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The method of global R* and its applications

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The global R^* operation is a powerful method for computing renormalisation group functions. This technique, based on the principle of *infrared rearrangement*, allows to express all the ultraviolet counterterms in terms of massless propagator integrals. In this talk we present the main features of global R^* and its application to the renormalisation of QCD. By combining this approach with the use of the program Forcer for the evaluation of the relevant Feynman integrals, we renormalise for the first time QCD at five loops in covariant gauges.

13th International Symposium on Radiative Corrections (Applications of Quantum Field Theory to Phenomenology)
25-29 September, 2017
St. Gilgen, Austria

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PoS(RADCOR2017)004

1. Introduction

Renormalisation group functions, such as beta functions and anomalous dimensions, are crucial objects in quantum field theories, that are necessary to define consistently Green functions and amplitudes beyond tree level. The computation of multiloop anomalous dimensions is therefore important both for theoretical studies and for phenomenological applications. In this respect, a central role is played by the renormalisation of QCD, which has a long history, starting with the one-loop calculation and the groundbreaking discovery of asymptotic freedom [1–4]. Impressive progress in this field pushed state-of-the-art calculations to five-loop level [5–23]. One of the main difficulties in the calculation of high order corrections is the dramatic growth of the number and of the complexity of Feynman diagrams, that can be tackled only with the use of highly efficient computational methods. In this talk we describe the main features of a particularly efficient technique to compute ultraviolet counterterms and renormalisation constants (RCs), based on the global R^* operation [12, 24–26], which highly simplifies the structure of Feynman integrals. We applied this method to the five-loop renormalisation of the QCD Lagrangian, obtaining all the RCs with complete dependence on the gauge fixing parameter ξ [27]. In this way, it was possible to confirm and extend the results of [23], which include up to the linear terms in the expansion around the Feynman gauge result, $\xi = 0$.

2. Infrared rearrangements

The idea of *infrared rearrangement* (IRR) [28] is at the basis of the method of global R^*. This trick follows from the properties of dimensional regularisation and minimal subtraction [29, 30], which ensure that, for every Feynman diagram Γ, its ultraviolet (UV) counterterm $Z(\Gamma)$ is a polynomial in *all* the masses [31]. This fact implies that counterterms of logarithmically divergent diagrams are *independent* on masses. As a consequence, rearranging the configuration of external momenta or (external and/or internal) masses doesn't change the UV counterterm of a logarithmically divergent Feynman diagram, but it can simplify drastically the structure of the associated integral. Notice that requiring logarithmic divergence is not restrictive, because the superficial degree of divergence of any diagram can always be reduced to zero by taking the appropriate number of derivatives with respect to external momenta and masses.

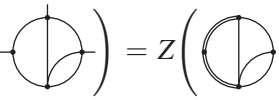
The specific type of rearrangement operation is not fixed a priori and in principle every diagram can be modified in a different way. A convenient choice is to reduce Feynman diagrams to "one-mass" tadpoles, which consists in two steps:

i) Selection of an external vertex attached to two propagators. These internal lines are modified with the introduction of an internal mass.

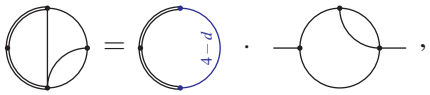
ii) Nullification of all the external momenta of the diagram.

For example, this procedure can be used to determine the vertex counterterm

$$Z\left(\quad\right) = Z\left(\quad\right), \tag{2.1}$$



where the two propagators attached to the leftmost vertex are modified with the introduction of a mass, represented with a double line. While the UV counterterms of the two diagrams in eq. (2.1) are equal to each other, the tadpole is computed much more easily using the factorisation



(2.2)

where the one-loop tadpole has a massless propagator raised to non-integer power $4-d$. Note that the most complicated integral we have to compute in eq. (2.2) is a two-loop massless propagator, while the original diagram was a three-loop one. This fact is a particular case of a general theorem [32], which shows that in principle UV counterterms of L-loop diagrams are entirely determined in terms of $(L-1)$-loop massless propagators, known also as *p-integrals*.

One potential problem of infrared rearrangements is the generation of spurious infrared divergences that contaminate the singularities of UV origin. This issue occurs for example if we set to zero the internal mass in eq. (2.1). Of course, this rearrangement doesn't affect the UV pole, but the tadpole integral becomes scaleless and it vanishes in dimensional regularisation, as a consequence of the exact cancellation of UV and IR poles. Several strategies have been adopted to overcome the problem of IR singularities. One possibility consists in the introduction of an internal mass, which regulates potential IR divergences, in *all* the propagators [11, 33, 34]. After nullification of the external momenta, this operation transforms every L-loop Feynman diagram in a completely massive tadpole of the same loop order, which in general is not factorisable in the form of eq. (2.2). This method has been applied to the five-loop renormalisation of QCD in a series of recent works [18, 19, 21], culminating with the calculation of all the anomalous dimensions, expanded up to the linear order in the gauge fixing parameter ξ, that was presented at this conference [23].

A different strategy is based on the R^* operation [32,35,36], which generalises the Bogoliubov R-operation, by subtracting both IR and UV divergences of Feynman diagrams

$$R^*(\Gamma) = \widetilde{R} \circ R(\Gamma), \tag{2.3}$$

where the operations R and $\widetilde{R}$ generate recursively UV and IR counterterms, respectively. The power of R^* is that it can be used to cancel IR poles of rearranged Feynman integrals, thus removing every restriction on the use of IRR. However, in practice computations become demanding at high perturbative orders, because each Feynman diagram generates many IR counterterms, and the combinatorial growth of the number of diagrams makes it impossible to proceed with this method by hand. The important step of automating the R^* operation for generic Feynman diagrams [37] was crucial for obtaining the five-loop QCD beta function within this method [20].

The global R^* operation [25, 26] provides a very elegant solution to the computational issues related to the proliferation of IR counterterms, by implementing the subtraction of infrared singularities at the level of the whole rearranged Green functions. In this way we avoid the diagram-by-diagram recursive calculation of counterterms, which results in a more efficient approach.

3. Introduction to global R^*

In this section we describe the main features of the R^* operation, by discussing the renormali-

sation of the ghost-gluon vertex as illustrative example. We define the 1PI vertex

$$\Gamma_\mu^{abc}(p,q) = -g_s f^{abc}\left[p_\mu \Gamma_p(p,q) + q_\mu \Gamma_q(p,q)\right], \tag{3.1}$$

where p is the momentum of the outgoing ghost, q the momentum of the gluon and g_s is the QCD coupling constant. At tree level the functions $\Gamma_p^{\text{tree}} = 1$ and $\Gamma_q^{\text{tree}} = 0$ are fixed by the Feynman rule and it is convenient to introduce the notation

$$\Gamma_p(p,q) = 1 + \delta\Gamma_p(p,q), \qquad \Gamma_q(p,q) = \delta\Gamma_q(p,q), \tag{3.2}$$

to distinguish the tree level contribution from the loop corrections. The renormalisation constant $Z_1^{ccg} = 1 + \delta Z_1^{ccg}$ satisfies the conditions

$$\begin{aligned} &K_\varepsilon\left[Z_1^{ccg}\left(1+\delta\Gamma_p^B(p,q)\right)\right] = 0, \\ &K_\varepsilon\left[Z_1^{ccg}\delta\Gamma_q^B(p,q)\right] = 0, \end{aligned} \tag{3.3}$$

where the operator K_ε extracts the pole part in the Laurent expansion in the dimensional regulator $\varepsilon = \frac{4-d}{2}$ and the superscript "B" indicates the use of bare lagrangian parameters g_0, ξ_0. The first identity of eq. (3.3) determines Z_1^{ccg} order-by-order in perturbation theory

$$\delta Z_1^{ccg} = -K_\varepsilon\left[Z_1^{ccg}\,\delta\Gamma_p^B(p,q)\right]. \tag{3.4}$$

The first point that we want to discuss is the determination of a global IR counterterm for the whole vertex function, which becomes IR singular for $p,q \to 0$. In this limit IR poles cancel the UV singularities exactly, because all the integrals become scaleless. The key observation is that Z_1^{ccg} is independent on the value of p and q and therefore eq. (3.4) holds also at zero momenta, provided IR singularities are subtracted by $\widetilde{R}$

$$\delta Z_1^{ccg} = -K_\varepsilon\left[Z_1^{ccg}\,\widetilde{R}\left(\delta\Gamma_p^B(0,0)\right)\right]. \tag{3.5}$$

Within minimal subtraction Z_1^{ccg} and $\widetilde{R}\left(\delta\Gamma_p^B(0,0)\right)$ contain only poles in ε. This fact implies that we can drop the operator K_ε from the equation above and we derive the *IR subtracted* vertex

$$\widetilde{R}\left(\delta\Gamma_p^B(0,0)\right) = -\frac{\delta Z_1^{ccg}}{Z_1^{ccg}}. \tag{3.6}$$

$$\delta\Gamma_p^M(g,p,M) =$$

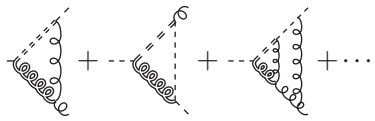


Figure 1: Global infrared rearrangement of the ghost-gluon vertex.

The second ingredient of the procedure is the choice of a convenient infrared rearrangement. As discussed in sec. 2, we want to reduce the complexity of the calculation to $(L-1)$-loop massless propagators. This result is achieved in global R^* by applying the rearrangement to one-mass tadpoles, which follows steps i) and ii) of sec. 2, to *all* the diagrams contributing to Γ_μ^{abc}. After

masses have been introduced in all the diagrams, as described in step i), we define a *globally rearranged* vertex $\Gamma_p^M(p,q,M)$ as the sum of rearranged diagrams, depicted in Fig. 1. The genuine UV divergence of this object is the same as $\Gamma_p(p,q)$, therefore it is renormalised by δZ_1^{ccg}, while subdivergences are modified. In particular eq. (3.4) becomes

$$\delta Z_1^{ccg} = -K_\varepsilon \left[\delta\Gamma_p^{M,B}(p,q,M) + \delta Z_1^{ccg} \cdot \delta\Gamma_p^B(p,q) \right], \tag{3.7}$$

where, in the spirit of the R-operation, the subtraction of the divergent subgraphs including the massive vertex is achieved by the counterterm δZ_1^{ccg} and the remaining reduced graph consists of the massless vertex $\delta\Gamma_p^B(p,q)$, as shown in the two-loop example of Fig. 2.

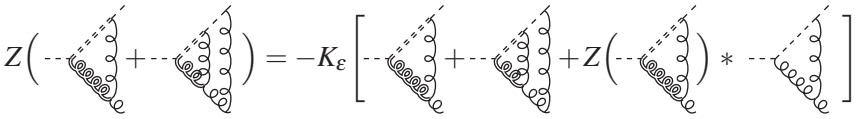


Figure 2: Pattern of divergences of the rearranged vertex $\Gamma_p^M(p,q,M)$.

We proceed with step ii) of the rearrangement, namely the nullification of external momenta in eq. (3.7). We evaluate the leading behaviour of the massive vertex $\delta\Gamma_p^{M,B}(p,q,M)$ in the limit $p,q \to 0$ by applying the well-known *hard mass expansion* (see the books [38, 39] and references therein)

$$\delta\Gamma_p^{M,B}(p,q,M) \underset{M \gg p,q}{\longrightarrow} \delta\Gamma_p^{M,B}(0,0,M) + \delta\Gamma_p^{M,B}(0,0,M) \cdot \delta\Gamma_p^B(p,q) + \mathscr{O}\left(\frac{1}{M^2}\right), \tag{3.8}$$

where $\delta\Gamma_p^{M,B}(0,0,M)$ by construction is a sum of one-mass tadpole diagrams. Finally we cancel the IR poles of the vertex function $\delta\Gamma_p^B(p,q)$ at vanishing external momenta by using the globally subtracted vertex $\widetilde{R}(\delta\Gamma_p^B(0,0))$, eq. (3.6). In conclusion, by introducing eqs. (3.8) and (3.6) in eq. (3.7) we conclude

$$\delta Z_1^{ccg} = -K_\varepsilon \left[\frac{\delta\Gamma_p^{M,B}(0,0,M)}{Z_1^{ccg}} - \frac{\left(\delta Z_1^{ccg}\right)^2}{Z_1^{ccg}} \right]. \tag{3.9}$$

We are now able to determine Z_1^{ccg} by expanding eq. (3.9) order-by-order in perturbation theory. The necessary ingredients to compute δZ_1^{ccg} at L loops are Z_1^{ccg} at $(L-1)$ loops and $\delta\Gamma_p^{M,B}(0,0,M)$ at L loops, that is anyway factorisable in $(L-1)$-loop massless propagators, as in the example in eq. (2.2). We calculated these integrals up to four loops with the program `Forcer` [40] and therefore we were able to determine the RC Z_1^{ccg} at five-loop level.

4. The five-loop renormalisation of QCD

We complete the renormalisation of QCD to five loops within the global R^* method introduced in sec. 3. Besides the ghost-gluon vertex RC, the remaining quantities that we have to determine are the wave function renormalisations of the ghost (Z_3^c), of the fermion (Z_2) and of the gluon (Z_3). All the other RCs are fixed by Ward identities

$$Z_g = \frac{Z_1^{ccg}}{Z_3^c\sqrt{Z_3}} = \frac{Z_1^{\psi\psi g}}{Z_2\sqrt{Z_3}} = \frac{Z_1^{3g}}{\left(\sqrt{Z_3}\right)^3} = \frac{\sqrt{Z_1^{4g}}}{Z_3}, \tag{4.1}$$

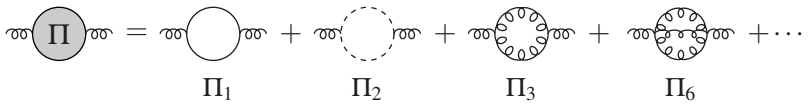


Figure 3: The different contributions to the gluon self energy.

where Z_1^i is the RC of the vertex i and Z_g is the coupling constant renormalisation.

The calculation of the ghost and the fermion wave function renormalisation follows the steps leading to eq. (3.9), described in sec. 3, and we get

$$\delta Z_3^c = -K_\varepsilon \left\{ \frac{Z_3^c}{Z_1^{ccg}} \left[\Pi^B(0,M) - \frac{\delta Z_3^c}{Z_3^c} \left(\delta\Gamma_p^{M,B}(0,0,M) + \delta Z_1^{ccg} \right) \right] \right\}, \tag{4.2}$$

$$\delta Z_2 = -K_\varepsilon \left\{ \frac{Z_2}{Z_1^{\psi\psi g}} \left[\Sigma^B(0,M) - \frac{\delta Z_2}{Z_2} \left(\delta\Lambda^{M,B}(0,0,M) + \delta Z_1^{\psi\psi g} \right) \right] \right\}. \tag{4.3}$$

Here $\Pi(0,M)$ is the ghost self-energy at zero momentum, where the mass M was introduced in the vertex of the incoming ghost by the infrared rearrangement. Similarly $\Sigma(0,M)$ is the rearranged fermion self-energy, while $\delta\Gamma_p^M$ and $\delta\Lambda^M$ are respectively the ghost-gluon vertex and the quark-gluon vertex with masses inserted, that arise in the hard mass expansion of the rearranged self-energies, as in eq. (3.8). All the quantities appearing in eq. (4.2) and in eq. (4.3) are either one-mass tadpoles, or QCD renormalisation constants, therefore we could compute Z_3^c and Z_2 up to five loops with the help of `Forcer`.

The calculation of the gluon wave function renormalisation within the global R^* method is conceptually more complicated. We won't describe here the derivation of Z_3, which will be given in [41], but we will only comment on the main differences with respect to the procedure adopted for Z_1^{ccg}, Z_3^c and Z_2. In general, infrared rearranging gluon correlators requires to modify several types of vertices with the insertion of a mass. This problem doesn't occur in the rearrangement of correlators with external ghosts or fermions, where it is always possible to modify a uniquely defined vertex in all the diagrams, because ghosts and fermions undergo a single type of interaction. In the case of external gluons we distinguish the contributions of the different interactions and we rearrange them separately. As shown in Fig. 3, for the gluon self-energy we have[1]

$$\begin{aligned} \Pi^{\mu\nu;ab} &= i \int d^d x\, e^{iq\cdot x} \langle 0|T\left(A^{\mu,a}(x)A^{\nu,b}(0)\right)|0\rangle_{1\text{PI}} \\ &= i \sum_{i=1,2,3,6} \int d^d x\, e^{iq\cdot x} \langle 0|T\left(O_i^{\mu,a}(x)A^{\nu,b}(0)\right)|0\rangle_{1\text{PI}} \equiv \sum_{i=1,2,3,6} \Pi_i^{\mu\nu;ab}, \end{aligned} \tag{4.4}$$

where $O_1^{\mu,a}$, $O_2^{\mu,a}$, $O_3^{\mu,a}$ and $O_6^{\mu,a}$ identify the operators[2] that couple the external gluon $A^{\mu,a}(x)$ respectively via the quark-gluon, ghost-gluon, tri- and four-gluon interactions of QCD, *e.g.* $O_1^{\mu,a} = g\overline{\psi}\gamma^\mu T^a \psi$. Each contribution Π_i is rearranged by introducing a mass in O_i: in the case of O_1, O_2 and O_3 this is straightforward. For O_6 we can't directly apply steps i) and ii) of sec. 2, which require to identify two propagators attached to an external leg, because here the external gluon

[1] Note that the right part of eq. (4.4) does not include all diagrams with both external gluons being coupled with one and the same vertex. Such contributions in any case are set to zero in dimensionally regulated massless QCD.

[2] Colour and Lorentz indices will be suppressed to simplify notations.

is connected to three lines. The solution to this first issue is to split the four-gluon operator into three-point vertices with the introduction of an auxiliary field, as shown in Fig. 4. This procedure

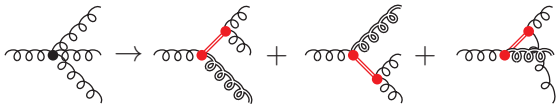


Figure 4: Splitting the four-point vertex into two product of two three-point vertices.

generates two classes of diagrams with different structure of subdivergences, as depicted in Fig. 5. Diagrams of the "special" type have subdivergences associated to the internal vertex of the auxiliary field, while "non-special" diagrams have only the singularity of the external vertex. The properties of special and non-special diagrams will be discussed in detail in [41]. Beyond tree level, vertices O_1, O_2, O_3 and O_6 (both special and non-special contributions) are mixing among each other under renormalisation

$$O_i^R = \sum_j z_{ij} O_j, \tag{4.5}$$

where O_i^R denote the renormalised vertices and z_{ij} is the renormalisation matrix. Note that the sum over j is not restricted to the QCD operators, O_1, O_2, O_3 and O_6, but it must include *all* the operators that cancel the UV divergences of the vertices. We determined the set of the required

Figure 5: The classes of special and non-special diagrams generated by the modified four gluon vertex.

operators by analysing the structure of the vertices at three and four loops and we identified two new types of three-gluon interactions, named O_4 and O_5, and six different four-gluon interactions, $O_7 \ldots O_{12}$ that can appear up to three loops [41]. Indeed the number of four-gluon operators that can be constructed in a general gauge group increases with the loop order and we limited our study to the structures that are relevant for the renormalisation of the gluon self-energy to five loops. In this way, renormalisation of operators O_i in eq. (4.4) generates a mixing of the different contributions Π_i dictated by the matrix z_{ij}, as shown in Fig. 6 where Π_1 mixes into Π_i, with $i = 6 \ldots 12$. Note however that it is not necessary to compute all the 144 matrix elements of z_{ij} to renormalise eq. (4.4), because summing over the index $i = 1,2,3,6$ we have

$$\sum_{i=1,2,3,6} z_{ij} = \begin{cases} Z_1^{(j)} & j = 1,2,3,6 \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

where $Z_1^{(j)}$ are the RCs of the QCD vertices O_j. These features were crucial to determine the RC Z_3 from one-mass tadpoles $\Pi_i(0,M)$, that were computed with `Forcer` to five loops.

5. Conclusions

With the global R^* operation we were able to determine all the RCs of the QCD Lagrangian in terms of one-mass tadpoles, that are computed at L-loop order, by evaluating $(L-1)$-loop massless

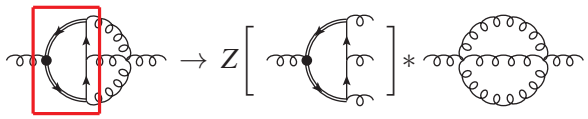


Figure 6: Mixing between fermion and four-gluon operator in the rearranged gluon self-energy.

propagators. Crucially, the program `Forcer` allowed us to calculate these integrals to four-loop order with high efficiency and to determine [27]

- the RCs Z_1^{ccg}, $Z_1^{\psi\psi g}$, Z_3^c and Z_2 to five loops with complete dependence on ξ,
- Z_3 to five loops, at linear order in ξ.

Because of eqs. (4.1) and of the independence of Z_g on the parameter ξ, we derived the complete renormalisation of QCD to five loops in general covariant gauges. Results were checked by verifying the explicit cancellation of the linear dependence on ξ in the ratio

$$Z_g = \frac{Z_1^{ccg}}{Z_3^c\sqrt{Z_3}}, \tag{5.1}$$

as well as the consistency of $Z_1^{\psi\psi g}$ with the Ward identities, eq. (4.1). We compared also with the five-loop results in Feynman gauge [17, 20, 23] and with those expanded up linear order in ξ of [23], finding agreement. In Landau gauge, we verified that $Z_1^{ccg} = 1$ [42, 43] and that Z_3^c, Z_3, Z_2 and Z_g agree with the results in the limit of large number of fermions [44–46].

While global R^* is extremely efficient from the computational point of view, it requires an elaborate analysis of the pattern of UV subdivergences of the rearranged diagrams. It is interesting and worthwhile to study R^* in different contexts, where other techniques become very demanding, such as the determination of the anomalous dimensions of twist two operators.

Acknowledgement

This work is part of the HEPGAME project, supported by the ERC advanced grant 320651. The work of F.H. was supported by the ERC advanced grants 320651 and 320389. The work of K.G.C. was supported by the Deutsche Forschungsgemeinschaft through CH1479/1-1 and by the German Federal Ministry for Education and Research BMBF through Grant No. 05H15GUCC1.

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