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Sub-gauge Conditions for the Gluon Propagator Singularities in Light-Cone Gauge

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Perturbative QCD calculations in the light-cone gauge have long suffered from the ambiguity associated with the regularization of the poles in the gluon propagator. Using the functional integral method, we re-derive the known sub-gauge conditions for the θ-function gauges and identify the sub-gauge condition for the principal value (PV) regularization of the gluon propagator's light-cone poles.

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1. Introduction

A popular gauge used in Quantum Chromodynamics (QCD) is the light-cone gauge due to the absence in this gauge of ghost fields. If the light-cone gauge is naively imposed to the QCD lagrangian, the resulting gluon propagator presents an unregulated singularity, the light-cone singularity. It is then a common procedure to choose an ad hoc prescription to regulate it. However, one should wonder whether such prescription is consistent with the quantization of the theory. In ref. [1] we have shown within the functional integration formalism that a proper way to quantize the theory in the light-cone gauge is to impose also appropriate sub-gauge conditions. Different sub-gauge conditions generate different prescriptions of the light cone singularity.

Let us introduce two light-cone vectors η^μ and $\tilde{\eta}^\mu$ such that $x\cdot\eta = x^+ = \frac{x^0+x^3}{\sqrt{2}}$ and $x\cdot\tilde{\eta} = x^- = \frac{x^0-x^+}{\sqrt{2}}$. In the $A^+=0$ gauge the propagator is

$$D^{\mu\nu}(x,y) \equiv \langle 0|\mathrm{T}A^\mu(x)A^\nu(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-i}{k^2+i\varepsilon}\left[g^{\mu\nu} - \frac{k^\mu\eta^\nu + k^\nu\eta^\mu}{k^+}\right] \tag{1.1}$$

Propagator (1.1) has light-cone singularity $\frac{1}{k^+}$ with no prescription and is not consistent with path integral formalism and the Fadeev-Popov method. Indeed, in order to perform functional integration of correlation functions, one has to make sure that the gauge is properly fixed to avoid integration over gauge orbits. Once the light-cone gauge is fixed, the lagrangian is still invariant under x^- independent gauge transformation and until the Lagrangian is not fully fixed the functional integration cannot be performed. In the next section we will show how to identify appropriate sub-gauge conditions.

The most commonly used regularization prescriptions for the $k^+=0$ pole of the gluon light-cone gauge propagator are:

- θ-function sub-gauges [2, 3, 4]:

$$D_1^{\mu\nu}(x,y) \equiv \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-i}{k^2+i\varepsilon}\left[g^{\mu\nu} - \frac{k^\mu\eta^\nu}{k^+-i\varepsilon} - \frac{k^\nu\eta^\mu}{k^++i\varepsilon}\right], \tag{1.2}$$

$$D_2^{\mu\nu}(x,y) \equiv \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-i}{k^2+i\varepsilon}\left[g^{\mu\nu} - \frac{k^\mu\eta^\nu}{k^++i\varepsilon} - \frac{k^\nu\eta^\mu}{k^+-i\varepsilon}\right]. \tag{1.3}$$

  The name stems from the fact that the classical field of a point (color) charge moving along the $x^-=0$ light cone is proportional to $A^\mu_\perp \sim \theta(-x^-)$ in the first case and $A^\mu_\perp \sim \theta(x^-)$ in the second case.

- Principal value (PV) sub-gauge [5]

$$D_{PV}^{\mu\nu}(x,y) \equiv \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-i}{k^2+i\varepsilon}\left[g^{\mu\nu} - \left(k^\mu\eta^\nu + k^\nu\eta^\mu\right)\mathrm{PV}\left\{\frac{1}{k^+}\right\}\right]. \tag{1.4}$$

- Mandelstam–Leibbrandt (ML) prescription [6, 7]

$$D_{ML}^{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-i}{k^2+i\varepsilon}\left[g^{\mu\nu} - \frac{k^\mu\eta^\nu + k^\nu\eta^\mu}{k^++i\varepsilon k^-}\right]. \tag{1.5}$$

Each prescription corresponds to a specific choice of sub-gauge condition. This is true for all the above mentioned prescription except for the ML prescription for which no sub-gauge condition could be identified using functional integration formalism.

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2. θ-Function Sub-Gauges

In the case of temporal $A^0 = 0$ gauge the gluon propagator and the prescription for regulating the singularity at $k^0 = 0$ in it are obtained by imposing a sub-gauge condition at a specific point in time: $\vec{\partial}\cdot\vec{A}(t_0,\vec{x}) = 0$. Motivated by the $A^0 = 0$ gauge example, we impose the following sub-gauge condition:

$$\partial_{\perp\mu} A^{\mu}_{\perp}(x^+, x^- = \sigma, \vec{x}_\perp) = 0\ . \tag{2.1}$$

In other words, we require that the transverse divergence of the gauge field vanishes at $x^- = \sigma$ with the value of σ not specified yet. (In the $A^0 = 0$ gauge the corresponding time t_0 at which the sub-gauge condition is specified remains arbitrary.) Clearly, Eq. (2.1) is not the only sub-gauge choice that can be made. For example, an alternative gauge choice is to require that the four-divergence is zero at a generic point in x^-, $\partial_\mu A^\mu(x^+, x^- = \sigma, \vec{x}_\perp) = 0$. However, this sub-gauge choice is not supported by the functional integral calculation [1].

In the functional integral formalism the propagator is obtained by applying functional derivatives of the generating functional with respect to the sources,

$$\begin{aligned}\langle 0|\mathrm{T}A_\mu(x)A_\nu(y)|0\rangle &= -\left[\frac{\delta}{\delta J^\mu(x)}\frac{\delta}{\delta J^\nu(y)}\, e^{-\frac{1}{2}\int d^4x' d^4y'\, J^\alpha(x') D_{\alpha\beta}(x',y') J^\beta(y')}\right]\Bigg|_{J=0} \\ &= -\left[\frac{\delta}{\delta J^\mu(x)}\frac{\delta}{\delta J^\nu(y)}\left(\frac{Z[J]}{Z[0]}\right)\right]\Bigg|_{J=0},\end{aligned} \tag{2.2}$$

where $D_{\mu\nu}(x,y)$ is the gluon propagator and $Z[J]$ is the generating functional. To arrive at the expression for the gluon propagator $D_{\mu\nu}(x,y)$ (with regularizations for all the poles in momentum space) using the functional integration for constructing the generating functional used in (2.2), one has to take special care of the surface terms arising from integration by parts and of the gauge conditions. In what follows we will consider the x^+ variable as time, and will define the initial and final conditions at the light-cone times x_i^+ and x_f^+ respectively. It will be implied that x_i^+ is large and negative while x_f^+ is large and positive. In addition we assume that the system is localized in space but not in time: since now x^+ is our time variable, instead of the "standard" assumption that all fields go to zero as $|\vec{x}| \to \infty$, we will assume that the fields go to zero as $|\vec{x}_\perp| \to \infty$. As will become apparent below, careful treatment will be needed of the functional integral at the boundaries in x^+ and x^- directions.

The generating functional for an Abelian gauge theory in the light-cone gauge with the sub-gauge condition (2.1) is

$$\begin{aligned}Z[J] = &\lim_{\xi_1,\xi_2\to 0}\int \mathscr{D}A_i\,\mathscr{D}A_f\,\Psi_0(A_i)\Psi_0^*(A_f) \\ &\times \int_{A(x_i^+,x^-,\vec{x}_\perp)=A_i}^{A(x_f^+,x^-,\vec{x}_\perp)=A_f} \mathscr{D}A_\mu \exp\left\{ i\int_{x_i^+}^{x_f^+} dx^+ \int dx^-\, d^2x_\perp \left[\mathscr{L}_0(A) + \mathscr{L}_{fix}(A) + J_\mu A^\mu\right]\right\}\end{aligned} \tag{2.3}$$

with

$$\mathscr{L}_0(A) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) \tag{2.4}$$

and the gauge and sub-gauge fixing terms

$$\mathscr{L}_{fix}(A) = -\frac{1}{2\xi_1} A_\mu \eta^\mu \eta^\nu A_\nu - \frac{1}{2\xi_2} \left(\vec{\partial}_\perp \cdot \vec{A}_\perp\right)^2 \delta(x^- - \sigma). \tag{2.5}$$

The generating functional in Eq. (2.3) can also be thought of as describing the Abelian part of a non-Abelian theory such as gluodynamics. Notice that, as discussed above, in the generating functional (2.3) we have used the light-cone coordinates with x^+ as the time direction. As is usually done, we have exponentiated the gauge conditions and the parameters ξ_1 and ξ_2 will be sent to zero at the end of the calculation.

In Eq. (2.3) $\Psi_0(A)$ represents the vacuum wave function in the A_μ-representation. In the light-cone gauge it is

$$\Psi_0(A) = \exp\left\{\frac{1}{2}\int dx^- d^2x_\perp A^\mu \sqrt{-(\partial^+)^2} A_\mu\right\}. \tag{2.6}$$

The expression Eq. (2.6) can be obtained by starting with the vacuum wave function in the $A^0 = 0$ gauge

$$\Psi_0(A) = \exp\left\{-\frac{1}{2}\int d^3x A^i \sqrt{-\vec{\nabla}^2}\left[\delta^{ij} - \frac{\partial^i\partial^j}{\vec{\nabla}^2}\right] A^j\right\}, \tag{2.7}$$

(with $\vec{\nabla} = (\partial_x, \partial_y, \partial_z)$ and $i, j = 1,2,3$ only in this formula) and performing an ultra-boost along the $+z$ direction to change the gauge into the $A^+ = 0$ gauge and the wave function Eq. (2.7) into Eq. (2.6).

It is known that one of the advantages of using axial-type gauge conditions is the absence of ghost fields. However, now, in addition to the light-cone gauge, we have a sub-gauge condition Eq. (2.1) which introduces a non trivial determinant, leading to a ghost field $c(x)$ localized at $x^- = \sigma$:

$$\det\left[\partial^\perp_\mu \mathscr{D}^\mu_\perp(x^- = \sigma)\right] = \int \mathscr{D}\bar{c}\,\mathscr{D}c \exp\left\{-i\int dx^+ d^2x_\perp \bar{c}\,\partial^\perp_\mu \mathscr{D}^\mu_\perp c(x^- = \sigma)\right\}, \tag{2.8}$$

where $\mathscr{D}^{ab}_\mu \equiv \partial_\mu \delta^{ab} + g f^{acb} A^c_\mu$ is the covariant derivative and $\bar{c}(x)$ is the complex conjugate ghost field. Just like in Feynman gauge, the ghost field is needed only in the non-Abelian case. The ghost field does not affect the gluon propagator in question. The propagator of this ghost field, along with the ghost-gluon vertices, depend only on transverse momenta, and are independent of k^-. Because of that it appears that ghost loops are zero in calculations using dimensional regularization. Therefore, in Eq. (2.3) and in the subsequent analysis we omit ghost contributions arising from sub-gauge conditions.

In order to put Eq. (2.3) in the same form as the first line of Eq. (2.2), we will adopt the following standard procedure of "completing the square". First we perform a shift of the gauge field $A^\mu \to A^\mu + a^\mu$ and obtain

$$Z[J] = \lim_{\xi_1,\xi_2\to 0}\int \mathscr{D}A_i\,\mathscr{D}A_f\,\Psi_0(A_i)\Psi_0^*(A_f)\,\Psi_0(a_i)\Psi_0^*(a_f)$$

$$\times\exp\left\{\int dx^- d^2x_\perp\left(A^\mu_i\sqrt{-(\partial^+)^2}\,a_{i\mu} + A^\mu_f\sqrt{-(\partial^+)^2}\,a_{f\mu}\right)\right\}\int_{A(x_i^+,x^-,\vec{x}_\perp)=A_i}^{A(x_f^+,x^-,\vec{x}_\perp)=A_f} \mathscr{D}A_\mu \exp\left\{i\int_{x_i^+}^{x_f^+} dx^+\right.$$

$$\times \int dx^- \, d^2x_\perp \Bigg[\mathscr{L}_0(A) + \mathscr{L}_{fix}(A) + \mathscr{L}_0(a) + \mathscr{L}_{fix}(a) + J^\mu A_\mu + J^\mu a_\mu - (\partial_\mu A_\nu)(\partial^\mu a^\nu)$$

$$+(\partial_\mu A_\nu)(\partial^\nu a^\mu) - \frac{1}{\xi_1} A_\mu \, \eta^\mu \, \eta^\nu \, a_\nu - \frac{1}{\xi_2} \left(\vec{\partial}_\perp \cdot \vec{A}_\perp \right) \left(\vec{\partial}_\perp \cdot \vec{a}_\perp \right) \delta(x^- - \sigma) \Bigg] \Bigg\} . \tag{2.9}$$

In arriving at Eq. (2.9) we have done an integration by parts in (parts of) the vacuum wave functions, discarding the two-dimensional boundary integral which is outside the precision of the approximation that was used in deriving Eq. (2.6). We now perform integration by parts in the terms linear in a^μ in the rest of the expression to arrive at

$$Z[J] = \lim_{\xi_1, \xi_2 \to 0} \int \mathscr{D}A_i \, \mathscr{D}A_f \, \Psi_0(A_i) \Psi_0^*(A_f) \, \Psi_0(a_i) \, \Psi_0^*(a_f)$$

$$\times \exp \left\{ \int dx^- d^2x_\perp \left(A_i^\mu \sqrt{-(\partial^+)^2} \, a_{i\,\mu} + A_f^\mu \sqrt{-(\partial^+)^2} \, a_{f\,\mu} \right) \right\} \int\limits_{A(x_i^+, x^-, \vec{x}_\perp) = A_i}^{A(x_f^+, x^-, \vec{x}_\perp) = A_f} \mathscr{D}A_\mu$$

$$\times \exp \Bigg\{ i \int_{x_i^+}^{x_f^+} dx^+ \int dx^- \, d^2x_\perp \Bigg[\mathscr{L}_0(A) + \mathscr{L}_{fix}(A) + \mathscr{L}_0(a) + \mathscr{L}_{fix}(a) + J^\mu A_\mu + J^\mu a_\mu +$$

$$+ A_\nu \left[\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right] a_\mu - \frac{1}{\xi_1} A_\mu \, \eta^\mu \, \eta^\nu \, a_\nu + \frac{1}{\xi_2} A_{\perp\mu} (\partial_\perp^\mu \partial_\perp^\nu a_{\perp\nu}) \, \delta(x^- - \sigma) \Bigg]$$

$$\times - i \int d\sigma_\mu \left[A_\nu (\partial^\mu a^\nu) - A_\nu (\partial^\nu a^\mu) \right] \Bigg\} . \tag{2.10}$$

where $d\sigma^\mu = \pm (d^2x_\perp \, dx^+ \, \tilde{\eta}^\mu + d^2x_\perp \, dx^- \, \eta^\mu + d\sigma_\perp^\mu)$ is the integration measure over the 3-dimensional surface of our four-dimensional space-time. Here $d\sigma_\perp^\mu$ is the integration measure over the surface at $x_\perp \to \infty$. The choice of a plus or minus in each of the terms depends on which boundary one is integrating over.

In order to "complete the square" we need to eliminate all the terms linear in A^μ in Eq. (2.10). Starting from the 4-dimensional volume integration terms we have to choose a^μ such that

$$A_\nu \left[\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right] a_\mu - \frac{1}{\xi_1} A_\mu \, \eta^\mu \, \eta^\nu \, a_\nu + \frac{1}{\xi_2} A_{\perp\mu} (\partial_\perp^\mu \partial_\perp^\nu a_{\perp\nu}) \, \delta(x^- - \sigma) + J_\mu A^\mu = 0 \tag{2.11}$$

for any A^μ. Solving for a^μ we get

$$a^\mu(x) = i \int d^4y D^{\mu\nu}(x,y) \, J_\nu(y) \tag{2.12}$$

where $D^{\mu\nu}(x,y)$ is the Green function found from

$$\left[\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu - \frac{1}{\xi_1} \eta^\mu \eta^\nu + \frac{1}{\xi_2} \partial_\perp^\mu \partial_\perp^\nu \, \delta(x^- - \sigma) \right] D_{\nu\rho}(x,y) = i \, \delta_\rho^\mu \, \delta^{(4)}(x-y). \tag{2.13}$$

The boundary conditions for Eq. (2.13) are obtained by requiring that the 3-dimensional *surface* integration terms linear in A^μ should also vanish in the exponent of Eq. (2.10),

$$\int dx^- \, d^2x_\perp \left(A_i^\mu \sqrt{-(\partial^+)^2} \, a_{i\,\mu} + A_f^\mu \sqrt{-(\partial^+)^2} \, a_{f\,\mu} \right)$$

$$- i \int d\sigma_\mu \left[A_\nu (\partial^\mu a^\nu) - A_\nu (\partial^\nu a^\mu) \right] = 0. \tag{2.14}$$

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Note that the condition Eq. (2.14) eliminates all the boundary term dependent on a^μ from the exponent of Eq. (2.10) (and not just the terms linear in A^μ). More precisely, for a^μ satisfying Eq. (2.14) one gets

$$\Psi_0(a_i)\Psi_0^*(a_f)\exp\Big\{\int dx^- d^2x\Big(A_i^\mu\sqrt{-(\partial^+)^2}a_{i\mu}+A_f^\mu\sqrt{-(\partial^+)^2}a_{f\mu}\Big)\Big\} \tag{2.15}$$
$$\times\exp\left\{-\frac{i}{2}\int d\sigma_\mu\Big(a_\nu(\partial^\mu a^\nu)-a_\nu(\partial^\nu a^\mu)\Big)-i\int d\sigma_\mu\Big(A_\nu(\partial^\mu a^\nu)-A_\nu(\partial^\nu a^\mu)\Big)\right\}=1.$$

With this in mind one can readily show that after using a^μ satisfying Eqs. (2.12), Eq. (2.13) and Eq. (2.14) in Eq. (2.10) the generating functional becomes

$$Z[J]=Z[0]\exp\left\{-\frac{1}{2}\int d^4x d^4y J_\mu(x)D^{\mu\nu}(x,y)J_\nu(y)\right\}. \tag{2.16}$$

From (2.16) we see that $D^{\mu\nu}(x,y)$ is indeed the gluon propagator, as defined in (2.2), obtained in the light-cone gauge with the sub-gauge condition (2.1).

We conclude that to find the gluon propagator we need to solve Eq. (2.13) and verify that the solution leads to a^μ satisfying Eq. (2.14).

For any $x^-\neq\sigma$ the general solution of Eq. (2.13) is

$$D^{\mu\nu}(x,y)|_{x^-\neq\sigma}=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{-i}{k^2}\left[g^{\mu\nu}-\frac{k^\mu\eta^\nu+k^\nu\eta^\mu}{k^+}\right], \tag{2.17}$$

where the regularization of the $k^2=0$ and $k^+=0$ poles is not specified on purpose, since the remaining uncertainty in this solution is solely due to the freedom to regulate these poles in various ways. Integrating Eq. (2.13) over x^- in an infinitesimal interval centered at σ and assuming that $D^{\mu\nu}$ is continuous we see that for $x^-=\sigma$ (and $y^-\neq\sigma$) the solution of (2.13) has to satisfy the following condition

$$\partial_\mu^\perp\partial_\rho^\perp D^{\rho\nu}(x,y)|_{x^-=\sigma}=0\,. \tag{2.18}$$

(One also obtains continuity of $\partial_- D_{+\rho}$ at $x^-=\sigma$.) The continuity of $D^{\mu\nu}$ implies that its value at $x^-=\sigma$ is fixed by Eq. (2.17), such that we can write

$$D^{\mu\nu}(x,y)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{-i}{k^2}\left[g^{\mu\nu}-\frac{k^\mu\eta^\nu+k^\nu\eta^\mu}{k^+}\right] \tag{2.19}$$

for all x^- with the only remaining freedom in this result being due to unspecified regularization of the $k^2=0$ and $k^+=0$ poles. In fact one may still have different regularizations (or linear combinations thereof) of the $k^2=0$ and $k^+=0$ poles for $x^->\sigma$ and $x^-<\sigma$ in Eq. (2.19). With the help of a direct calculation one can see that no regularization of the $k^2=0$ and $k^+=0$ poles in Eq. (2.17) would lead to Eq. (2.18) for an arbitrary finite value of σ and for all $x^+,\vec{x}_\perp$. This leaves $\sigma=\pm\infty$ as the only possibilities.

Let us first establish the Feynman prescription for the $k^2=0$ pole in Eq. (2.19). Picking up the $x^+=x_i^+$ and $x^+=x_f^+$ surfaces in Eq. (2.14) and using a^μ from Eq. (2.12) with the Green function from Eq. (2.19) (with $k^2 \to k^2 + i\varepsilon$) while keeping in mind that $a^+ = 0$ in Eq. (2.12) and $A^+ = 0$ due to $\xi_1 \to 0$ limit in Eq. (2.10) yields

$$\int dx^- d^2x_\perp A^\mu_\perp(x_i^+)\Big(\sqrt{-(\partial^+)^2} + i\partial^+\Big)\, a^\perp_\mu(x_i^+) \int d^4y\, dx^- d^2x_\perp A^\mu_\perp(x_i^+) \\ = \times \int \frac{d^4k}{(2\pi)^4}\frac{2k^+\theta(k^+)}{k^2+i\varepsilon}\Big(g^{\mu\nu}_\perp - \frac{k^\mu_\perp \eta^\nu}{k^+}\Big)\, e^{-ik^+(x^- - y^-) - ik^-(x_i^+ - y^+) + i\vec{k}_\perp\cdot(\vec{x}_\perp - \vec{y}_\perp)} = 0 \tag{2.20}$$

and

$$\int dx^- d^2x_\perp A_\perp(x_f^+)^\mu\Big(\sqrt{-(\partial^+)^2} - i\partial^+\Big)\, a^\perp_\mu(x_f^+) = -\int d^4y\, dx^- d^2x_\perp A^\mu_\perp(x_f^+) \\ \times \int \frac{d^4k}{(2\pi)^4}\frac{2k^+\theta(-k^+)}{k^2+i\varepsilon}\Big(g^{\mu\nu}_\perp - \frac{k^\mu_\perp \eta^\nu}{k^+}\Big)\, e^{-ik^+(x^- - y^-) - ik^-(x_f^+ - y^+) + i\vec{k}_\perp\cdot(\vec{x}_\perp - \vec{y}_\perp)} = 0. \tag{2.21}$$

To prove the validity of Eqs. (2.20) and (2.21), it is enough to observe that the direction of the k^- -contour closure is determined by the fact that $x_i^+ - y^+ < 0$ and $x_f^+ - y^+ > 0$ for all y^+, since x_i^+ is the initial and therefore the smallest x^+ value, while x_f^+ the final and therefore the largest x^+ value in the 4-volume considered. Eqs. (2.20) and (2.21) are zero independent of the regularization prescription for the $k^+ = 0$ pole, and hence do not allow us to fix this prescription. Note also that other regularizations of the $k^2 = 0$ pole would not satisfy both Eqs. (2.20) and (2.21).

We now write

$$D^{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-i}{k^2+i\varepsilon}\Big[g^{\mu\nu} - \frac{k^\mu\eta^\nu + k^\nu\eta^\mu}{k^+}\Big] \tag{2.22}$$

and directly face the need to regulate the $k^+ = 0$ pole as the only remaining ambiguity in the expression. Substituting Eq. (2.22) into Eq. (2.18) yields

$$\partial^\mu_\perp \int \frac{d^4k}{(2\pi)^4}\, \frac{e^{-ik^+(\sigma - y^-) - ik^-(x^+ - y^+) + i\vec{k}_\perp\cdot(\vec{x}_\perp - \vec{y}_\perp)}}{k^2+i\varepsilon}\Big(k^\nu_\perp + \frac{k^2_\perp \eta^\nu}{[k^+]}\Big) = 0 \tag{2.23}$$

where we have indicated with $[k^+]$ the prescription to be determined. Once again we see that for finite σ it is impossible to satisfy Eq. (2.23) and hence Eq. (2.18).

Since σ can not be finite, we consider $\sigma = +\infty$ first. In such case we need to close the k^+-integration contour in the lower half-plane. Before doing the calculation, it is already clear that our best chance of getting zero on the left-hand-side of Eq. (2.23) is to put $[k^+] = k^+ - i\varepsilon$, such that the light-cone pole would not contribute to the integral.

Using the following Fourier transform

$$\int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{1}{k^2+i\varepsilon}\Big(k^\nu_\perp + \frac{k^2_\perp\eta^\nu}{k^+ - i\varepsilon}\Big) \\ = \frac{(x-y)^\nu_\perp}{2\pi^2[(x-y)^2 - i\varepsilon]^2} + \eta^\nu\Big[\frac{(x^- - y^-)}{\pi^2[(x-y)^2 - i\varepsilon]^2} - i\delta^{(2)}(\vec{x}_\perp - \vec{y}_\perp)\delta(x^+ - y^+)\theta(y^- - x^-)\Big] \tag{2.24}$$

we see that using $[k^+] = k^+ - i\varepsilon$ satisfies Eq. (2.23) for $\sigma = +\infty$ since Eq. (2.24) is zero for $x^- = +\infty$. With this result we rewrite Eq. (2.22) as

$$D^{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-i}{k^2+i\varepsilon}\Big[g^{\mu\nu} - \frac{k^\mu\eta^\nu}{k^+ - i\varepsilon} - \frac{k^\nu\eta^\mu}{k^+}\Big]. \tag{2.25}$$

It may seem that there is still an unregulated pole at $k^+ = 0$ in the last term of the square brackets in Eq. (2.25). However, regularization of this last term can be fixed using the symmetry of the gluon propagator, $D^{\mu\nu}(x,y) = D^{\nu\mu}(y,x)$. This yields

$$D^{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-i}{k^2+i\varepsilon} \left[g^{\mu\nu} - \frac{k^\mu \eta^\nu}{k^+ - i\varepsilon} - \frac{k^\nu \eta^\mu}{k^+ + i\varepsilon} \right]. \tag{2.26}$$

The derivation is similar for the case of $\sigma = -\infty$. We employ

$$\begin{aligned} &\int \tfrac{d^4k}{(2\pi)^4} \tfrac{1}{k^2+i\varepsilon} \left(k_\perp^\nu + \tfrac{k_\perp^2 \eta^\nu}{k^+ + i\varepsilon} \right) e^{-ik\cdot(x-y)} \\ &= \tfrac{(x-y)_\perp^\nu}{2\pi^2[(x-y)^2 - i\varepsilon]^2} + \eta^\nu \left[\tfrac{(x^- - y^-)}{\pi^2[(x-y)^2 - i\varepsilon]^2} + i\delta^{(2)}(\vec{x}_\perp - \vec{y}_\perp)\delta(x^+ - y^+)\theta(x^- - y^-) \right] \end{aligned} \tag{2.27}$$

and observe that Eq. (2.27) is zero for $x^- = -\infty$. Thus Eq. (2.23) is satisfied for $[k^+] = k^+ + i\varepsilon$ and $\sigma = -\infty$.

To summarize, we obtain the following two sub-gauge conditions and the corresponding gluon propagators for $\sigma = \pm\infty$ [2, 3, 4]:

- Light-cone gauge gluon propagator for the sub-gauge condition $\vec{\partial}_\perp \cdot \vec{A}_\perp(x^- = +\infty) = 0$

$$D_1^{\mu\nu}(x,y) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-i}{k^2+i\varepsilon} \left[g^{\mu\nu} - \frac{k^\mu \eta^\nu}{k^+ - i\varepsilon} - \frac{k^\nu \eta^\mu}{k^+ + i\varepsilon} \right]; \tag{2.28}$$

- Light-cone gauge gluon propagator for the sub-gauge condition $\vec{\partial}_\perp \cdot \vec{A}_\perp(x^- = -\infty) = 0$

$$D_2^{\mu\nu}(x,y) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-i}{k^2+i\varepsilon} \left[g^{\mu\nu} - \frac{k^\mu \eta^\nu}{k^+ + i\varepsilon} - \frac{k^\nu \eta^\mu}{k^+ - i\varepsilon} \right]. \tag{2.29}$$

As a consistency check, we now need to show that when using the propagators (2.28) or (2.29), Eq. (2.14) is satisfied along the $x^- = \pm\infty$ surfaces, along with the $x_\perp = \infty$ boundary. (We have checked the $x^+ = x_i^+$ and $x^+ = x_f^+$ surfaces when deriving Feynman regularization in Eqs. Eq. (2.20) and Eq. (2.21).) Eq. (2.14) is trivially satisfied at the $x_\perp = \infty$ boundary, since we assumed initially that the system is localized in $x_\perp$ and all fields vanish when $x_\perp \to \infty$. We are left only with the $x^- = \pm\infty$ surfaces to consider, for which Eq. (2.14) reduces to

$$-i \int dx^+ d^2x_\perp \left[A_\nu(\partial^- a^\nu) - A_\nu(\partial^\nu a^-) \right] \Big|_{x^- = -\infty}^{x^- = +\infty} = 0. \tag{2.30}$$

Let us demonstrate that Eq. (2.30) is indeed valid for the case of $\vec{\partial}_\perp \cdot \vec{A}_\perp(x^- = +\infty) = 0$ sub-gauge. (The argument for the $\vec{\partial}_\perp \cdot \vec{A}_\perp(x^- = -\infty) = 0$ sub-gauge is constructed by analogy.) The a^μ-shift is (cf. Eq. (2.12))

$$a_1^\mu(x) = i \int d^4y D_1^{\mu\nu}(x,y) J_\nu(y) \,. \tag{2.31}$$

We now plug Eq. (2.31) into Eq. (2.30) and use Eq. (2.28) to integrate over k^+. Note that, just like in Eqs. Eq. (2.24) and Eq. (2.27), picking up the $k^2 = 0$ pole of the k^+-integral would give us a contribution which goes to zero as $x^- \to \pm\infty$. (Those contributions are given by the first term on the right-hand side of Eq. (2.24) and Eq. (2.27) and by the first term in the square brackets of the right-hand side of Eq. (2.24) and Eq. (2.27).) Only picking the $k^+ = 0$ pole may give a term (akin to the last terms in the square brackets on the right-hand side of Eq. (2.24) and Eq. (2.27)) which may potentially violate Eq. (2.30). Therefore, we substitute Eq. (2.31) into Eq. (2.30) and use Eq. (2.28) to integrate over k^+ picking up the $k^+ = 0$ poles only. Keeping in mind the $A^+ = 0$ gauge condition we write (see Ref. [1] for the details of the calculation)

$$\begin{aligned} & -i\int dx^+ d^2x_\perp \left[A_\nu(\partial^- a_1^\nu) - A_\nu(\partial^\nu a_1^-)\right]\Big|_{x^-=-\infty}^{x^-=+\infty} \\ & = \int d^4y\, dx^+\, d^2x_\perp J_\mu(y) \int \frac{d^2k_\perp\, dk^-}{(2\pi)^3}\, e^{-ik^-(x^+-y^+)+i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{y}_\perp)}\, \frac{-1}{k_\perp^2}\vec{\partial}_\perp\cdot\vec{A}_\perp(x^-=+\infty)\left(k^-\eta^\mu + k_\perp^\mu\right) \\ & = 0, \end{aligned} \tag{2.32}$$

Note that the contribution of the $k^2 = 0$ pole is independent of the regularization prescription for the $k^+ = 0$ pole: hence the conclusion of Appendix A of Ref. [1] is valid for all $k^+ = 0$ pole prescriptions.

Note that a 4-divergence sub-gauge condition, $\partial_\mu A^\mu(x^- = +\infty) = 0$, would not have led to zero in Eq. (2.32), and therefore does not correspond to propagator Eq. (2.28). For further reasons detailing why this is not a valid sub-gauge condition of the light-cone gauge see Appendix B of Ref. [1].

We have thus verified that a^μ from Eq. (2.12) with either one of the propagators Eq. (2.28) and Eq. (2.29) satisfies Eq. (2.14), while the propagators $D_1^{\mu\nu}(x,y)$ and $D_2^{\mu\nu}(x,y)$ solve Eq. (2.13) with $\sigma = \pm\infty$ respectively. Therefore, Eq. (2.16) is also verified, with $D_1^{\mu\nu}(x,y)$ and $D_2^{\mu\nu}(x,y)$ being valid light-cone gauge propagators satisfying corresponding sub-gauge conditions.

It is also easy to explicitly check that propagators $D_1^{\mu\nu}$ and $D_2^{\mu\nu}$ themselves respect the sub-gauge conditions

$$\begin{aligned} \partial_\mu^\perp D_1^{\mu\nu}(x,y)\Big|_{x^-=+\infty} &= 0\,, \\ \partial_\mu^\perp D_2^{\mu\nu}(x,y)\Big|_{x^-=-\infty} &= 0\,. \end{aligned} \tag{2.33}$$

Propagators (2.28) and (2.29) were already obtained by different procedures in [2, 3, 4]. We observe that in Ref. [4] the propagators (2.28) and (2.29) were obtained by imposing an additional sub-gauge condition, $A^-(x^- = \pm\infty) = 0$, while in the above procedure we showed that it is sufficient to assume that $\lim\limits_{x^-\to\infty}[A^-(x^-)/x^-] = 0$ (see Appendix **??**).

3. PV Sub-Gauge

In this section we will determine the sub-gauge condition that reproduces Principal Value (PV) prescription Eq. (1.4) for the k^+ pole in light-cone propagator. To this end, we will adopt the same procedure we used to arrive at propagators (2.28) and (2.29) with sub-gauge conditions $\vec{\partial}_\perp\cdot\vec{A}_\perp(x^- = +\infty) = 0$ and $\vec{\partial}_\perp\cdot\vec{A}_\perp(x^- = -\infty) = 0$ respectively, but in reverse order.

In the previous section we have assumed a sub-gauge condition Eq. (2.1), performed a shift of the field $A^\mu \to A^\mu + a^\mu$ in the generating functional, and made sure that the a^μ-dependent surface terms vanish (that is, Eq. (2.14) is satisfied) for the generating functional to reduce to the form given in (2.16).

As we do not know *a priori* the sub-gauge condition that reproduces the light-cone propagator with $k^+ = 0$ pole regulated by PV prescription, we consider from the start the propagator with the PV prescription and deduce the needed sub-gauge condition in order to put the generating functional in the form (2.16). In practical terms, we have to show that Eq. (15) is satisfied if we regulate the $k^+ = 0$ pole of the light-cone propagator with the PV prescription.

The gauge field propagator in the $A^+ = 0$ light-cone gauge with the PV-prescription is

$$D^{\mu\nu}_{PV}(x,y) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-i}{k^2+i\varepsilon}\left[g^{\mu\nu} - \left(k^\mu\eta^\nu + k^\nu\eta^\mu\right)\mathrm{PV}\left\{\frac{1}{k^+}\right\}\right] \tag{3.1}$$

where

$$\mathrm{PV}\left\{\frac{1}{k^+}\right\} \equiv \frac{1}{2}\left(\frac{1}{k^+-i\varepsilon} + \frac{1}{k^++i\varepsilon}\right). \tag{3.2}$$

Knowing the propagator means we know the shift field a^μ (cf. Eq. (2.12)),

$$a^\mu_{PV} = i\int d^4y\, D^{\mu\nu}_{PV}(x,y)\, J_\nu(y). \tag{3.3}$$

Let us plug the shift field Eq. (3.3) into Eq. (2.14) obtaining

$$\int dx^-\, d^2x_\perp \left(A^\mu_i\sqrt{-(\partial^+)^2}\,a^{PV}_{i\mu} + A^\mu_f\sqrt{-(\partial^+)^2}\,a^{PV}_{f\,\mu}\right)$$
$$-i\int d\sigma_\mu\left[A_\nu(\partial^\mu a^\nu_{PV}) - A_\nu(\partial^\nu a^\mu_{PV})\right] = 0 \tag{3.4}$$

and require that the latter is satisfied everywhere along the boundary of the four-dimensional space-time volume. Eq. (3.4) is satisfied at the $x^+ = x^+_i$ and $x^+ = x^+_f$ boundaries irrespective of the regularization of the $k^+ = 0$ pole, as follows from Eqs. Eq. (2.20) and Eq. (2.21). The boundary at $x_\perp \to \infty$ is also automatically satisfied, since we assumed from the start that all fields vanish as $x_\perp \to \infty$. We are only left with the boundary at $x^- = \pm\infty$. By analogy to Eq. (2.32) we evaluate the contributions of the $x^- = \pm\infty$ boundaries by neglecting the residues of $k^2 = 0$ pole in the propagator which vanish at those boundaries (see Ref. [1] for the details of the calculation):

$$0 = -i\int dx^+\, d^2x_\perp\left[A_\nu(\partial^- a^\nu_{PV}) - A_\nu(\partial^\nu a^-_{PV})\right]\Big|^{x^-=+\infty}_{x^-=-\infty}$$
$$= \int d^4y\, dx^+\, d^2x_\perp\, J_\mu(y)\int \frac{d^2k_\perp\, dk^-}{2(2\pi)^3} e^{-ik^-(x^+-y^+)+i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{y}_\perp)}\frac{-1}{k^2_\perp}\left(k^-\eta^\mu + k^\mu_\perp\right)$$
$$\times\left[\vec{\partial}_\perp\cdot\vec{A}_\perp(x^-=+\infty) + \vec{\partial}_\perp\cdot\vec{A}_\perp(x^-=-\infty)\right]. \tag{3.5}$$

We see that for the boundary condition in Eq. (3.5) to be satisfied, i.e. for the boundary term to vanish, one has to have the following sub-gauge condition:

$$\vec{\partial}_\perp\cdot\vec{A}_\perp(x^-=+\infty) + \vec{\partial}_\perp\cdot\vec{A}_\perp(x^-=-\infty) = 0. \tag{3.6}$$

We have thus arrived at the sub-gauge condition which leads to the k^+ pole in the gluon propagator regulated with the PV prescription. We can check the validity of the PV-sub-gauge condition (3.6) explicitly by using the PV-propagator:

$$\partial^{\perp}_{\mu} D^{\mu\nu}_{PV}(x,y)\Big|_{x^-=+\infty} + \partial^{\perp}_{\mu} D^{\mu\nu}_{PV}(x,y)\Big|_{x^-=-\infty} = 0. \tag{3.7}$$

In Ref. [1] we showed that the PV sub-gauge condition (3.6) is consistent with reproducing the classical gluon field generated by two ultrarelativistic quarks propagating along two parallel light-cones, whereas a stronger condition

$$\vec{A}_{\perp}(x^- = +\infty) + \vec{A}_{\perp}(x^- = -\infty) = 0, \tag{3.8}$$

while still satisfying Eq. (3.5) does not allow one to construct the classical field of the color charges at the non-Abelian level. Therefore, it is Eq. (3.6) which appears to be the correct sub-gauge condition in the PV case.

4. conclusion

We have shown that the ambiguity associated with the regularization of the poles of the light-cone gauge gluon propagator can be eliminated by fixing the residual gauge freedom using a sub-gauge condition. We saw that this is indeed the case for the θ-function sub-gauges and for the PV sub-gauge. As it is explicitly shown in Ref. [1], within the functional integration formalism it is not clear whether it is possible to find suitable sub-gauge conditions that justify Mandelstam-Leibbrandt prescription.

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