with total momentum $p^2 = M^2$, of two massive scalar particles of mass m is given by

$$\Phi_b(k,p) = \frac{\mathrm{i}}{[(p/2+k)^2 - m^2 + \mathrm{i}\varepsilon]}\frac{\mathrm{i}}{[(p/2-k)^2 - m^2 + \mathrm{i}\varepsilon]}\int \frac{d^4k'}{(2\pi)^4}\mathrm{i}K(k,k',p;\alpha)\Phi_b(k',p). \quad (2.1)$$

Here i$K(k,k',p;\alpha)$ is the two-particle interaction kernel that contains all the irreducible diagrams, k is their relative momentum and α a dimensionless coupling constant. To solve Eq. (2.1) we used the method developed in [11], based on writing the BSA by means of the NIR and project the BSE on the light-front. The BSA for $s-$wave in Eq. (2.1) is written in NIR as:

$$\Phi_b(k,p) = \mathrm{i}\int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{g_b(\gamma',z';\kappa^2)}{[\gamma' + m^2 - \frac{M^2}{4} - k^2 - p.kz' - \mathrm{i}\varepsilon]^3}, \tag{2.2}$$

where $B(n) = 2m - M > 0$ is the binding energy of the nth state. By substituting Eq. (2.2) into Eq. (2.1) and integrating over k^- on both sides, one obtains a generalized integral equation for the Nakanishi weight function which is solved by standard numerical methods. The valence LF wave function, calculated in terms of the Nakanishi weight function [22], is shown in Fig. 1. A

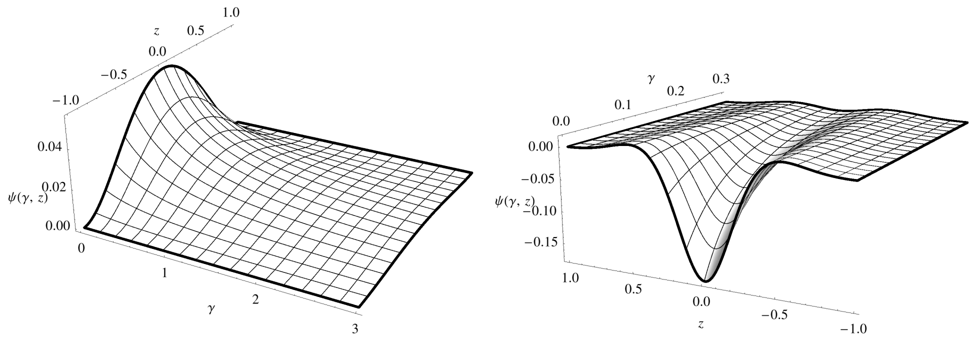

Figure 1: The valence LF wave function are shown for the ground and first excited states, $B(0) = 1.9m$ (left frame) and $B(1) = 0.22m$ (right frame), respectively; obtained for the first order of the ladder approximation with the mass of the exchange particle given by $\mu = 0.1\,m$ and α =6.437. Units of $m = 1$.

comparison between the eigenvalues calculated in Euclidean and Minkowiski space can be found in [22]. It's worthwhile to emphasize that the method using the NIR gives not only access to the binding energy but also to the wave function itself.

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