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# Iterated integrations of complete elliptic integrals

**Ettore Remiddi***
*Dip. di Fisica, Universit di Bologna, and INFN, Sezione di Bologna, I-40126 Bologna*
*E-mail:* ettore.remiddi@bo.infn.it

**Lorenzo Tancredi**
*CERN Theory Division, CH-1211, Geneva 23, Switzerland*
*E-mail:* lorenzo.tancredi@cern.ch

We study an elliptic generalization of multiple polylogarithms that appears naturally in the computation of the imaginary part of the two-loop massive sunrise graph with equal masses. The newly introduced functions fulfil non-homogeneous second order differential equations. As an important result, we introduce a concept of weight associated to the action of the second order differential operator and show how to classify the relations between the functions bottom up in their weight.

*13th International Symposium on Radiative Corrections (Applications of Quantum Field Theory to Phenomenology)*
*24-29 September, 2017*
*St. Gilgen, Austria*

*Speaker.



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PoS(RADCOR2017)009

## 1. Introduction

We consider the equal mass two-loop sunrise amplitude in $d$-continuous dimensions, see

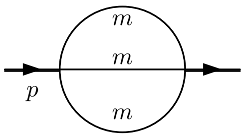


**Figure 1:** The two-loop equal mass sunrise graph.

Fig. 1, depending on the external momentum $p$. It is known that the amplitude develops an imaginary part in $p^2 = p_0^2 - \vec{p}^2 = u$ for timelike $p$ and $u \geq 9m^2$ or $W \geq 3m$, with $W = \sqrt{u}$. That imaginary part is equal, up to an overall constant irrelevant here, to the 3-body phase space at energy $W$, which in $d$ dimension, again up to an overall constant, reads

$$\Phi(d,W) = \int_{4m^2}^{(W-m)^2} \frac{db}{\sqrt{R_4(u,b)}} \left( \frac{R_4(u,b)}{ub} \right)^{\frac{d-2}{2}} , \tag{1.1}$$

where $R_4(b,u)$ is the fourth-order polynomial in $b$

$$R_4(b,u) = b(b-4m^2)((W-m)^2-b)((W+m)^2-b) . \tag{1.2}$$

At $d=2$ one finds (anticipating the notation which will be used later)

$$\begin{aligned} \Phi(2,W) = I_0(u) &= \int_{4m^2}^{(W-m)^2} \frac{db}{\sqrt{R_4(u,b)}} \\ &= \frac{2}{\sqrt{(\sqrt{u}+3m)(\sqrt{u}-m)^3}} K\left( \frac{(\sqrt{u}-3m)(\sqrt{u}+m)^3}{(\sqrt{u}+3m)(\sqrt{u}-m)^3} \right) , \end{aligned} \tag{1.3}$$

where $K(x)$ is the complete elliptic integral of the first kind. In the following, we will find also the related elliptic integral, also real for $W \geq 3m$, which corresponds to the second period of the associated elliptic curve

$$\begin{aligned} J_0(u) &= \int_0^{4m^2} \frac{db}{\sqrt{-R_4(u,b)}} \\ &= \frac{2}{\sqrt{(\sqrt{u}+3m)(\sqrt{u}-m)^3}} K\left( 1 - \frac{(\sqrt{u}-3m)(\sqrt{u}+m)^3}{(\sqrt{u}+3m)(\sqrt{u}-m)^3} \right) . \end{aligned} \tag{1.4}$$

By expanding $\Phi(d,W)$ in $d$ around $d=2$, one obtains the family of integrals

$$\int_{4m^2}^{(W-m)^2} \frac{db}{\sqrt{R_4(u,b)}} \times \left\{ \ln b,\ \ln(b-4m^2),\ \ln b\ \ln(b-4m^2),\ \ldots \right\} .$$

As an almost obvious generalization, we introduce the functions

$$EG^{(n)}(k,u) = \int_{b_i}^{b_j} \frac{db\ b^k}{\sqrt{R_4(u,b)}} g^{(n)}(u,b) , \tag{1.5}$$

where $k \geq 0$ is a non-negative integer, $(b_i, b_j)$ are any two of the four roots of $\sqrt{R_4(u,b)}$

$$\left\{0, \quad 4m^2, \quad (W-m)^2, \quad (W+m)^2 \right\} ,$$

and finally $g^{(n)}(u,b)$ is a polylogarithm in $b$ of weight $n$, whose *alphabet* corresponds to the above four roots of $\sqrt{R_4(u,b)}$, *i.e.* a polylogarithm which upon differentiation produces only the denominators

$$\left\{ \frac{1}{b}, \quad \frac{1}{b-4m^2}, \quad \frac{1}{b-(W-m)^2}, \quad \frac{1}{b-(W+m)^2} \right\}. \tag{1.6}$$

The functions defined in Eq. (1.5) are an obvious elliptic generalization of multiple polylogarithms and we will refer to them for simplicity as *elliptic polylogarithms* from now on, in spite of the fact that the name is already used in the literature with different meaning. Clearly, the two functions $I_0(u), J_0(u)$ seen above correspond to $EG^{(0)}(0,u)$ with $n = 0$ and $k = 0$ for a suitable choice of the end points $b_i, b_j$.

In this talk we will study the properties of the above *elliptic polylogarithms* $EG^{(n)}(k,u)$, by deriving the differential equations that they fulfill and then (in a kind of *reverse engineering* procedure) by solving them with the usual Euler's variation of constants approach. As a byproduct, we will establish a number of (somewhat unexpected) identities between the new functions and products of polylogarithms and complete elliptic integrals. The literature on elliptic polylogarithms is by now very vast, and similar functions to the ones considered here have been considered in [1–12], using very different approaches. This contribution relies heavily on Ref. [13], to which we refer for more details.

## 2. The differential equations

We start from the integration by parts (*ibp*) identities

$$\int_{b_i}^{b_j} db \frac{d}{db}\left( \sqrt{R_4(u,b)} b^k g^{(n)}(u,b) \right) = 0 ,$$

which are obvious as $\sqrt{R_4(u,b)}$ vanishes at both the end points. Working out explicitly the derivatives one finds that:

*i*) the derivative of the first factor, $(\sqrt{R_4(u,b)} b^k)$, can be written as a polynomial in $(u,b)$ divided by $\sqrt{R_4(u,b)}$, without affecting the factor $g^{(n)}(u,b)$, *i.e.* a combination, with coefficients depending on $u$, of the integrands appearing in the definition of functions $EG^{(n)}(k',u)$ with various values of $k'$ and the same $g^{(n)}(u,b)$;

*ii*) the derivative of the the polylogarithm $g^{(n)}(u,b)$, is itself a polylogarithm (or, more in general, a combination of polylogarithms) of lower weight, say $g^{(n-1)}(u,b)$, divided by one of the four denominators listed in Eq.(1.6); by writing the square root in the first factor as $\sqrt{R_4(u,b)} = R_4(u,b)/\sqrt{R_4(u,b)}$ one sees from Eq.(1.2) that those denominators are always compensated by one of the factors appearing in the definition of of $R_4(u,b)$ Eq.(1.2), so that, again, one is left with a combination of terms corresponding to the integrands appearing in the definition of some $EG^{(n-1)}(k'',u)$.

The identities are therefore linear combinations, with coefficients depending on $u$, of the functions $EG^{(n)}(k,u)$ with various values of $k$ (but the same weight $n$) and of functions $EG^{(n-1)}(k',u)$ (of lower weight) set equal to zero. One finds that the identities (used recursively, when needed) allow to express *all* the $EG^{(n)}(k,u)$ (for any non-negative integer $k$) in terms of *three master integrals* of weight $n$

$$EG^{(n)}(0,u), \quad EG^{(n)}(1,u), \quad EG^{(n)}(2,u),$$

and of $EG$'s of lower weight. As a consequence, one can write, for any non-negative integer $s$,

$$\int_{b_i}^{b_j} db \left( \sqrt{R_4(u,b)} b^s g^{(n)}(u,b) \right) = \sum_{k=0,1,2} a_s(k,u) EG^{(n)}(k,u),$$

where the $a_s(k,u)$ are suitable polynomials in $u$. One can now take the derivative in $u$ of the *l.h.s.* of the previous equation, then the $u$-derivative of its *r.h.s.*, and finally write that the two derivatives are equal. The *l.h.s.* gives (when expressing everything in terms of the three above master integrals)

$$\begin{aligned} \frac{d}{du} \int_{b_i}^{b_j} db \left( \sqrt{R_4(u,b)}\; b^s\; g^{(n)}(u,b) \right) &= \int_{b_i}^{b_j} db \frac{d}{du} \left( \sqrt{R_4(u,b)}\; b^s\; g^{(n)}(u,b) \right) \\ &= \sum_{k=0,1,2} \left( b_s(k,u) EG^{(n)}(k,u) + c_s(k,u) EG^{(n-1)}(k,u) \right) \;, \end{aligned}$$

where the $b_s(k,u), c_s(k,u)$ are suitable polynomials in $u$. Similarly, the derivative of the *r.h.s.* is

$$\begin{aligned} &\frac{d}{du} \sum_{k=0,1,2} \left( a_s(k,u) EG^{(n)}(k,u) \right) = \\ &\qquad \sum_{k=0,1,2} \left( \frac{d}{du} a_s(k,u) \right) EG^{(n)}(k,u) + \sum_{k=0,1,2} a_s(k,u) \frac{d}{du} EG^{(n)}(k,u) \;. \end{aligned}$$

The equality of the two derivatives gives a linear first order equation for a combination of the three master integrals; one can take three such equations, corresponding to three different values of $s$, and rewrite those three equations as

$$\begin{aligned} \frac{d}{du} EG^{(n)}(k,u) = &\sum_{k'=0,1,2} B(k,k',u) EG^{(n)}(k',u) \\ &+ \sum_{k'=0,1,2} C(k,k',u) EG^{(n-1)}(k',u) \;, \end{aligned}$$

*i.e.* a system of three linear differential equations for the three master integrals, where $B(k,k',u)$, and $C(k,k',u)$ are rational functions of $u$. It is to be noted that the nine coefficients of the homogeneous part $B(k,k',u)$ are *universal*, *i.e.* are the same for all the possible choices of $g^{(n)}(u,b)$, while the coefficients of the inhomogeneous part, the $C(k,k',u)$, depend on the actual value of $g^{(n)}(u,b)$. All the coefficients are rational expressions in $u$, with the simple poles

$$\frac{1}{u}, \quad \frac{1}{u-m^2}, \quad \frac{1}{u-9m^2}.$$

As an extension of an old result due to A. Sabry (1962) [14], it is convenient to introduce a new set of master integrals with the change of basis

$$
\begin{aligned}
EG_0^{(n)}(u) &= EG^{(n)}(0,u) \,, \\
EG_1^{(n)}(u) &= EG^{(n)}(1,u) - \frac{u+3m^2}{3} EG^{(n)}(0,u) \,, \\
EG_2^{(n)}(u) &= EG^{(n)}(2,u) - (u+3m^2) EG^{(n)}(1,u) + \frac{(u+3m^2)^2}{3} EG^{(n)}(0,u) \,.
\end{aligned} \tag{2.1}
$$

We stress here the change of notation from the original basis of functions $EG^{(n)}(k,u)$, to the new basis $EG_k^{(n)}(u)$. In the new basis the equations decouple into

$$
\frac{d}{du} EG_1^{(n)}(u) = \sum_{k=0,1,2} C_{1k}(u) EG_k^{(n-1)}(u)
$$

and

$$
\begin{aligned}
\frac{d}{du} EG_0^{(n)}(u) &= B_{00}(u) EG_0^{(n)}(u) + B_{02}(u) EG_2^{(n)}(u) + \sum_{k=0,1,2} C_{0k}(u) EG_k^{(n-1)}(u) \,, \\
\frac{d}{du} EG_2^{(n)}(u) &= B_{20}(u) EG_0^{(n)}(u) + B_{22}(u) EG_2^{(n)}(u) + \sum_{k=0,1,2} C_{2k}(u) EG_k^{(n-1)}(u) \,,
\end{aligned} \tag{2.2}
$$

where the coefficients $B_{ij}(u)$, $C_{ij}(u)$ have the same structure as the $B(k,k',u)$, and $C(k,k',u)$ above; again, the coefficients of the homogeneous part, $B_{ij}(u)$, are *universal*, *i.e.* are the same independently of the polylogarithm $g^{(n)}(u,b)$ entering in the definition of $EG^{(n)}(k,u)$ Eq.(1.5). Note that $EG_0^{(n)}(u), EG_2^{(n)}(u)$ do not appear in the equation for $EG_1^{(n)}(u)$ and $EG_1^{(n)}(u)$ does not appear in the equations for $EG_0^{(n)}(u)$, $EG_2^{(n)}(u)$, while the $EG_k^{(n-1)}(u)$ appear in all the equations as inhomogeneous terms.

A two-by-two linear first order system implies a second order linear equation for a single function; for the system (2.2), considering for simplicity only the homogeneous part of the system, the second order equation for $EG_0(u)$, corresponding to $n=0$ in the notation of (1.5), is

$$
D\left(u, \frac{d}{du}\right) EG_0(u) = 0 \,, \tag{2.3}
$$

with

$$
\begin{aligned}
D\left(u, \frac{d}{du}\right) = \Bigg\{ &\frac{d^2}{du^2} + \left[\frac{1}{u} + \frac{1}{u-m^2} + \frac{1}{u-9m^2}\right] \frac{d}{du} \\
&+ \frac{1}{m^2} \left[ -\frac{1}{3u} + \frac{1}{4(u-m^2)} + \frac{1}{12(u-9m^2)} \right] \Bigg\} \,,
\end{aligned} \tag{2.4}
$$

and the accompanying function $EG_2(u)$ is given by

$$
EG_2(u) = \left[ -\frac{2}{3} u(u-m^2)(u-9m^2) \frac{d}{du} + \left( -\frac{1}{3}u^2 + \frac{14}{3} m^2 u + m^4 \right) \right] EG_0(u) \,. \tag{2.5}
$$

In particular, the functions $I_0(u)$, $J_0(u)$, defined in Eq.s(1.3,1.4), satisfy Eq.(2.3), with the accompanying functions $I_2(u)$, $J_2(u)$, given by Eq.(2.5).

As a by-product, if a function $f(u)$ satisfies the equation

$$D\left(u, \frac{d}{du}\right) f(u) = 0 \ ,$$

one has

$$f(u) = c_1 I_0(u) + c_2 J_0(u) \ ,$$

where $c_1, c_2$ are two constants to be fixed by the boundary conditions.

## 3. Solving the equations

The differential equation for $EG_1^{(n)}(u)$ is trivial and can be solved by quadrature. The system of two differential equations for $EG_0^{(n)}(u), EG_2^{(n)}(u)$ can be solved by the Euler's method of the variation of constants, which requires the knowledge of the two pairs of independent solutions of the homogeneous equation. In our case, the homogeneous solutions are $I_0(u), I_2(u)$ and $J_0(u), J_2(u)$, where $I_0(u), J_0(u)$ are the complete elliptic integrals of first kind seen at the beginning, while $I_2(u), J_2(u)$, which can be obtained from Eq.(2.5), are in general linear combinations of complete elliptic integrals of first and second kind. Euler's method requires also the Wronskian of the system, $W_s(u)$ which in this case is found to be constant and equal to $\pi$

$$W_s(u) = I_0(u)J_2(u) - I_2(u)J_0(u) = \pi \ .$$

In the second order formalism, if the inhomogeneous equation is written as

$$D\left(u, \frac{d}{du}\right) F(u) = N(u) \ ,$$

the solution *à la Euler* reads

$$F(u) = \left[c_1 - \int_{u_0}^{u} \frac{dv}{W(v)} N(v) J_0(v)\right] I_0(u) + \left[c_2 + \int_{u_0}^{u} \frac{dv}{W(v)} N(v) I_0(v)\right] J_0(u) \qquad (3.1)$$

where $W(v)$ is the Wronskian of the 2nd order equation

$$\begin{aligned} W(u) &= I_0(u)\frac{d}{du}J_0(u) - J_0(u)\frac{d}{du}I_0(u) \\ &= -\frac{3\pi}{2u(u-m^2)(u-9m^2)} \ . \end{aligned} \qquad (3.2)$$

Note that $W(u)$ is in general *not* identical to $W_s(u)$. Eq. (3.1) is a sum of terms

$$\Big(I_0(u), J_0(u)\Big) \times \Big(F^{(n)}(u), \ldots\Big)$$

where the functions $F^{(n)}(u)$ are different types of iterative integrals. The simplest instance of the functions $F^{(n)}(u)$ is given by

$$F^{(n)}(u) = I_k^{[n]}(p_n, p_{n-1}, .., p_1; u)$$

where $I_k^{[n]}(p_n,..,p_1;u)$ is the (repeated) integration of $I_k(u)$ or $J_k(u)$ times (simple) rational factors

$$I_k^{[n]}(p_n,p_{n-1},..,p_1;u) = \int_{u_0}^{u}\frac{du_n}{u_n-p_n}\int_{u_0}^{u_n}\frac{du_{n-1}}{u_{n-1}-p_{n-1}}..\int_{u_0}^{u_2}\frac{du_1}{u_1-p_1}I_k(u_1)\ , \tag{3.3}$$

or a similar formula with $J_k(u_1)$, with $k=0$ or $k=2$. More in general (and in the more interesting cases), the rightmost term in the above formula can also be a product of elliptic integrals, $I_k(v)I_{k'}(v)$, $I_k(v)J_{k'}(v)$, *etc.*

Eq.(3.3) reminds closely the definition of the Generalized-Goncharov polylogarithms

$$G_k^{[n]}(p_n,p_{n-1},..,p_1;u) = \int_{u_0}^{u}\frac{du_n}{u_n-p_n}\int_{u_0}^{u_n}\frac{du_{n-1}}{u_{n-1}-p_{n-1}}..\int_{u_0}^{u_2}\frac{du_1}{u_1-p_1}\mathbf{1}\ , \tag{3.4}$$

where the rightmost factor is $\mathbf{1}$ (usually not written, of course) instead of $I_k(u_1)$ as in (3.3). When using the Goncharov polylogarithms one can define

$$G^{[0]}(u) = 1\ , \qquad G^{[-1]}(u) = 0 \tag{3.5}$$

so that

$$\frac{d}{du}G^{[0]}(u) = 0\ ;$$

therefore, they satisfy the obvious relation

$$\frac{d}{du}G^{[n]}(p_n,p_{n-1},..,p_1;u) = \frac{1}{u-p_n}\ G^{[n-1]}(p_{n-1},..,p_1;u)$$

for any non-negative integer $n$. A similar formula can be written also for the $I_k^{[n]}(p_n,p_{n-1},..,p_1;u)$,

$$\frac{d}{du}I_k^{[n]}(p_n,p_{n-1},..,p_1;u) = \frac{1}{u-p_n}\ I_k^{[n-1]}(p_{n-1},..,p_1;u)$$

but it is valid only for $n>1$, because the derivative of the rightmost factor, $I_k(u)$ *does not vanish*

$$\frac{d}{du}I_k(u) \neq 0\ ,$$

but according to Eq.s(2.2) is a combination of $I_0(u)$ and $I_2(u)$. The same applies of course to $J_k(u)$, $I_k(v)I_{k'}(v)$, *etc.*

If we refer to the number of integrations $n$ as the weight of the functions $I_k^{[n]}$, clearly at weight $n=1$ the structure is non-trivial due to the presence of the elliptic integration kernels. The properties of the functions $I_k^{[1]}, J_k^{[1]}$ are easily investigated with the by now familiar *ibp* approach, *i.e.* by considering all the *integration by parts* identities generated by a relation of the form

$$\int^{u} dv\frac{d}{dv}\Big(X(v)\Big) = X(u)\ ,$$

where $X(v)$ stands for all possible products of the form

$$\left(1,\ v^n,\ \frac{1}{v^n},\ \frac{1}{(v-m^2)^n},\ \frac{1}{(v-9m^2)^n}\right)\times\Big(I_k(v)\ ,\ J_k(v)\Big)\ ,$$

where $n$ is a positive integer. Note that, in general, it is not enough to consider integrals over the $I_k(u)$ and $J_k(u)$ with factors of $1/(v-p_j)^n$. Indeed, one finds that *all* the above (indefinite) integrals can be expressed in terms of the just four *master integrals*

$$\int^u dv \left(1,\ \frac{1}{v},\ \frac{1}{v-m^2},\ \frac{1}{v-9m^2}\right) \times \left( I_0(v) \right) ,$$

which involve only $I_0(v)$ (plus terms in $I_k(u)$ arising from the end-point contributions).

A similar (but somewhat more complicated and interesting) pattern appear in the repeated integration of *higher order* products, $I_k(u)I_{k'}(u),\ \ I_k(u)J_{k'}(u),$ where it is essential to make use of the Wronskian (3) to re-express all products of functions in terms of a subset of linear independent ones.

## 4. Some results and Conclusion

A typical result of the *reverse engineering* is the *identity*

$$\begin{aligned} \int_{4m^2}^{(W-m)^2} \frac{db}{\sqrt{R_4(u,b)}} \ln(b) &= \frac{2}{3} \ln(u-m^2) \int_{4m^2}^{(W-m)^2} \frac{db}{\sqrt{R_4(u,b)}} \\ &= \frac{2}{3} \ln(u-m^2)\, I_0(u) , \end{aligned}$$

a result which can be established by writing explicitly Eq.s(2.2) for the integral

$$\int_{4m^2}^{(W-m)^2} \frac{db}{\sqrt{R_4(u,b)}} \ln(b)$$

and then solving the resulting equation *à la Euler*. The result can also be obtained by means of the operator $D(u,d/du)$, Eq.(2.4), by checking, with an explicitly calculation, the validity of the relation

$$D\left(u, \frac{d}{du}\right) \left( \int_{4m^2}^{(W-m)^2} \frac{db}{\sqrt{R_4(u,b)}} \ln(b) - \frac{2}{3} \ln(u-m^2)\, I_0(u) \right) = 0 .$$

Similarly, one can obtain

$$\begin{aligned} \int_{4m^2}^{(W-m)^2} \frac{db}{\sqrt{R_4(u,b)}} \ln(b-4m^2) &= \left( \frac{1}{2} \ln(u-9m^2) + \frac{1}{6} \ln(u-m^2) \right) \int_{4m^2}^{(W-m)^2} \frac{db}{\sqrt{R_4(u,b)}} \\ &- \frac{\pi}{2} \int_0^{4m^2} \frac{db}{\sqrt{-R_4(u,b)}} , \end{aligned}$$

(note the different integration range in the last term), *etc.*

As a conclusion, the investigation of the properties of the integrals $EG_k^{(n)}(u)$, Eq.s(2.1,1.5) brought us to introduce the second order differential operator $D(u,d/du)$, Eq.(2.4), which can be regarded as a *destruction operator* of the weight $n$ of the functions $EG_k^{(n)}(u)$ (it decreases by one the weight $n$ when acting on those functions), while the Euler's formula for the solution of the so obtained equation provides with the corresponding *creation operator*, which increases by one the same weight. As a result, one can in particular establish a new class of otherwise unexpected identities.

As a final remark, our approach is built up explicitly on top of the (elliptic) curve associated to the physical 3-body phase space

$$R_4(b,u) = b(b-4m^2)((W-m)^2-b)((W+m)^2-b) \; .$$

Still, our method is more general and can be used to study similar functions stemming from different elliptic curves, or even associated to higher genus surfaces. In the latter case, we expect classes of functions which fulfil higher order differential equations, see for example the three-loop massive banana graph [15]. While of course the algebra of these functions will be more complicated, conceptually there is no obstruction in applying the methods described here in order to simplify the corresponding special functions. This will be studied in a future publication.

## Acknowledgements

The speaker (ER) acknowledges the financial support of DESY for the attendance of the RADCOR conference 2017.

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