PROCEEDINGS OF SCIENCE

Computing the Inverse Mellin Transform of Holonomic Sequences using Kovacic's Algorithm

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We describe how the extension of a solver for linear differential equations by Kovacic's algorithm helps to improve a method to compute the inverse Mellin transform of holonomic sequences. The method is implemented in the computer algebra package HarmonicSums.

13th International Symposium on Radiative Corrections
24-29 September, 2017
St. Gilgen, Austria

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[†]This work has been supported in part by the Austrian Science Fund (FWF) grant SFB F50 (F5009-N15).

PoS(RADCOR2017)001

1. Introduction

There have been several methods proposed to compute the inverse Mellin transform of special sequences, for instance in [14] an algorithm (using rewrite rules) to compute the inverse Mellin transform of harmonic sums was stated. This algorithm was extended in [3] to generalized harmonic sums such as S-sums and cyclotomic sums. A different approach to compute inverse Mellin transforms of binomial sums was described in [4]. In [2] a method to compute the inverse Mellin transform of general holonomic sequences was described. That method uses holonomic closure properties and was implemented in the computer algebra package `HarmonicSums` [1, 3, 5, 6, 7]. In the frame of the method a linear differential equation has to be solved. So far the differential equations solver of `HarmonicSums` was only able to find d'Alembertian solutions [8]. Recently the solver was generalized and therefore more general inverse Mellin transforms can be computed.

In the following we repeat important definitions and properties (compare [2, 4, 11]). Let $\mathbb{K}$ be a field of characteristic 0. A function $f = f(x)$ is called *holonomic* (or *D-finite*) if there exist polynomials $p_d(x), p_{d-1}(x), \ldots, p_0(x) \in \mathbb{K}[x]$ (not all p_i being 0) such that the following holonomic differential equation holds:

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0. \tag{1.1}$$

We emphasize that the class of holonomic functions is rather large due to its closure properties. Namely, if we are given two such differential equations that contain holonomic functions $f(x)$ and $g(x)$ as solutions, one can compute holonomic differential equations that contain $f(x)+g(x)$, $f(x)g(x)$ or $\int_0^x f(y)dy$ as solutions. In other words any composition of these operations over known holonomic functions $f(x)$ and $g(x)$ is again a holonomic function $h(x)$. In particular, if for the inner building blocks $f(x)$ and $g(x)$ the holonomic differential equations are given, also the holonomic differential equation of $h(x)$ can be computed.
Of special importance is the connection to recurrence relations. A sequence $(f_n)_{n\geq 0}$ with $f_n \in \mathbb{K}$ is called holonomic (or *P-finite*) if there exist polynomials $p_d(n), p_{d-1}(n), \ldots, p_0(n) \in \mathbb{K}[n]$ (not all p_i being 0) such that a holonomic recurrence

$$p_d(n) f_{n+d} + \cdots + p_1(n) f_{n+1} + p_0(n) f_n = 0 \tag{1.2}$$

holds for all $n \in \mathbb{N}$ (from a certain point on). In the following we utilize the fact that holonomic functions are precisely the generating functions of holonomic sequences: if $f(x)$ is holonomic, then the coefficients f_n of the formal power series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

form a holonomic sequence. Conversely, for a given holonomic sequence $(f_n)_{n\geq 0}$, the function defined by the above sum (i.e., its generating function) is holonomic (this is true in the sense of formal power series, even if the sum has a zero radius of convergence). Note that given a holonomic differential equation for a holonomic function $f(x)$ it is straightforward to construct a holonomic recurrence for the coefficients of its power series expansion. For a recent overview of this holonomic machinery and further literature we refer to [11]. An additional property of

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holonomic functions was given for example in [2] and [4]: if the Mellin transform of a holonomic function

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x)dx = F(n). \tag{1.3}$$

of a holonomic function is defined i.e., the integral $\int_0^1 x^n f(x)dx$ exists, then it is a holonomic sequence. Conversely, if the Mellin transform $\mathbf{M}[f(x)](n)$ of a function $f(x)$ is holonomic, then also the function $f(x)$ is holonomic. In this article we will report an extension of `HarmonicSums` based on Kovacic's algorithm [12] that supports the user to calculate the inverse Mellin transform in terms of iterated integrals that exceed the class of d'Alembertian solutions.

The paper is organized as follows. In Section 2 we revisit the method to compute the inverse Mellin transform of holonomic functions from [2], while in Section 3 we explain the generalization of the method and in Section 4 we give some examples.

2. The Inverse Mellin Transform of Holonomic Sequences

In the following, we deal with the following problem:
Given a holonomic sequence $F(n)$.
Find, whenever possible, a holonomic function $f(x)$ such that for all $n \in \mathbb{N}$ (from a certain point on) we have

$$\mathbf{M}[f(x)](n) = F(n).$$

In [2] a procedure was described to compute a differential equation for $f(x)$ given a holonomic recurrence for $\mathbf{M}[f(x)](n)$. Given this procedure the following method to compute the inverse Mellin transform of holonomic sequences was proposed in [2]:

1. Compute a holonomic recurrence for $\mathbf{M}[f(x)](n)$.
2. Use the method mentioned above to compute a holonomic differential equation for $f(x)$.
3. Compute a linear independent set of solutions of the holonomic differential equation (using `HarmonicSums`).
4. Compute initial values for $\mathbf{M}[f(x)](n)$.
5. Combine the initial values and the solutions to get a closed form representation for $f(x)$.

In our applications we usually apply this method on expressions in terms of nested sums, however as long as there is a method to compute the holonomic recurrence for a given expression (i.e., item 1 can be performed) this proposed method can be used. Another possible input would be a holonomic recurrence together with sufficient initial values. Note that until recently `HarmonicSums` could find all solutions of holonomic differential equations that can be expressed in terms of iterated integrals over hyperexponential alphabets [4, 9, 10, 13]; these solutions are called d'Alembertian solutions [8]. Hence as long as such solutions were sufficient to solve the differential equation in item 3 we succeeded to compute $f(x)$. In case d'Alembertian solutions do not suffice to solve the differential equation in item 3, we have to extend the solver for differential equations.

3. Beyond d'Alembertian solutions of linear differential equations

Until recently only d'Alembertian solutions of linear differential equations could be found in using `HarmonicSums` (compare [2]), but in order to treat more general problems the differential equation solver had to be extended. In [12] an algorithm to solve second order linear homogeneous differential equations is described. We will refer to this algorithm as Kovacic's algorithm.
Consider the *holonomic* differential equation ($p_i(x) \in \mathbb{C}[x]$)

$$p_2(x)f''(x)+p_1(x)f'(x)+p_0(x)f(x)=0. \tag{3.1}$$

Kovacic's algorithm decides whether (3.1)

- has a solution of the form $e^{\int \omega}$ where $\omega \in \mathbb{C}(x)$;
- has a solution of the form $e^{\int \omega}$ where ω is algebraic over $\mathbb{C}(x)$ of degree 2 and the previous case does not hold;
- all solutions are algebraic over $\mathbb{C}(x)$ and the previous cases do not hold;
- has no such solutions;

and finds the solutions if they exist. Note that the solutions Kovacic's algorithm can find are called Liouvillian solutions [10]. In case Kovacic's algorithm finds a solution, it is straightforward to compute a second solution which will again be Liouvillian. This algorithm was implemented in `HarmonicSums`.

Example 1. Consider the following differential equation:

$$\left(4x\left(40-891x+1701x^2\right)+9x^2\left(32-376x+459x^2\right)D_x+18x^3\left(4-31x+27x^2\right)D_x^2\right)f(x)=0,$$

with the implementation of Kovacic's algorithm in `HarmonicSums` we find the following two solutions:

$$\begin{aligned} f_1(x) &= -\frac{\sqrt{10-6\sqrt{1-x}-x}\sqrt[6]{2+2\sqrt{1-x}-x}}{\sqrt{(1-x)^3}\sqrt[3]{x^5}(-4+27x)}, \\ f_2(x) &= -\frac{\sqrt{10+6\sqrt{1-x}-x}\sqrt[6]{2-2\sqrt{1-x}-x}}{\sqrt{(1-x)^3}\sqrt[3]{x^5}(-4+27x)}. \end{aligned}$$

3.1 Composing solutions

Suppose we are given the linear differential equation ($q_i, p_i \in \mathbb{C}[x]; d>0$)

$$\left(q_d(x)D_x^d+\cdots+q_0(x)\right)f(x) = 0, \tag{3.2}$$

which factorizes linearly into d first-order factors. Then this yields d linearly independent solutions of the form

$$f_1(x), f_1(x)\int\frac{f_2(x)}{f_1(x)}dx, f_1(x)\int\frac{f_2(x)}{f_1(x)}\int\frac{f_3(x)}{f_2(x)}dxdx,\ldots, f_1(x)\int\frac{f_2(x)}{f_1(x)}\cdots\int\frac{f_d(x)}{f_{d-1}(x)}dx\cdots dx,$$

where the f_i are hyperexponential functions (i.e., $\frac{D_x f_i(x)}{f_i(x)} \in \mathbb{K}(x)^*$). These solutions are also called d'Alembertian solutions of (3.2), compare [13, 8].

Now suppose that a given differential equation does not factorize linearly, but contains in between second-order factors, which can be solved, e.g., by Kovacic's algorithm. Let the following differential equation correspond to a second order factor:

$$\left(p_2(x)D_x^2 + p_1(x)D_x + p_0(x)\right) f(x) \;=\; 0, \tag{3.3}$$

then we can compose the solutions of the first order and second order factors as follows. Let $s(x)$ be solution of (3.2) and let $g_1(x)$ and $g_2(x)$ be solutions of (3.3). Then

$$s(x),\ s(x)\int \frac{g_1(x)}{s(x)}dx \text{ and } s(x)\int \frac{g_2(x)}{s(x)}dx$$

are solutions of

$$\left(p_2(x)D_x^2 + p_1(x)D_x + p_0(x)\right)\left(q_d(x)D_x^d + \cdots + q_0(x)\right) f(x) = 0.$$

In addition, if we define $w(x) := p_2(x)(g_1'(x)g_2(x) - g_1(x)g_2'(x))$ then

$$g_1(x),\ g_2(x) \text{ and } g_1(x)\int s(x)w(x)g_2(x)dx - g_2(x)\int s(x)w(x)g_1(x)dx$$

are solutions of

$$\left(q_d(x)D_x^d + \cdots + q_0(x)\right)\left(p_2(x)D_x^2 + p_1(x)D_x + p_0(x)\right) f(x) = 0.$$

4. Examples

Example 2. We want to compute the inverse Mellin transform of

$$f_n := \left(\frac{4}{27}\right)^n \binom{3n}{n}.$$

We find that

$$-2(3n+1)(3n+2)f_n + 9(n+1)(2n+1)f_{n+1} = 0,$$

which leads to the differential equation

$$(27x-4)f(x) + 9x(7x-4)f'(x) + 18x^2(x-1)f''(x) = 0,$$

for which we find with the help of Kovacic's algorithm the general solution

$$s(x) = c_1\frac{\sqrt[6]{2+2\sqrt{1-x}-x}}{\sqrt{1-x}\sqrt[3]{x^2}} + c_2\frac{\sqrt[6]{2-2\sqrt{1-x}-x}}{\sqrt{1-x}\sqrt[3]{x^2}},$$

for some constants c_1 and c_2. In order to determine these constants we compute

$$\int_0^1 x^1 s(x)dx \;=\; \frac{1}{9}c_1\left(-3+\frac{8\pi}{\sqrt{3}}+8\log(2)\right)+\frac{1}{9}c_2\left(3+\frac{8\pi}{\sqrt{3}}-8\log(2)\right),$$

$$\int_0^1 x^2 s(x)dx = \frac{1}{486}c_1\left(-147+\frac{320\pi}{\sqrt{3}}+320\log(2)\right)+\frac{1}{486}c_2\left(147+\frac{320\pi}{\sqrt{3}}-320\log(2)\right).$$

Since $f_1 = 4/9$ and $f_2 = 80/243$ we can deduce that $c_1 = c_2 = \frac{\sqrt{3}}{4\pi}$ and hence

$$f_n = \frac{\sqrt{3}}{4\pi}\mathbf{M}\left[\frac{\sqrt[6]{2-2\sqrt{1-x}-x}+\sqrt[6]{2+2\sqrt{1-x}-x}}{\sqrt{1-x}x^{2/3}}\right](n).$$

Example 3. During the Computation of the inverse Mellin transform of

$$\sum_{i=1}^{n}\binom{3i}{i}\frac{1}{i}$$

we have to solve the following differential equation:

$$0 = 27(27x-4)f(x)+\left(4131x^2-2160x+16\right)f'(x)+9x\left(351x^2-298x+16\right)f''(x) +18(x-1)x^2(27x-4)f^{(3)}(x).$$

We are able to find the general solution of that differential equation using `HarmonicSums`:

$$c_1\frac{1}{27x-4}+c_2\frac{\int_0^x\frac{1}{\left(1-\sqrt{1-\tau}\right)^{2/3}\sqrt[3]{1+\sqrt{1-\tau}}\sqrt{1-\tau}}d\tau}{27x-4}+c_3\frac{\int_0^x\frac{1}{\sqrt[3]{1-\sqrt{1-\tau}}\left(1+\sqrt{1-\tau}\right)^{2/3}\sqrt{1-\tau}}d\tau}{27x-4}.$$

Given this general solution we find:

$$\sum_{i=1}^{n}\binom{3i}{i}\frac{1}{i} = \left(\frac{27}{4}\right)^{n+1}\left(4\int_0^1\left(x^n-\frac{4^n}{27^n}\right)\frac{1}{27x-4}dx \right.$$
$$-\frac{\sqrt{3}}{\pi}\int_0^1\left(x^n-\frac{4^n}{27^n}\right)\frac{\int_0^x\frac{1}{\sqrt[3]{1-\sqrt{1-\tau}}\left(1+\sqrt{1-\tau}\right)^{2/3}\sqrt{1-\tau}}d\tau}{27x-4}dx$$
$$\left.-\frac{\sqrt{3}}{\pi}\int_0^1\left(x^n-\frac{4^n}{27^n}\right)\frac{\int_0^x\frac{1}{\sqrt[3]{1+\sqrt{1-\tau}}\left(1-\sqrt{1-\tau}\right)^{2/3}\sqrt{1-\tau}}d\tau}{27x-4}dx\right).$$

Finally, we list several examples that could be computed using `HarmonicSums`:

Example 4.

$$\binom{4n}{2n} = 16^n\frac{1}{2\sqrt{2}\pi}\int_0^1\frac{x^n(1+\sqrt{x}+\sqrt{x-1})}{\sqrt{\sqrt{x}+\sqrt{x-1}}\sqrt{1-x}x^{\frac{3}{4}}}dx,$$

$$\frac{1}{n\binom{4n}{2n}} = \frac{1}{16^n\sqrt{2}}\int_0^1\frac{x^n\left(1+\sqrt{x-1}+\sqrt{x}\right)}{\sqrt{\sqrt{x-1}+\sqrt{x}}\sqrt{1-x}x}dx,$$

$$\frac{1}{n\binom{3n}{n}} = \frac{\left(\frac{4}{27}\right)^n}{\sqrt{3}}\int_0^1\frac{x^n\left(1+\left(\sqrt{x-1}+\sqrt{x}\right)^{2/3}\right)}{\sqrt[3]{\sqrt{x-1}+\sqrt{x}}\sqrt{1-x}x}dx,$$

$$\sum_{i=1}^{n}\binom{3i}{i} = \frac{\sqrt{3}\left(\frac{27}{4}\right)^{n+1}}{\pi}\int_0^1 \frac{\left(x^n-\left(\frac{4}{27}\right)^n\right)\left(\left(\sqrt[6]{2-2\sqrt{1-x}-x}+\sqrt[6]{2+2\sqrt{1-x}-x}\right)\sqrt[3]{x}\right)}{\sqrt{1-x}(27x-4)}dx,$$

$$\sum_{i=1}^{n}\binom{4i}{2i}\frac{1}{i} = 16^{n+1}\left(\frac{\sqrt{2}}{4\pi}\int_0^1\frac{x^n-16^{-n}}{1-16x}\int_0^x\frac{1+\sqrt{y-1}+\sqrt{y}}{\sqrt{\sqrt{y-1}+\sqrt{y}}\sqrt{1-y}y^{3/4}}dydx-\int_0^1\frac{x^n-16^{-n}}{1-16x}dx\right)$$

$$\sum_{i=1}^{n}\binom{3i}{i}\frac{1}{i^2} = \left(\frac{27}{4}\right)^{n+1}\left(\frac{\sqrt{3}}{\pi}\int_0^1\frac{x^n-\left(\frac{4}{27}\right)^n}{27x-4}\int_0^x\frac{1}{y}\int_0^y\frac{\sqrt[3]{1-\sqrt{1-z}}+\sqrt[3]{1+\sqrt{1-z}}}{\sqrt{1-z}z^{2/3}}dz\,dy\,dx,\right.$$
$$\left.-4\int_0^1\frac{x^n-\left(\frac{4}{27}\right)^n}{27x-4}\log(x)\,dx-4\log\left(\frac{27}{4}\right)\int_0^1\frac{x^n-\left(\frac{4}{27}\right)^n}{27x-4}dx\right).$$

Acknowledgements

I want to thank C. Schneider, C. Raab and J. Blümlein for useful discussions.

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