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Cuts of Feynman integrals in Baikov representation*

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A systematic approach to compute cuts of Feynman Integrals, appropriately defined in d dimensions based on the Baikov representation, is presented. The information provided by these computations may be used to determine the class of functions needed to analytically express the full integrals. Differential equations for Master Integrals are also discussed.

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1. Introduction

It is almost seventy years from the time Feynman Integrals (FI) were first introduced [1, 2, 3] and forty-five years since the dimensional regularisation [4] set up the framework for an efficient use of loop integrals in computing scattering matrix elements, and still the frontier of multi-scale multi-loop integral calculations (maximal both in number of scales and number of loops) is determined by the planar five-point two-loop on-shell massless integrals [5, 6], recently computed[1]. On the other hand, in order to keep up with the increasing experimental accuracy as more data is collected at the LHC, more precise theoretical predictions and higher loop calculations are required [8].

In the last years our understanding of the reduction of one-loop amplitudes to a set of Master Integrals (MI), a minimal set of FI that form a basis, either based on unitarity methods [9, 10, 11] or at the integrand level via the OPP method [12, 13], has drastically changed the way one-loop calculations are preformed resulting in many fully automated numerical tools (some reviews on the topic are [14, 15, 16]), making the next-to-leading order (NLO) approximation the default precision for theoretical predictions at the LHC. In the recent years, progress has been made also towards the extension of these reduction methods for two-loop amplitudes at the integral [17, 18, 19, 20, 21, 22, 23, 24, 25, 26] as well as the integrand [27, 28, 29, 30, 31, 32] level. Two-loop MI are defined using the *integration by parts* (IBP) identities [33, 34, 35], an indispensable tool beyond one loop. Contrary to the one-loop case, where MI have been known for a long time already [36], a complete library of MI at two-loops is still missing. At the moment this is the main obstacle to obtain a fully automated NNLO calculation framework similar to the one-loop one, that will satisfy the precision requirements at the LHC [8].

Many methods have been introduced in order to compute MI [37]. The overall most successful one, is based on expressing the FI in terms of an integral representation over Feynman parameters, involving the two well-known Symanzik Polynomials U and F [38]. The introduction of the sector decomposition [39, 40, 41, 42, 43] method resulted in a powerful computational framework for the numerical evaluation of FI, see for instance `SecDec` [7]. An alternative is based on Mellin-Barnes representation [44, 45], implemented in [46][2]. Nevertheless, the most successful method to calculate multi-scale multi-loop FI is, for the time being, the differential equations (DE) approach [47, 48, 49, 50, 51], which has been used in the past two decades to calculate various MI at two-loops and beyond. Following the work of refs. [52, 53, 54], there has been a building consensus that the so-called *Goncharov Polylogarithms* (GPs) form a functional basis for many MI. The so-called canonical form of DE, introduced by Henn [55], manifestly results in MI expressed in terms of GPs [3]. Nevertheless the reduction of a given DE to a canonical form is by no means fully understood. First of all, despite recent efforts [57, 58, 59], and the existence of *sufficient* conditions that a given MI can be expressed in terms of GPs, no criterion, with practical applicability, that is at the same time *necessary and sufficient* has been introduced so far. Moreover, it is well known that

[1]Complete results, including physical region kinematics, are presented in [6]. Notice that numerical codes, like for instance `SecDec` [7], can reproduce analytic results only at Euclidean region kinematics; results for physical region kinematics are not supported due to poor numerical convergence.

[2]See also https://mbtools.hepforge.org

[3]For an alternative method in the single scale case see also ref. [56]

PoS(RADCOR2017)006

when for instance enough internal masses are introduced, MI are not anymore expressible in terms of GPs, and in fact a new class of functions involving elliptic integrals is needed [60, 61, 62, 63, 64].

In this contribution we present results [65] on FI based on Baikov representation [66, 67, 68, 69, 70, 71]. In Section 2, we present a consistent definition of the integration limits which will be important for the computation of cut integrals in d dimensions. In Section 3 we present a novel approach to obtain DE from the Baikov representation. In Section 4 we introduce the definition of the cut integral in Baikov representation[4], that satisfies the same DE and the same IBP identities as the uncut one [78], and we conjecture that computing the corresponding maximally cut integral [79] we may have a necessary and sufficient condition for the expression of the uncut integral in terms of GPs and when applied to the whole family of MI on the possibility to obtain a canonical form.

2. The Baikov representation

An L-loop Feynman Integral with $E+1$ external lines can be written in the form

$$F_{\alpha_1\ldots\alpha_N} = \int\left(\prod_{i=1}^{L}\frac{d^d k_i}{i\pi^{d/2}}\right)\frac{1}{D_1^{\alpha_1}\ldots D_N^{\alpha_N}} \tag{2.1}$$

with $N = \frac{L(L+1)}{2}+LE$, α_i arbitrary integers, and D_a, $a = 1,\ldots,N$, inverse Feynman propagators,

$$D_a = \sum_{i=1}^{L}\sum_{j=i}^{M}\underset{a}{A^{ij}}s_{ij}+f_a = \sum_{i=1}^{L}\sum_{j=i}^{L}\underset{a}{A^{ij}}k_i\cdot k_j+\sum_{i=1}^{L}\sum_{j=L+1}^{M}\underset{a}{A^{ij}}k_i\cdot p_{j-L}+f_a,\quad a=1,\ldots,N \tag{2.2}$$

where $q_i = k_i, (i = 1,\ldots,L)$ the loop momenta and $q_{L+i} = p_i, (i = 1,\ldots,E)$, the independent external momenta, $M = L+E$, $s_{ij} = q_i\cdot q_j$ and f_a depend on external kinematics and internal masses. $\underset{a}{A^{ij}}$ can be understood as an $N\times N$ matrix, with a running obviously from 1 to N and with (ij) taking also N values as $i = 1,\ldots,L$ and $j = i,\ldots,M$. The elements of the matrix $\underset{a}{A^{ij}}$ are integer numbers taken from the set $\{-2,-1,0,+1,+2\}$. This matrix is characteristic of the corresponding Feynman graph and can, in a loose sense, be associated with the 'topology' of the graph. Then, by projecting each of the loop momenta $q_i = k_i, (i = 1,\ldots,L)$ with respect to the space spanned by the external momenta involved plus a transverse component (for details see [70]), we may write

$$F_{\alpha_1\ldots\alpha_N} = C_N^L\left(G(p_1,\ldots,p_E)\right)^{(-d+E+1)/2}\int\frac{dx_1\ldots dx_N}{x_1^{\alpha_1}\ldots x_N^{\alpha_N}}P_N^L(x_1-f_1,\ldots,x_N-f_N)^{(d-M-1)/2} \tag{2.3}$$

with

$$C_N^L = \frac{\pi^{-L(L-1)/4-LE/2}}{\prod_{i=1}^{L}\Gamma\left(\frac{d-M+i}{2}\right)}\det\left(A_{ij}^{a}\right) \tag{2.4}$$

and

$$P_N^L(x_1,x_2,\ldots,x_N) = G(k_1,\ldots,k_L,p_1,\ldots,p_E)\Big|_{s_{ij}=\sum\limits_{a=1}^{N}A_{ij}^a x_a\ \&\ s_{ji}=s_{ij}}$$

with G representing the Gram determinant, $G(q_1,\ldots,q_n) = \det(q_i\cdot q_j)$ and A_{ij}^a is the inverse of the topology matrix $\underset{a}{A^{ij}}$. An alternative derivation of the Baikov representation for one- and two-loop FI as well as the *loop-by-loop representation*, can be found in Appendix A of ref. [65]. The derivation of the Baikov representation can easily be implemented in a computer algebra code[5].

[4]For related work see [72, 73, 74, 75, 76, 77]

[5]A `Mathematica` script, `Baikov.m`, is provided as an attachment in ref. [65]

PoS(RADCOR2017)006

We conclude this section by elaborating on the limits of the x_a-integrations in Eq. (2.3). In order to simplify the discussion, let us start with a generic one-loop configuration defined by

$$x_1 = k^2 - m_1^2, \quad x_2 = (k+p_1)^2 - m_2^2, \quad \ldots, \quad x_N = (k+p_1+\ldots+p_{N-1})^2 - m_N^2$$

Then consider the generic integral ($\alpha_i \geq 0$),

$$F_{\alpha_1 \cdots \alpha_N} = C_N^1 G(p_1,\ldots,p_{N-1})^{(N-d)/2} \int \frac{dx_1 \ldots dx_N}{x_1^{\alpha_1} \ldots x_N^{\alpha_N}} P_N^{1\,(d-N-1)/2} \tag{2.5}$$

$$C_N^1 = \frac{\pi^{-(N-1)/2}}{\Gamma\left(\frac{d-N+1}{2}\right)} \left(\frac{1}{2}\right)^{N-1} \tag{2.6}$$

It is easy to verify that P_N^1 is a polynomial that is quadratic in the variables x_a [68], and that obviously when $\alpha_N = 0$, the external momentum p_{N-1} decouples, so that

$$\begin{aligned} F_{\alpha_1 \ldots \alpha_{N-1} 0} &= C_N^1 G(p_1,\ldots,p_{N-1})^{(N-d)/2} \int \frac{dx_1 \ldots dx_{N-1}}{x_1^{\alpha_1} \ldots x_{N-1}^{\alpha_{N-1}}} \int_{x_N^-}^{x_N^+} dx_N P_N^{1\,(d-N-1)/2} \\ &= C_{N-1}^1 G(p_1,\ldots,p_{N-2})^{(N-1-d)/2} \int \frac{dx_1 \ldots dx_{N-1}}{x_1^{\alpha_1} \ldots x_{N-1}^{\alpha_{N-1}}} P_{N-1}^{1\;\;(d-(N-1)-1)/2} \end{aligned} \tag{2.7}$$

where $P_N^1\left(x_N^+\right) = P_N^1\left(x_N^-\right) = 0$ and

$$\begin{aligned} \int_{x_N^-}^{x_N^+} dx_N P_N^{1\,(d-N-1)/2} &= \frac{2\pi^{1/2}\Gamma\left(\frac{d-N+1}{2}\right)}{\Gamma\left(\frac{d-N+2}{2}\right)} G(p_1,\ldots,p_{N-1})^{(d-N)/2} G(p_1,\ldots,p_{N-2})^{(N-1-d)/2} \\ &\times P_{N-1}^{1\;\;(d-(N-1)-1)/2} \end{aligned}$$

using $P_N^1 = \frac{1}{4} G(p_1,\ldots,p_{N-2})\left(x_N^+ - x_N\right)\left(x_N - x_N^-\right)$ and $\left(x_N^+ - x_N^-\right)^2 = 16 \frac{G(p_1,\ldots,p_{N-1})}{G(p_1,\ldots,p_{N-2})^2} P_{N-1}^1$. This can be repeated straightforwardly for all variables except $x_1 = k^2 - m_1^2$ whose integration limits are simply derived from the $k-$modulus integration limits. The generalisation to the two-loop case is straightforward, with the integration at each step performed over the $x-$variables involving a given external momentum, and the last ones derived by the corresponding k_1- and k_2-modulus integration limits. We have checked both analytically and numerically that the limits, as defined above, reproduce the known results for several examples at one and two loops.

3. Deriving differential equations

Differential equations are usually written in terms of external kinematical invariants, $s_{ij} = (p_i + p_j)^2$ and internal masses, m_i^2. In the standard approach, since the integral in the momentum-space representation is not an explicit function of the kinematical invariants, derivatives with respect to external momenta, $p_j^\mu \frac{\partial}{\partial p_i^\mu}$, are used. In Baikov representation though, the dependence on external kinematical invariants and internal masses is explicit. Indeed in Eq. (2.3), it is easy to identify two terms that depends on the external kinematics and/or masses, namely the overall factor $G(p_1,\ldots,p_E)^{(-d+E+1)/2}$ and the Baikov polynomial P_N^L itself. The differentiation of the first factor causes no problem since the result is expressed in terms of the original integral. For the Baikov polynomial this is not so, since the derivative introduces a different integrand that is not directly expressible in terms of FI, Eq. (2.1). To be more specific, let us denote by X a generic kinematical variable, for instance a Mandelstam invariant $X=(p_i+p_j)^2$ or an internal mass $X=m_i^2$. Then

$$\begin{aligned}\frac{\partial}{\partial X}F_{\alpha_1\ldots\alpha_N} &= \left(\frac{-d+E+1}{2}\right)\left(\frac{1}{G}\frac{\partial G}{\partial X}\right)F_{\alpha_1\ldots\alpha_N} \\ &+ C_N^L G^{(-d+E+1)/2}\int\frac{dx_1\ldots dx_N}{x_1^{\alpha_1}\ldots x_N^{\alpha_N}}P_N^{L\,(d-M-1)/2}\left[\left(\frac{d-M-1}{2}\right)\frac{1}{P_N^L}\frac{\partial P_N^L}{\partial X}\right]\end{aligned} \tag{3.1}$$

where G is used for $G(p_1,\ldots,p_E)$. Based on the fact that the derivatives $\frac{\partial P_N^L}{\partial X},\frac{\partial P_N^L}{\partial x_a}$ are polynomials in x_a, the idea is to turn the derivative with respect to X into derivatives with respect to x_a. This can be achieved by the equation, known as the syzygy equation [80, 81],

$$b\frac{\partial P_N^L}{\partial X}+\sum_a c_a\frac{\partial P_N^L}{\partial x_a}=0 \tag{3.2}$$

with b and c_a being polynomials in x_a.

Assuming that a solution of this equation has been found such that b is independent of x_a (eventually depending on external kinematics and internal masses and not identical to zero), we have

$$\begin{aligned}\frac{\partial}{\partial X}F_{\alpha_1\ldots\alpha_N} &= \left(\frac{-d+E+1}{2}\right)\frac{1}{G}\frac{\partial G}{\partial X}F_{\alpha_1\ldots\alpha_N} \\ &+ C_N^L G^{(-d+E+1)/2}\int\frac{dx_1\ldots dx_N}{x_1^{\alpha_1}\ldots x_N^{\alpha_N}}\left(-\sum_a\frac{c_a}{b}\frac{\partial}{\partial x_a}P_N^{L\,(d-M-1)/2}\right)\end{aligned} \tag{3.3}$$

Then integrating by parts the second term in the rhs of the above equation and assuming that surface terms are vanishing (a standard assumption through Baikov representation) we get

$$\begin{aligned}\frac{\partial}{\partial X}F_{\alpha_1\ldots\alpha_N} &= \left(\frac{-d+E+1}{2}\right)\frac{1}{G}\frac{\partial G}{\partial X}F_{\alpha_1\ldots\alpha_N} \\ &+ C_N^L G^{(-d+E+1)/2}\int dx_1\ldots dx_N P_N^{L\,(d-M-1)/2}\left\{\sum_a\frac{\partial}{\partial x_a}\left(\frac{c_a}{b}\frac{1}{x_1^{\alpha_1}\ldots x_N^{\alpha_N}}\right)\right\}\end{aligned} \tag{3.4}$$

The term in the curly bracket is easily seen to be a sum of terms of the form $\frac{1}{x_1^{\alpha'_1}\ldots x_N^{\alpha'_N}}$. The powers α'_a depend on the actual form of the solution of the syzygy equation, Eq. (3.2). The result is as expected

$$\frac{\partial}{\partial X}F_{\alpha_1\ldots\alpha_N}=\sum_i R_i\,F_{\alpha_1^{(i)}\ldots\alpha_N^{(i)}} \tag{3.5}$$

with coefficients R_i that are rational functions of the space-time dimension d, the external kinematics and the internal masses. The rhs of the above equation contains integrals that are in general not MI. We have verified in numerous examples, that after applying a standard IBP reduction to MI for the rhs of the above equation, the resulting differential equations for the MI are the same as those obtained with the standard approach. It is still interesting to note that the initial form, Eq. (3.5), is generally not.

PoS(RADCOR2017)006

4. Cutting Feynman Integrals

Cutting FI in the Baikov representation has a very natural definition. Indeed we define an $n-$cut as follows

$$F_{\alpha_1\ldots\alpha_N}|_{n\times\text{cut}} \equiv C_N^L (G)^{(-d+E+1)/2} \left(\prod_{a=n+1}^{N} \int dx_a\right)\left(\prod_{c=1}^{n} \oint_{x_c=0} dx_c\right) \frac{1}{x_1^{\alpha_1}\ldots x_N^{\alpha_N}} P_N^{L\,(d-M-1)/2} \tag{4.1}$$

where the Baikov variables $\{x_a : a = 1,...,N\}$ have been divided in two subsets, containing n cut propagators and $(N-n)$ uncut ones. The cut operation defined above is operational in any space-time dimension d and for any FI given by Eq. (2.1). Notice that the definition of the cut, Eq. (4.1), is not identical to the traditional unitarity cut, see for instance Section 8.4 of ref. [82], due to the lack of the θ-function constraint on the energy, and therefore it is not directly related to the discontinuity of the FI [83, 84].

Let us now consider a set of MI, $F_i \equiv F_{\alpha_1^{(i)}\ldots\alpha_N^{(i)}}$, $i = 1,...,I$, satisfying a system of DE, with respect to variables X_j,

$$\frac{\partial}{\partial X_j} F_i = \sum_{l=1}^{I} M_{il}^{(j)} F_l \tag{4.2}$$

with matrices $M^{(j)}$ depending on kinematical variables, internal masses, and the space-time dimension, d. Since the derivation of DE in Section 3 is insensitive to the cut operation, as defined in Eq. (4.1), we may immediately write[6]

$$\frac{\partial}{\partial X_j} F_i|_{n\times\text{cut}} = \sum_{l=1}^{I} M_{il}^{(j)} F_l|_{n\times\text{cut}} \tag{4.3}$$

with $F|_{n\times\text{cut}}$ representing an arbitrary $n-$cut: in other words, the cut integrals satisfy the same DE as the uncut ones[7]. Of course for a given $n-$cut many of the MI that are not supported on the corresponding cut vanish identically. Nevertheless, Eq. (4.3) remains valid. Especially for the maximally cut integrals defined so that n is equal to the number of propagators (with $\alpha_i > 0$) of the integral, all integrals not supported on the cut vanish and the resulting DE is restricted to its homogeneous part. Evaluating the maximally cut MI provides therefore a solution to the homogeneous equation [78, 79]. Non-maximally cut integrals, on the other hand, can resolve non-homogenous parts of the DE as well [78].

One important implication is that cut and uncut integrals, although very different in many respects, as for instance their structure in $\varepsilon-$expansion ($\varepsilon \equiv (4-d)/2$), they are expressed in terms of the same class of functions[8]. This is particularly important if we want to know *a priori* if a system of DE can be solved, for instance, in terms of Goncharov Polylogarithms, or if the solution contains a larger class of functions including, for instance Elliptic Integrals.

Several results of maximally cut MI, expressed either in terms of Polylogarithmic functions or in terms of Elliptic Integrals, can be found in the Appendix B of ref. [65] as well as in refs. [73, 76].

[6] Care should be taken in defining the DE so that no symmetries of MI are used that may be violated by the corresponding $n-$cut.

[7] See also ref. [85, 86, 80] for related considerations.

[8] See also related discussion in ref. [19], section 3.4.1.

5. Discussion and Outloook

In this contribution we have presented properties of Feynman integrals in Baikov representation. We have shown how to determine the limits of integration and how to obtain DE with respect to external kinematics and internal masses. Then we provided a definition of a cut integral, operational in d dimensions, and show that a cut integral satisfies the same system of DE as the uncut, original integral.

Based on the fact that cut integrals satisfy the same system of DE as the full, uncut integrals we have verified that their analytic expressions are given in terms of the same class of functions, such as Goncharov Polylogarithms or Elliptic Integrals. We have therefore arrived at the conclusion that in a family of MI satisfying a given system of DE, the study of the maximally cut integrals for all its members can provide a *necessary and sufficient* criterion for the existence of a canonical form of the DE, and in the case when such a canonical form does not exist, it provides solutions of the homogeneous parts of the system of DE (see also ref. [78, 79]). An application of these ideas to non-planar pentabox integrals will be discussed elsewhere.

Baikov representation is well suited for these considerations, drastically simplifying the computation of cut integrals for arbitrary external momenta and internal masses. It is still an open question if it can also be used to actually compute the MI. To this end, an algorithm, allowing the resolution of singularities in ε, needs to be devised. It remains to be seen if this is possible and more importantly what kind of integral representations for the individual terms in this expansion such an algorithm produces.

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