QCD vacuum energy in 5 loops

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We report about analytic calculation of the five-loop contribution to the anomalous dimension of the QCD vacuum energy. The result completes recent series of works [1–7] devoted to the renormalization of QCD at the five-loop level.

13th International Symposium on Radiative Corrections (Applications of Quantum Field Theory to Phenomenology)
25-29 September, 2017
St. Gilgen, Austria

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1. Introduction

Last year saw a remarkable development in the program of renormalization of the QCD Lagrangian: the $\beta$-function, the quark anomalous dimension as well as anomalous dimensions of all relevant quantum fields had been computed at the five-loop level [2–7] for the case of a generic gauge group. Essentially every result has been successfully cross-checked by (at least) two independent calculations! It is worthwhile to note, that at the 4-loop level it took almost 8 years before the pioneering result for the QCD $\beta$-function [8] was independently confirmed [9].

In our talk we describe analytic calculation of the five-loop contribution to the anomalous dimension of the QCD vacuum energy. The result provides the last missing piece for the complete renormalization of the QCD Lagrangian in 5 loops.

2. Preliminaries

Our starting point is the QCD Lagrangian with $n_f$ quark flavors written in terms of (bare) fields, coupling constant $g_B$ and quark masses $m_B^f$:

$$L_{\text{full QCD}} = -\frac{1}{4} (G_{\mu\nu}^B)^2 + \bar{\psi}_B (i\hat{\nabla} - m_B^B) \psi_B - E_B^0,$$  \hspace{0.5cm} (2.1)

where the $m_B^f$ stands for a (diagonal) matrix of the bare quark masses $m_B^f$ (with $f$ running from 1 to $n_f$). The bare coupling constant, quark masses, gluon, quark and ghost fields are related to the renormalized ones as follows:

$$g_B = \sqrt{Z_3 g_1}, \quad m_B^f = Z_m m_B^f, \quad A_0^{\alpha\mu} = \sqrt{Z_3} A^{\alpha\mu}, \quad \psi_0^f = \sqrt{Z_2} \psi^f, \quad c_0^a = \sqrt{Z_3} c^a.$$ \hspace{0.5cm} (2.2)

$E_0(\mu)$ is the renormalized (density of) vacuum energy

$$E_0^B = \mu^{-2\varepsilon} \left( E_0(\mu) - Z_0^{di} \sum_f m_f^4 + Z_0^{nd} \sum_{f,f'} m_f^2 m_{f'}^2 \right), \quad \varepsilon = (4-D)/2,$$ \hspace{0.5cm} (2.3)

while $Z_0^{di}$ and $Z_0^{nd}$ are the corresponding renormalization constants (RCs) [10].

In terms of the renormalized fields the Lagrangian reads\footnote{For simplicity we set the 't Hooft mass $\mu = 1$ in eqs. (2.1), 2.2 and (2.4).}.

$$L_{\text{full QCD}} = -\frac{1}{4} Z_3 (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2} g Z_1 g^{\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) (A_\mu - A_\nu)^a$$
$$- \frac{1}{4} g^2 Z_2 Z_1^g (A_\mu \times A_\nu)^2 - \frac{1}{2} g Z_3 Z_1 (\partial_\nu A_\mu)^2 + Z_2 Z_1 (\partial_\nu c) (\partial_\nu c) + g Z_2 Z_1 \partial_\mu \bar{c} (A_\mu \times c)$$
$$+ Z_2 \sum_{f=1}^{n_f} \bar{\psi}_f i \hat{\nabla} \psi_f + g Z_1 \bar{\psi}_f A_\mu \psi_f - Z_1 \bar{\psi}_f \psi_f - E_0^B.$$ \hspace{0.5cm} (2.4)

All the RCs appearing in (2.4) have been recently computed at 5 loops except for $Z_0^{di}$ and $Z_0^{nd}$. The latter are (partially) known at 4 loops (see [11] and references therein).
The anomalous dimension of the renormalized vacuum energy reads:

$$\hat{\gamma}_0(m) = \mu^2 \frac{d}{d \mu^2} E_0 = (4 \gamma_n - \varepsilon) \hat{Z}_0(m) + \left( - \varepsilon + \beta \right) a_s \frac{d}{da_s} \hat{Z}_0(m)$$

$$= \left( \sum_f m_f^4 \right) \gamma_0^{di}(a_s) + \left( \sum_{f \neq f'} m_f^2 m_{f'}^2 \right) \gamma_0^{nd}(a_s), \quad (2.5)$$

where $a_s = \frac{g^2}{4\pi} = \frac{a_s}{\pi}$ and

$$\hat{Z}_0(m) \equiv Z_0^d \sum_f m_f^4 + Z_0^{nd} \sum_{f \neq f'} m_f^2 m_{f'}^2.$$ 

For the case of just one massive quark we have $\hat{\gamma}_0 = m_q^4 \gamma_0^{di}(a_s)$ and $\gamma_0^{nd} = 0.$

3. Calculation and results

We have computed the 5-loop contribution to the $\hat{\gamma}_0$ within the massless approach (that is with the use of the global $R^*$-operation [12–14], the FORM [15, 16] program BAICER [17] and the computer facilities of the KIT) for a general case of arbitrary many quark flavors with different masses. Our results for

$$\gamma_0^{di} = \sum_{i \geq 0} \left( \gamma_0^{di} \right)_i \left( \frac{\alpha_s}{4\pi} \right)^i \quad \text{and} \quad \gamma_0^{nd} = \sum_{i \geq 2} \left( \gamma_0^{nd} \right)_i \left( \frac{\alpha_s}{4\pi} \right)^i \quad (3.1)$$

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$$\left( \gamma_0^{di} \right)_0 = \{ -dR \}, \quad \left( \gamma_0^{di} \right)_1 = \frac{dR}{16\pi^2} \{ -4C_F \}, \quad (3.2)$$

$$\left( \gamma_0^{di} \right)_2 = \frac{dR}{16\pi^2} \left\{ C_F^3 \left[ \frac{131}{2} - 48\zeta_3 \right] + C_F C_A \left[ -\frac{109}{2} + 24\zeta_3 \right] + 10C_F T_f n_f + 48C_F T_f \right\}, \quad (3.3)$$

$$\left( \gamma_0^{di} \right)_3 = \frac{dR}{16\pi^2} \left\{ C_F^3 \left[ -\frac{2942}{3} + 48\zeta_3 + 288\zeta_4 + 160\zeta_5 \right] + C_F T_f^2 n_f \left[ \frac{10912}{243} - \frac{128}{3}\zeta_3 \right] \right.$$

$$+ C_F T_f n_f C_A \left[ -\frac{256}{9} + C_F^2 T_f \left[ \frac{562}{3} + \frac{32}{3}\zeta_3 - 160\zeta_4 \right] \right.$$

$$+ C_F T_f C_A \left[ \frac{2644}{243} + 16\zeta_3 + 128\zeta_4 \right] + C_F^2 T_f \left[ -64 \right] \right.$$ 

$$+ \left. C_F T_f C_A \left[ \frac{5888}{9} + 352\zeta_3 - 160\zeta_5 \right] + C_F^2 C_A \left[ \frac{3584}{3} - \frac{3304}{3}\zeta_3 + 32\zeta_4 + 720\zeta_5 \right] \right) \right\}, \quad (3.4)$$
\[ (\gamma_0^{di})_4 = \frac{dR}{16\pi^2} \left\{ C_F^4 \left[ \frac{787555}{48} - 9470\zeta_3 + 5568\zeta_3^2 + 432\zeta_4 - 5232\zeta_5 - 1200\zeta_6 - 3024\zeta_7 \right] 
+ C_F T_f^3 n_f^3 \left[ \frac{1492}{27} + \frac{832}{27} \zeta_3 - \frac{256}{3} \zeta_4 \right] 
+ C_F T_f^3 n_f^3 \left[ -\frac{256}{9} \right] 
+ C_F T_f^3 n_f^3 C_A \left[ \frac{103547}{486} - \frac{10064}{9} \zeta_3 - 40\zeta_4 + \frac{2816}{3} \zeta_5 \right] 
+ C_F T_f^3 n_f^3 C_A \left[ -\frac{14080}{9} - \frac{8704}{3} \zeta_3 - 96\zeta_3^2 + 528\zeta_4 + 640\zeta_5 - 400\zeta_6 \right] 
+ C_F T_f^3 n_f^3 C_A \left[ -\frac{27373}{18} - \frac{3244}{3} \zeta_4 - 1024\zeta_4^2 - 1700\zeta_4 + 4624\zeta_5 + 1600\zeta_6 \right] 
+ C_F T_f^3 [2360 + 3136\zeta_3 - 4480\zeta_5] \right\} 
+ \frac{\left[ C_F T_f^3 n_f^3 C_A \left[ \frac{1388131}{972} + \frac{13004}{9} \zeta_3 + 2176\zeta_3^2 - 2000\zeta_4 - \frac{17512}{3} \zeta_5 + 1600\zeta_6 \right] 
+ C_F T_f^3 A T_f \left[ -\frac{31160}{9} - 8544\zeta_4 + 480\zeta_4^2 - 1584\zeta_4 + 9920\zeta_5 + 1200\zeta_6 + 2240\zeta_7 \right] 
+ C_F T_f^2 n_f^2 C_A \left[ \frac{744661}{486} + \frac{466}{9} \zeta_3 - 1032\zeta_3^2 + 2376\zeta_4 + \frac{5444}{3} \zeta_5 - 2300\zeta_6 - 294\zeta_7 \right] 
+ C_F T_f^2 C_A \left[ 9456 + 2896\zeta_3 + 584\zeta_3^2 - 1452\zeta_4 - \frac{17360}{3} \zeta_5 + 1100\zeta_6 + \frac{10997}{3} \zeta_7 \right] 
+ C_F T_f A \left[ -\frac{287954}{9} - \frac{22844}{3} \zeta_3 - 6088\zeta_3^2 + 5764\zeta_4 + 27296\zeta_5 - 6500\zeta_6 - 9184\zeta_7 \right] 
+ C_F^2 A^2 \left[ \frac{23242925}{972} - \frac{70940}{9} \zeta_3 - 1820\zeta_3^2 + 1719\zeta_4 + 14402\zeta_5 - 2150\zeta_6 + 14182\zeta_7 \right] 
+ C_F^3 A \left[ -\frac{28785743}{3888} + \frac{167719}{27} \zeta_3 + 2318\zeta_3^2 - \frac{9280}{3} \zeta_4 - \frac{17512}{3} \zeta_5 + 4675\zeta_6 - \frac{14525}{6} \zeta_7 \right] \right\} 
+ n_f \frac{d^4\bar{\psi} \gamma^a \bar{\psi}}{dR} \left[ -824 - 5584\zeta_3 + 2304\zeta_3^2 + 720\zeta_4 - 3680\zeta_5 + 7056\zeta_7 \right] 
+ \frac{d^4\bar{\psi} \gamma^a \bar{\psi}}{dR} \left[ -2816\zeta_3 + 2688\zeta_3^2 + 8320\zeta_5 - 5320\zeta_7 \right] 
+ \frac{d^4\bar{\psi} \gamma^a \bar{\psi}}{dR} \left[ 352 + 3608\zeta_3 - 11136\zeta_3^2 - 360\zeta_4 + 2492\zeta_7 \right] \right\}, \quad (3.5)
\[
(\gamma_{0}^{nd})_{4} = \frac{dR}{16\pi^{2}} \left\{ C_{F} T_{3}^{e} n_{f}^{2} \left[ -\frac{256}{9} + C_{F}^{2} T_{3}^{e} n_{f} \left[ -\frac{18176}{9} + 1920\zeta_{3} \right]
\right.ight.
\]
\[
+ C_{F} T_{3}^{e} n_{f} C_{A} \left[ -\frac{14080}{9} - \frac{8704}{3} \zeta_{3} - 96\zeta_{3}^{2} + 528\zeta_{3} + 640\zeta_{5} - 400\zeta_{6} \right]
\right.
\]
\[
+ C_{F}^{2} T_{3}^{e} C_{A} \left[ -\frac{31160}{9} + 8544\zeta_{3} + 480\zeta_{3}^{2} - 1584\zeta_{4} - 9920\zeta_{5} + 1200\zeta_{6} + 2240\zeta_{7} \right]
\]
\[
+ C_{F} T_{f} C_{A}^{2} \left[ 9456 + 2896\zeta_{3} + 584\zeta_{3}^{2} - 1452\zeta_{4} - \frac{17360}{3} \zeta_{5} + 1100\zeta_{6} + \frac{10997}{3} \zeta_{7} \right]
\]
\[
+ \left\{ d_{F}^{abcd} d_{F}^{abcd} \right\} \left[ -2816\zeta_{3} + 2688\zeta_{3}^{2} + 8320\zeta_{8} - 5320\zeta_{7} \right].
\]
\] (3.8)

Here \(\zeta\) is the Riemann zeta-function (with \(\zeta_{1} = 1.2020569, \ldots, \zeta_{4} = 1.0823232, \ldots, \zeta_{6} = 1.0173431, \ldots, \zeta_{7} = 1.0083943\ldots\). \(C_{F}\) and \(C_{A}\) are the quadratic Casimir operators of the quark \(T_{a}^{b} T_{a}^{b}\) and the adjoint \(C_{A}^{b} C_{A}^{b}\) representations of the Lie algebra. \(n_{f}\) stands for the number of quark flavors, \(d_{q}\) is dimension of the quark representation of the gauge group and \(T_{f}\) refers to the trace normalization \(\text{tr}(T_{a}^{b} T_{b}^{b}) = T_{f} \delta_{ab}\). The higher order group invariants are defined according to [8, 18].

Note that if a color structure contributes to \(\gamma_{0}^{nd}\) then the same color structure also appears in \(\gamma_{0}^{di}\) with an identical coefficient. This is a direct consequence of our way of separating \(\gamma_{0}(m)\) into two pieces in eq. (2.5).

For the QCD with the colour group SU(3) we have

\[
\gamma_{0}^{di} = \frac{1}{16\pi^{2}} \left\{ -3 - 4\frac{\alpha_{s}}{\pi} \right. + \left(\frac{\alpha_{s}}{\pi}\right)^{2} \left[ -\frac{313}{24} + \frac{5}{4} n_{f} + 2 \zeta_{3} \right]
\]
\[
+ \left(\frac{\alpha_{s}}{\pi}\right)^{3} \left[ -\frac{14251}{432} + \left(\frac{341}{486} - \frac{2}{3} \zeta_{3}\right) n_{f}^{2} + \frac{231}{2} \zeta_{3}
\right.
\]
\[
+ n_{f} \left[ \frac{303061}{3072} - \frac{882061}{432} \zeta_{3} + \frac{20083}{288} \zeta_{3}^{2} - \frac{124511}{192} \zeta_{4}
\right.
\]
\[
- \frac{11543}{3} \zeta_{5} + \frac{632375}{576} \zeta_{6} + \frac{36883}{32} \zeta_{7}
\]
\[
+ n_{f} \left[ \frac{6286061}{62208} + \frac{593}{864} \zeta_{3} - \frac{2327}{144} \zeta_{3}^{2} + \frac{50867}{576} \zeta_{4} + \frac{1571}{144} \zeta_{5} - \frac{27125}{288} \zeta_{6} - \frac{147}{16} \zeta_{7}
\right.
\]
\[
+ n_{f} \left[ -\frac{530837}{373248} - \frac{4817}{432} \zeta_{3} + \frac{179}{96} \zeta_{4} + \frac{83}{9} \zeta_{5}\right]
\]
\[
+ n_{f}^{2} \left( \frac{373}{3456} + \frac{13}{216} \zeta_{3} - \frac{1}{16} \zeta_{4}\right) \}.
\]

\[
\gamma_{0}^{nd} = \frac{1}{16\pi^{2}} \left\{ 6 \left(\frac{\alpha_{s}}{\pi}\right)^{2} + \left(\frac{\alpha_{s}}{\pi}\right)^{3} \left[ \frac{176}{3} + 33\zeta_{3} - 15\zeta_{5} + n_{f} \left(\frac{4}{9}\right) \right]
\right.
\]
\[
+ \left(\frac{\alpha_{s}}{\pi}\right)^{4} \left[ -\frac{14147}{24} + \frac{36691}{72} \zeta_{3} + \frac{967}{16} \zeta_{3}^{2} - \frac{4851}{32} \zeta_{4} + \frac{6890}{9} \zeta_{5} + \frac{3675}{32} \zeta_{5} + \frac{3829}{12} \zeta_{7}
\right.
\]
\[
+ n_{f} \left[ -\frac{779}{27} - \frac{9}{8} \zeta_{3} + \frac{99}{16} \zeta_{4} + \frac{15}{2} \zeta_{5} - \frac{75}{16} \zeta_{6} - \frac{1}{18} n_{f}^{2}\right] \} \}.
\]
or, numerically,
\[
\gamma_0^{d_i} = \frac{-3}{16\pi^2} \left( 1 + 1.333 a_s + (3.546 - 0.4167 n_f) a_s^2 \\
+ (6.069 - 4.8170 n_f + 0.03327 n_f^2) a_s^3 \\
+ (-14.658 - 26.779 n_f + 1.0816 n_f^2 + 0.000038 n_f^3) a_s^4 \right)
\]
\[
= 1. + 1.333 a_s + 1.04585 a_s^2 - 21.636 a_s^3 - 136.384 a_s^4 
\text{if } n_f = 6
\]

\[
\gamma_0^{n_d} = \frac{3}{8\pi^2} \left( a_s^2 + (13.7968 - 0.07407 n_f) a_s^3 + (128.339 - 8.2703 n_f - 0.0093 n_f^2) a_s^4 \right)
\]

4. Applications

4.1 Mixing of all scalar operators of dimension 4

Along with the $\beta$-function and quark mass anomalous dimension $\gamma_m$ the vacuum anomalous dimension $\hat{\gamma}_0$ lead to a complete description of the renormalization mixing of the operators

\[
O_1 = G_{\mu\nu}^a G_{\mu\nu}^a, \quad O_{ij}^{\mu\nu} = m_i \overline{\psi}_j \psi_j, \quad O_6^{ij} = m_i^2 m_j^2. \tag{4.1}
\]

The corresponding matrix of anomalous dimensions reads [10, 19, 20]

\[
\mu^2 \frac{d}{d\mu^2} O_1 = - \left( a_s \frac{\partial}{\partial a_s} \beta \right) O_1 + 4 \left( a_s \frac{\partial}{\partial a_s} \gamma_m \right) \sum_i O_{ij}^{ii} + 4 a_s \frac{\partial}{\partial a_s} \hat{g}_0,
\]

\[
\mu^2 \frac{d}{d\mu^2} O_{ij}^{\mu\nu} = - m_i \frac{\partial}{\partial m_j} \hat{g}_0,
\]

\[
\mu^2 \frac{d}{d\mu^2} O_6^{ij} = 4 \gamma_m O_6^{ij}.
\] \tag{4.2}

Here

\[
\mu^2 \frac{d}{d\mu^2} a_s = a_s \beta(a_s) \quad \text{and} \quad \mu^2 \frac{d}{d\mu^2} m_i = m_i \gamma_m(a_s). \tag{4.3}
\]

4.2 Quadratic mass corrections to $R(s)$

As $\hat{\gamma}_0$ fully describes the mixing of $m\overline{\psi}\psi$ to $m_4$, our result could be effectively employed to find the quartic mass corrections to $R(s)$ at order $\alpha_s^4$. At orders $\alpha_s^2$ and $\alpha_s^3$ it was done in [20] and [21] respectively.

4.3 Application for the RG optimized perturbative theory

$\hat{\gamma}_0$ as well as the (perturbatively computed with all quark massive) VEV $\overline{\psi}\psi$ are main ingredients of the so-called RG optimized perturbation theory [22, 23] as applied to the chiral condensate (in the massless limit!). Using the previous 4-loop $\hat{\gamma}_0$ and the 3-loop value of $\left< \overline{\psi}\psi \right>$ the authors of [24] arrived at

\[
- \left< \overline{\psi}\psi \right>^{1/3} (2 \text{GeV}) = 281 \pm 4 \pm 7 \text{ MeV} \tag{4.4}
\]

which is in agreement to other independent determinations. It would be useful to upgrade the analysis by one more order of PT.
5. Conclusions

We have computed the QCD vacuum anomalous dimension at 5 loops and, thus, have finished the program of the 5-loop renormalization of the QCD Lagrangian.

The authors are grateful to the Institut für Theoretische Teilchenphysik of Karlsruher Institut für Technologie (KIT) for support and kind permission to use its computer facilities.

The work of P. A. Baikov was supported in part by the grant RFBR 17-02-00175A of the Russian Foundation for Basic Research. The work of K. G. Chetyrkin was supported by the German Federal Ministry for Education and Research BMBF through Grant No. 05H15GUCC1.

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