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Cylindrical symmetry: An aid to calculating the zeta-function in 3 + 1 dimensional curved space

Gopinath Kamath [1]
Department of Mathematics, Indian Institute of Technology Tirupati,
Renigunta Road, Tirupati 517 506,India
E-mail: gkamath01865@gmail.com

The spherically symmetric Schwarzschild solution is a staple of textbooks on general relativity; not so perhaps, the static but cylindrically symmetric ones, though they were obtained almost contemporaneously by H. Weyl, Ann.Phys.Lpz.**54** (1917) 117 and T. Levi-Civita, Atti Acc. Lincei Rend. **28** (1919) 101. A renewed interest in this subject in C.S. Trendafilova and S.A. Fulling , Eur.J.Phys. **32** (2011) 1663 – to which the reader is referred to for more references – motivates this work; thus, we rework the Antonsen-Bormann idea – arXiv:hep-th/9608141v1 – that was originally intended to compute the heat kernel in curved space, to determine – following D.McKeon and T.Sherry, Phys.Rev.D**35** (1987) 3584 – the zeta-function associated with the Lagrangian density for a massive real scalar field theory in 3 + 1 dimensional stationary curved space, the metric for which is cylindrically symmetric. As a calculation, it pays to use a metric characterised by the parameters j,k with $j = -4$ and $k = -4$, j,k being integer solutions to $2(j+k) = -jk$. Importantly, this enables – unlike the obvious solution $j = 2,\ k = -1$, an easy evaluation of the momentum integrals implied in the Schwinger expansion for the zeta-function. Happily, the work reported here is easy to go through – relative to that presented by the author at ICHEP2014 with the Schwarzschild metric, and this contrast will be taken up in some detail.

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Two integrals and their evaluation

Consider the integral

$$(\pi)^2 I = \int_r e^{-s(1-u)p^2} K(r,p) e^{-usr^2}\, , K(r,p) = \frac{\left(r^2 - \vec{r}\cdot\vec{p}\right)\left(p^2 - \vec{p}\cdot\vec{r}\right)}{\left(\vec{r}-\vec{p}\right)^4} \tag{1}$$

with $\int_r \equiv \int_{-\infty}^{\infty} d^2 r$, s and u being non-negative constants with the latter less than 1; with standard methods one easily gets

$$2\pi\, I = e^{-sp^2}\left(2p^2 + z\left(e^{az}-1\right) - p^2 \int_0^z dt\, \frac{e^{at}}{t}\right), z = \frac{1}{us}, a = \frac{p^2}{z^2} \tag{2}$$

As a second example with $m = s(1-u), n = us(1-w)$ let's calculate

$$(\pi)^3 J = \int_{rj} e^{-mp^2} K(r,j,p) e^{-nr^2} e^{-uswj^2}\, , K(r,j,p) = \frac{r^2 - \vec{r}\cdot\vec{p}}{\left(\vec{r}-\vec{p}\right)^2}\,\frac{j^2 - \vec{r}\cdot\vec{j}}{\left(\vec{r}-\vec{j}\right)^2}\,\frac{p^2 - \vec{p}\cdot\vec{j}}{\left(\vec{p}-\vec{j}\right)^2} \tag{3}$$

with $\int_r \equiv \int_{-\infty}^{\infty} d^2 r$,as before and s, u and w being non-negative constants with each of the latter two less than 1.

With two Feynman parameters a, b one gets with $h = a + b + n, \vec{k} = a\ \vec{p} + b\ \vec{j},$

$$\int_r \frac{r^2 - \vec{r}\cdot\vec{p}}{\left(\vec{r}-\vec{p}\right)^2}\,\frac{j^2 - \vec{r}\cdot\vec{j}}{\left(\vec{r}-\vec{j}\right)^2}\, e^{-nr^2} = \frac{\pi}{2}\int_0^\infty \frac{da db}{h^3}\, e^{-ap^2 - bj^2}\, e^{\frac{k^2}{h}}\, N(p,j) \tag{4}$$

where

$$N(p,j) = h\,\vec{j}\cdot\vec{p} + 2\vec{j}\cdot\vec{k}\,\vec{k}\cdot\vec{p} + 2j^2\left(h + k^2 - h\vec{k}\cdot\vec{p}\right) - \frac{1}{h}\left(4h\,\vec{j}\cdot\vec{k} + 2k^2\,\vec{j}\cdot\vec{k}\right)$$

The inconvenience inherent to the integration over a, b in (4) suggests a rewrite of eq.(3) as

$$(\pi)^3 J = \int_{rja} e^{-mp^2} K(r,a,j,p) e^{-nr^2} e^{-uswj^2}\ ,\ K(r,a,j,p) = \frac{r^2 - \vec{r}\cdot\vec{p}}{\left(\vec{r}-\vec{p}\right)^2}\,\frac{j^2 - \vec{a}\cdot\vec{j}}{\left(\vec{a}-\vec{j}\right)^2}\,\frac{p^2 - \vec{p}\cdot\vec{j}}{\left(\vec{p}-\vec{j}\right)^2}\,\delta^{(2)}\left(\vec{r}-\vec{a}\right)$$

and the integral representation for the delta-function now helps to obtain

$$R \equiv \int_{ra} \frac{r^2 - \vec{r}\cdot\vec{p}}{\left(\vec{r}-\vec{p}\right)^2}\,\frac{j^2 - \vec{a}\cdot\vec{j}}{\left(\vec{a}-\vec{j}\right)^2}\,\delta^{(2)}\left(\vec{r}-\vec{a}\right) e^{-nr^2} = 2\pi\, e^{-np^2}\left(I_1 + I_2\right) \tag{5}$$

where

$$I_1 = \int_0^\infty \frac{db}{g}\, e^{ib\left(n^2p^2 + i\varepsilon\right)} L(p,j)\ ,\ I_2 = -\frac{1}{4m}\int_0^\infty \frac{db}{q}\, e^{r\left(n^2p^2 + i\varepsilon\right)} M(p,j)$$

with b a Feynman parameter,

$$2i\vec{s} = \left(\vec{j} - (1 - ibn)\vec{p}\right), i\vec{x} = \left(\vec{j} - (1 - rn)\vec{p}\right), nr = (1 + ibn), g = 4s^2, q = x^2$$

and

$$L(p,j) = A\left(\vec{j},\vec{p},\vec{s}\right) + e^{-\frac{ig}{b}} B\left(\vec{j},\vec{p},\vec{s}\right)\quad ,\ A\left(\vec{j},\vec{p},\vec{s}\right) = -\left(np^2 + \frac{2i}{g}\,\vec{p}\cdot\vec{s}\right)\vec{j}\cdot\vec{s} + \frac{i}{4}\,\vec{j}\cdot\vec{p}\, ,$$

$$B\left(\vec{j},\vec{p},\vec{s}\right) = -A\left(\vec{j},\vec{p},\vec{s}\right) - \frac{2}{b}\,\vec{p}\cdot\vec{s}\ \vec{j}\cdot\vec{s}\ ,\ M(p,j) = C\left(\vec{j},\vec{p},\vec{z}\right) + e^{\frac{q}{r}} D\left(\vec{j},\vec{p},\vec{z}\right)\, ,$$

$$C\left(\vec{j},\vec{p},\vec{x}\right) = -in\left(\vec{j}\cdot\vec{p} - \frac{2}{q}\,\vec{p}\cdot\vec{x}\ \vec{j}\cdot\vec{x}\right), D\left(\vec{j},\vec{p},\vec{x}\right) = -C\left(\vec{j},\vec{p},\vec{x}\right) + \frac{2}{r}\,\vec{j}\cdot\vec{x}\left(\frac{q}{r} + in\vec{p}\cdot\vec{x}\right) \tag{6}$$

As eq.(5) is easier to work with than (4) and the calculation of J needs just the evaluation of

$$2\pi e^{-np^2}\int_j (I_1+I_2)\frac{p^2-\vec{p}\cdot\vec{j}}{(\vec{p}-\vec{j})^2}e^{-uswj^2} \quad (7)$$

it pays to pause the calculation at this stage so as to net more dividends elsewhere with the approach adopted here.

A physical setting for I and J

Consider the following line element in 3 + 1 dimensional curved space

$$ds^2=\frac{a^4}{r^4}c^2dt^2-\frac{a^8}{r^8}dr^2-r^2d\theta^2-\frac{a^4}{r^4}dz^2 \quad (8)$$

with $r^2=x^2+y^2$, and a a constant. Eq.(8) yields a time – independent, cylindrically symmetric metric $g_{\mu\nu}$ whose non –zero elements in Cartesian coordinates are

$$g_{00}=\frac{a^4}{r^4},g_{11}=-\frac{a^8x^2}{r^{10}}-\frac{y^2}{r^2},g_{12}=-\frac{a^8xy}{r^{10}}+\frac{xy}{r^2},g_{22}=-\frac{a^8y^2}{r^{10}}-\frac{x^2}{r^2},g_{33}=-\frac{a^4}{r^4} \quad (9)$$

the others being zero. With $g_{\mu\nu}=\eta_{ab}e^a_\mu e^b_\nu$, $\eta_{ab}=diag(1-1-1-1)$ the vierbeins e^a_μ – repeated latin and greek indices are summed from 0 to 3 respectively – can now be worked out and one favours the following set for its calculational advantage :

$$e^0_0=0,\ e^1_0=\frac{ia^2}{r^2\sqrt{2}},e^2_0=-\frac{ia^2}{r^2\sqrt{2}},e^3_0=0\ ;\ e^0_1=\frac{ia^4x}{r^5},\quad e^1_1=0,\quad e^2_1=0,\quad e^3_1=\frac{y}{r}$$
$$e^0_2=\frac{ia^4y}{r^5},\quad e^1_2=0,\quad e^2_2=0,\quad e^3_2=-\frac{x}{r}\ ;\ e^0_3=0,\ e^1_3=\frac{a^2}{r^2\sqrt{2}},\ e^2_3=\frac{a^2}{r^2\sqrt{2}},\ e^3_3=0 \quad (10)$$

Their inverses e^μ_b got from $e^a_\mu e^\mu_b=\delta^a_b$ work to

$$e^0_0=0,\ e^1_0=-\frac{ixr^3}{a^4},e^2_0=-\frac{iyr^3}{a^4},\quad e^3_0=0\ ;\quad e^0_1=-\frac{ir^2}{a^2\sqrt{2}},\ e^1_1=0,\quad e^2_1=0,\quad e^3_1=\frac{r^2}{a^2\sqrt{2}}$$
$$e^0_2=\frac{ir^2}{a^2\sqrt{2}},\ e^1_2=0,\quad e^2_2=0,\quad e^3_2=\frac{r^2}{a^2\sqrt{2}};\quad e^0_3=0,\ e^1_3=\frac{y}{r},\quad e^2_3=-\frac{x}{r},\quad e^3_3=0 \quad (11)$$

and determine $g^{\mu\nu}=\eta^{ab}e^\mu_a e^\nu_b$ with $\eta^{ab}=diag(1-1-1-1)$. Eqs.(1) and (3) can now be motivated with the Lagrangian density for a real massive scalar field in 3 + 1 dimensional curved space namely,

$$L=\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}m^2\phi^2 \quad (12)$$

Following Antonsen and Bormann [1,2] the operator $B=-\partial_\mu(g^{\mu\nu}\partial_\nu)-m^2$ associated with (12) is first reworked as

$$\begin{aligned}B&=-\partial_\mu(\eta^{ab}e^\mu_a e^\nu_b\partial_\nu)-m^2=-\eta^{ab}\partial_a\partial_b-m^2-e^a_\mu\partial_a(e^\mu_b)\partial^b\\&=p_0^2-\vec{p}^2-m^2-e^a_\mu\partial_a(e^\mu_b)\partial^b\Rightarrow p_0^2+\vec{p}^2-m^2-e^a_\mu\partial_a(e^\mu_b)\partial^b\end{aligned} \quad (13)$$

in momentum space,with (13) being obtained from the transition to Euclidean space. With the vierbeins in eqs.(10) and (11) the last term in (13) labelled heretofore as H_2 works to $H_2=\frac{2i}{r^2}(xp_1+yp_2)$and with $H_0=p_0^2+p_3^2-m^2,I_0=p_1^2+p_2^2$ one now has $e^{-sB}=e^{-sH_0}e^{-s(I_0+H_2)}$ with $B=H_0+I_0+H_2$ and s a scalar. Following McKeon and Sherry [3] the zeta-function $\zeta(s)$ is now defined as

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty du\,u^{s-1}tr(e^{-Bu})=\frac{1}{\Gamma(s)}\int_0^\infty du\,u^{s-1}tr\left(e^{-uH_0}e^{-u(I_0+H_2)}\right) \quad (14)$$

and one can now write the Schwinger expansion[3,4] to the third order for the operator $e^{-u(I_0+H_2)}$; it is

$$e^{-(I_0+H_2)u} = e^{-I_0 u} + (-u)e^{-I_0 u}H_2 + \frac{(-u)^2}{2!}\int_0^1 dw e^{-u(1-w)I_0}H_2 e^{-uwI_0}H_2$$
$$+\frac{(-u)^3}{3!}\int_0^1 wdw\int_0^1 dw_1 e^{-u(1-w)I_0}H_2 e^{-uw(1-w_1)I_0}H_2 e^{-uww_1 I_0}H_2 + \ldots \tag{15}$$

The integrals I and J are the matrix elements of the third and fourth terms in eq.(15) respectively, they being determined as in Ref.3 and the second term likewise works to $L = -\frac{2ie^{-u\vec{p}^2}}{(2\pi)^2}\int_x \frac{(\vec{r}\cdot\vec{p})}{r^2+i\varepsilon} = 0$.With eq.(2),one now has the main result of this report – inspired by cylindrical symmetry in 3 + 1 dimensional curved space using the method of operator regularization[3,4] to 1- loop order – viz.

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty du\, u^{s-1} tr\left(e^{-u(H_0+p^2)}\left\{1+\frac{(-u)^2}{4\pi}\int_0^1 dw\left(2p^2 + z(e^{az}-1) - p^2\int_0^z dt\frac{e^{at}}{t}\right)\right\}\right) \tag{16}$$

with $uwz = 1, az^2 = p^2$ in eq.(16). Parenthetically,a similar derivation has been carried out elsewhere[5] but with the BTZ metric [6] in 2 + 1 dimensions.

Qualitatively, the calculation presented here is viable relative to that with the spherically symmetric Schwarzschild metric[7].Put differently, the counterpart of H_2 above but with the Schwarzschild metric is far too cumbersome to make an exact evaluation[8] of even the third term in eq.(15) in terms of matrix elements worthwhile; on the other hand, the exact evaluation of eq.(1) above and its further use to get eq.(16) here underlines our assertion that with cylindrical symmetry – and an appropriate choice of the vierbeins e^a_μ – an easier calculation happens. To emphasise this latter remark, note that eq.(8) is just one example of a more general cylindrically symmetric solution discussed recently by Trendafilova and Fulling[9] namely,

$$ds^2 = -r^{2a}dt^2 + r^{2(a+b)}dr^2 + r^2 d\theta^2 + r^{2b}dz^2 \tag{17}$$

with the parameters a, b being solutions of $ab = -(a+b)$. Continuing, it is easy to check that the form of H_2 with the appropriate e^a_μ for

$$ds^2 = \frac{r^2}{b^2}c^2dt^2 - \frac{r}{b}dr^2 - r^2d\theta^2 - \frac{b}{r}dz^2 \tag{18}$$

with b a constant in (18) has the same infirmity as that with the spherically symmetric metric of Schwarzschild[7]; in conclusion, eqs.(8) and (18) are just special cases of

$$ds^2 = r^j c^2dt^2 - r^{(j+k)}dr^2 - r^2d\theta^2 - r^k dz^2 \tag{19}$$

where $2(j+k) = -jk$,but the edge the former has over the latter for the derivation reported here spurs a second look at (16) using Ref.3 to determine the one-loop corrected propagator in curved space-time; this will be dealt with elsewhere.

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