We report on recent progress on the flavour non-singlet splitting functions in perturbative QCD. The exact four-loop (N^3LO) contribution to these functions has been obtained in the planar limit of a large number of colours. Phenomenologically sufficient approximate expressions have been obtained for the parts not exactly known so far. Both cases include results for the four-loop cusp and virtual anomalous dimensions which are relevant well beyond the evolution of non-singlet quark distributions, for which an accuracy of (well) below 1% has now been been reached.
1. Introduction

Up to power corrections, observables in $ep$ and $pp$ hard scattering can be schematically expressed as

$$O^{ep} = f_i \otimes c^0_i, \quad O^{pp} = f_i \otimes f_k \otimes c^0_{ik} \quad (1.1)$$

in terms of the respective partonic cross sections (coefficient functions) $c^0$ and the universal parton distribution functions (PDFs) $f_i(x, \mu^2)$ of the proton at a (renormalization and factorization) scale $\mu$ of the order of a physical hard scale, e.g., $M_H$ for the total cross section for the production of the Higgs boson. The dependence of the PDFs on the momentum fraction $x$ is not calculable in perturbative QCD; their scale dependence is governed by the renormalization-group evolution equations

$$\frac{d}{d \ln \mu^2} f_i(x, \mu^2) = \left[ P_{ik}(\alpha_s(\mu^2)) \otimes f_k(\mu^2) \right](x) \quad (1.2)$$

where $\otimes$ denotes the Mellin convolution. The splitting functions, which are closely related to the anomalous dimensions of twist-2 operators in the light-cone operator-product expansion (OPE), and the coefficient functions can be expanded in powers of the strong coupling $\alpha_s \equiv \alpha_s(\mu^2)/(4\pi)$,

$$P = a_s^2 P^{(0)} + a_s^3 P^{(1)} + a_s^4 P^{(2)} + a_s^5 P^{(3)} + \ldots, \quad (1.3)$$

$$c^a_i = a_s^a \left[ c^0_i(x) + a_s c^{(1)}_c(x) + a_s^2 c^{(2)}_c(x) + a_s^3 c^{(3)}_c(x) + \ldots \right]. \quad (1.4)$$

Together the first three terms in eqs. (1.3) and (1.4) provide the next-to-next-to-leading order (N$^2$LO) of perturbative QCD for the observables (1.1). This is now the standard approximation for many hard processes; see refs. [1–4] for the corresponding splitting functions.

Corrections beyond N$^3$LO are of phenomenological interest where high precision is required, such as in determinations of $\alpha_s$ from deep-inelastic scattering (DIS) (see refs. [5, 6] for the N$^3$LO corrections to the most important structure functions), and where the perturbation series shows a slow convergence, such as for Higgs production via gluon-gluon fusion calculated in ref. [7] at N$^3$LO. The size and structure of the corrections beyond N$^2$LO are also of theoretical interest.

Here we briefly report about considerable recent progress on the three four-loop (N$^3$LO) non-singlet splitting functions. We focus on the quantities $P^{(3)}_{\text{ns}}(x)$ for the evolution of flavour-differences $q_i \pm \bar{q}_i$ for quark and antiquark distributions; for more details see ref. [8].

2. Diagram calculations of fixed-$N$ moments

Two methods have been applied for obtaining Mellin moments of the quantities $P^{(3)}$ in eq. (1.3). Depending on the function, both can be used to determine the same even-$N$ or the odd-$N$ moments.

In the first one calculates, via the optical theorem and a dispersion relation in $x$, the unfactorized structure functions in DIS, as done at two and three loops in refs. [9–12]. The construction of the FORCER program [13] has facilitated the extension of those computations (which also provide moments of the coefficient functions) to four loops. For the hardest diagrams, the complexity of these computations rises quickly with $N$, hence only $N \leq 6$ has been covered completely so far [14]. Much higher $N$ can be accessed for simpler cases, e.g., values up to $N > 40$ have been reached for high-$n_f$ parts. These were sufficient to determine the complete $n_f^2$ and $n_f^3$ parts of the non-singlet splitting functions $P^{(3)}_{\text{ns}}(x)$ and the $n_f^3$ parts of the corresponding flavour-singlet quantities [15].
The increase of the complexity of the Feynman integrals with \( N \) is more benign for the second method based on the OPE which was applied to the present non-singlet cases at NLO in ref. \([16]\), see also ref. \([17]\). FORCER calculations in this framework have reached \( N = 16 \) for all contributions to the functions \( P_{ns}^{(3)} \), \( N = 18 \) for their \( n_f \) parts and \( N = 20 \) for the complete limit of a large number of colours \( n_c \) \([8]\). See refs. \([18–21]\) for earlier calculations of \( P_{ns}^{\pm(3)} \) at \( N \leq 4 \).

3. Towards all-\( N \) expressions

If the anomalous dimensions \( \gamma_{ns}(N) = -P_{ns}(N) \) at \( N^{n+2} \)LO are analogous to the lower orders, then they can be expressed in terms of harmonic sums \( S_{\vec{w}} \) \([22, 23]\) and denominators \( D_a^k \equiv (N+a)^{-k} \) as

\[
\gamma_{ns}^{(1)}(N) = \frac{2n+1}{\sum_{w=0}^{2n+1}} c_{00w} S_{\vec{w}}(N) + \sum_{a} \sum_{k=1}^{n} \sum_{w=0}^{2n+1-k} c_{akw} D_a^k S_{\vec{w}}(N) .
\]

The denominators at the calculated values of \( N \) indicate \( a = 0, 1 \) for \( \gamma_{ns}^{\pm} \), with coefficients \( c_{00w}, c_{akw} \) that are integer modulo low powers of \( 1/2 \) and \( 1/3 \). Sums up to weight \( w = 2n + 1 \) occur at \( N^3 \)LO.

Based on a conformal symmetry of QCD at an unphysical number of space-time dimensions \( D \), it has been conjectured that the \( \overline{MS} \) functions \( \gamma_{ns}(N) \) are constrained by ‘self-tuning’ \([24, 25]\),

\[
\gamma_{ns}(N) = \gamma_{0}(N + \sigma \gamma_{ns}(N) - \beta(a_s)/a_s))
\]

where \( \beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 - \ldots \) is the beta function, for its present status see refs. \([26, 27]\). The initial-state (PDF) and final-state (fragmentation-function) anomalous dimensions are obtained for \( \sigma = -1 \) and \( \sigma = 1 \), respectively, and the universal kernel \( \gamma_u \) is reciprocity respecting (RR), i.e., invariant under replacement \( N \rightarrow (1-N) \). Eq. (3.2) implies that the non-RR parts and the spacelike/timelike difference are inherited from lower orders. Hence ‘only’ \( \gamma_u \), which includes \( 2^{n-1} \) RR (combinations of) harmonic sums of weight \( w \), needs to be determined at four loops.

Present information, given by the even-\( N \) (odd-\( N \)) values \( N \leq 16 \) (15) of \( \gamma_{ns}^{+/-}(N) \) \((\gamma_{ns}^{-(3)}(N))\) and endpoint constraints (see below), is insufficient to determine the \( n = 3 \) coefficients in eq. (3.1).

However, \( \gamma_{ns}^{\pm} = \gamma_{ns} \) in the large-\( n_c \) limit, hence the known even-\( N \) and odd-\( N \) values can be used. Moreover, alternating sums do not contribute to \( \gamma_{ns}^{\pm} \) in this limit, leaving 1, 1, 2, 3, 5, 8, 13 = Fibonacci \((w)\) RR sums at weight \( w = 1, \ldots, 7 \) and a total of 87 basis functions for \( n = 3 \) in eq. (3.1).

Large-\( N \) and small-\( x \) limits provide more than 40 constraints on their coefficients. At large-\( N \), the non-singlet anomalous dimensions have the form \([33–35]\)

\[
\gamma_{ns}^{(n-1)}(N) = A_n \ln(N - B_n + N^{-1}) \{C_n \ln(N - \tilde{D}_n + \frac{1}{2} \ln(N)) + O(N^{-2})
\]

with \( \ln(N) \equiv \ln(N + \gamma_c) \), where \( \gamma_c \) denotes the Euler-Mascheroni constant. \( C_n \) and \( \tilde{D}_n \) are given by

\[
C(a_s) = (A(a_s))^2 , \quad \tilde{D}(a_s) = (A(a_s) - \beta(a_s)/a_s) ,
\]

in terms of lower-order information on the cusp anomalous dimension \( A(a_s) = A_1 a_s + A_2 a_s^2 + \ldots \) and the quantity \( B(a_s) = B_1 a_s + B_2 a_s^2 + \ldots \) sometimes called the virtual anomalous dimension.

The resummation of small-\( x \) double logarithms \([28–31]\) provides the four-loop coefficients of \( x^a \ln^b x \) at \( 4 \leq b \leq 6 \) and all \( a \) in the large-\( n_c \) limit (in full QCD, this holds only at even \( a \) for \( P_{ns}^+(x) \) and odd \( a \) for \( P_{ns}^-(x) \)). Moreover, a relation leading to a single-logarithmic resummation at \( a = 0 \),

\[
\gamma_{ns}^{+}(N) - \gamma_{ns}^{+}(N) + N - \beta(a_s)/a_s) = O(1) ,
\]
has been conjectured in ref. [32]. As far as it can be checked so far, this relation is found to be correct except for terms with $\zeta_2 = \pi^2/6$ that vanish in the large-$n_c$ limit.

Taking into account all the above information, it is possible to set up systems of Diophantine equations for the coefficients $c_{006}$, $c_{aik}$ of $\gamma^{\pm}_\text{ns}(N)$ in the large-$n_c$ limit that can be solved using the moments $1 \leq N \leq 18$, leaving the results of the diagram calculation at $N = 19, 20$ as checks.

### 4. All-$N$ anomalous dimension in the large-$n_c$ limit

The exact expressions for the new $n_f^0$ and $n_f^1$ parts cannot be shown here due to their length, they can be found in eq. (3.6) and (3.7) of ref. [8]. For the $n_f^2$ and $n_f^3$ terms see ref. [15]. The resulting large-$N$ coefficients $A_{L,A}$ and $B_{L,A}$ – the subscript $L$ indicates the large-$n_c$ limit – are found to be

\[
A_{L,A} = C_F n_c^3 \left( \frac{84278}{81} - \frac{88832}{81} \zeta_2 + \frac{20992}{27} \zeta_3 + 1804 \zeta_4 - \frac{352}{3} \zeta_2 \zeta_3 - 352 \zeta_5 - 32 \zeta_3^2 - 876 \zeta_6 \right) - C_F n_c n_f \left( \frac{39883}{81} - \frac{6692}{81} \zeta_2 + \frac{161252}{3} \zeta_3 + \frac{440}{3} \zeta_4 - \frac{256}{3} \zeta_2 \zeta_3 - 224 \zeta_5 \right) + C_F n_c n_f^2 \left( \frac{2119}{81} - \frac{608}{81} \zeta_2 + \frac{1280}{27} \zeta_3 - \frac{64}{3} \zeta_4 \right) - C_F n_f^3 \left( \frac{32}{81} - \frac{64}{27} \zeta_3 \right)
\]

(4.1)

and

\[
B_{L,A} = C_F n_c^3 \left( -\frac{1379569}{5184} + \frac{24211}{27} \zeta_2 - \frac{9803}{162} \zeta_3 - \frac{9382}{9} \zeta_4 + \frac{838}{9} \zeta_2 \zeta_3 + 1002 \zeta_3 + \frac{16}{3} \zeta_3^2 + 135 \zeta_6 - 80 \zeta_2 \zeta_5 + 32 \zeta_3 \zeta_4 - 560 \zeta_7 \right) + C_F n_c^2 n_f \left( \frac{353}{3} - \frac{85175}{162} \zeta_2 - \frac{137}{9} \zeta_3 + \frac{16186}{27} \zeta_4 - \frac{584}{9} \zeta_2 \zeta_3 - \frac{248}{3} \zeta_5 - \frac{16}{3} \zeta_3^2 - 144 \zeta_6 \right) - C_F n_c n_f^2 \left( \frac{127}{18} - \frac{5036}{81} \zeta_2 + \frac{932}{27} \zeta_3 + \frac{1292}{27} \zeta_4 - \frac{160}{9} \zeta_2 \zeta_3 - \frac{32}{3} \zeta_5 \right) - C_F n_f^3 \left( \frac{131}{81} - \frac{32}{81} \zeta_2 - \frac{304}{27} \zeta_3 + \frac{32}{27} \zeta_4 \right)
\]

(4.2)

The agreement of the four-loop cusp anomalous dimension (4.1) with the result obtained from the large-$n_c$ photon-quark form factor [36,37] provides a further non-trivial check of the determination of the all-$N$ expressions from the moments at $N \leq 18$, and hence also of the relations (3.1) – (3.5).

The maximum-weight $\zeta_3^2$ and $\zeta_6$ parts of eq. (4.1) also agree with the result obtained in planar $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory (MSYM) obtained before in ref. [38]. There is no such direct connection between the four-loop virtual anomalous dimension (4.2) and its counterparts in planar $\mathcal{N} = 4$ MSYM; see ref. [39] where the maximum-weight part of eq. (4.2) has been employed to derive the four-loop collinear anomalous dimension in planar $\mathcal{N} = 4$ MSYM.

The all-$N$ large-$n_c$ limit of $\gamma^{\pm}_\text{ns}(N)$ is compared in fig. 1 with the integer-$N$ QCD results at $N \leq 16$. As illustrated in the left panel, the former are a decent approximation to the latter for the individual $n_f^0$ contributions. However, as shown in the right panel, there are considerable cancellations between the these contributions. These cancellations are most pronounced for the physically relevant number of $n_f^0 = 5$ light quark flavours outside the large-$N$/large-$x$ region. Hence the large-$n_c$ suppressed contributions – indicated by the subscript $N$ below – need to be taken into account in phenomenological $N^3$LO analyses.
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Figure 1: The large-$n_c$ limit of the four-loop anomalous dimensions $\gamma_{\text{ns}}^{(3)}(N)$ (lines) compared to the QCD results for $\gamma_{\text{ns}}^{(3)}(N)$ at even $N$ and $\gamma_{\text{ns}}^{(3)}(N)$ at odd $N$ (points). Left: the $n_f$-independent contributions. Right: the results for physically relevant values of $n_f$. The values have been converted to an expansion in $\alpha_s$.

5. $x$-space approximations of the large-$n_c$ suppressed parts

With eight integer-$N$ moments known for both $P_{\text{ns}}^{+(3)}(x)$ and $P_{\text{ns}}^{-(3)}(x)$ and the large-$x$ and small-$x$ knowledge discussed in section 2, it is possible to construct approximate $x$-space expressions which are analogous to (but more accurate than) those used before 2004 at $N^2$LO, see refs. [40–43]. For this purpose an ansatz consisting of

- the two large-$x$ parameters $A_4$ and $B_4$ in eq. (3.3),
- two of three suppressed large-$x$ logs $(1-x) \ln^k(1-x)$, $k = 1, 2, 3$,
- one of ten two-parameter polynomials in $x$ that vanish for $x \to 1$,
- two of the three unknown small-$x$ logarithms $\ln^k x$, $k = 1, 2, 3$

is built for the large-$n_c$ suppressed $n_f^0$ and $n_f^1$ parts $P_{\text{ns}}^{(3)}_{N,0/1}$ of $P_{\text{ns}}^{+(3)}(x)$. This results in 90 trial functions, the parameters of which can be fixed from the eight available moments. Of these functions, two representatives $A$ and $B$ are then chosen that indicate the remaining uncertainty, see fig. 2.

This non-rigorous procedure can be checked by comparing the same treatment for the large-$n_c$ parts to our exact results. Moreover, the trial functions lead to very similar values for the next moment, e.g., $N = 18$ for $P_{\text{ns}}^{+(3)}$. The residual uncertainty at this $N$-value is a consequence of the width of the band at large $x$, which in turn is correlated with the uncertainties at smaller $x$. If the spread of the result $A$ and $B$ would underestimate the true remaining uncertainties, then a comparison with an additional analytic result at this next value of $N$ should reveal a discrepancy.
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Figure 2: About 90 trial functions for the $n_f$-independent contribution to the large-$n_c$ suppressed part of splitting function $P_{Ns}^{(3)}(x)$, multiplied by $x^{0.4}(1-x)$. The two functions chosen to represent the remaining uncertainty are denoted by $A$ and $B$ and shown by solid (blue) lines. Due to the factor $(1-x)$ the contribution $A_{N,4}$ to the four-loop cusp anomalous dimension can be read off at $x=1$.

We were able to extend the diagram computations of the $n_f$ parts of $P_{Ns}^{(3)}(x)$ to $N=18$ and find

$$P_{N,1}^{(3)}(N=18) = 195.88888792_b < 195.8888857_{\text{exact}} < 195.8888968_A. \quad (5.1)$$

A similar check for $P_{N,0}^{(3)}$ has been carried out by deriving a less accurate approximation using only seven moments and comparing the results to the now unused value at $N=16$.

The case of $P_{Ns}^{(3)}(x)$ has been treated in the same manner, but taking into account that only its leading small-$x$ logarithm is known up to now [29]. See ref. [8] for the (large-$N$ suppressed) additional $d^{abc}d_{abc}$ contribution $P_{Ns}^{(3)}(x)$ to the splitting function for the total valence quark PDF.

6. Numerical results for the cusp and virtual anomalous dimensions

Combining the exact large-$n_c$ results, the approximations for the remaining $n_f^0$ and $n_f^1$ contributions and the complete high-$n_f$ contributions of ref. [15], the four-loop cusp anomalous dimension for QCD with $n_f$ quark flavours are given by

$$A_4 = \frac{20702(2) - 5171.9(2) n_f + 195.5772 n_f^2 + 3.272344 n_f^3}{4}, \quad (6.1)$$

where the numbers in brackets represent a conservative estimate of the remaining uncertainty. The conversion of this result to an expansion in powers of $\alpha_s$ leads to

$$A_q(\alpha_s, n_f=3) = 0.42441 \alpha_s (1 + 0.72657 \alpha_s + 0.73405 a_s^2 + 0.6647(2) a_s^3 + \ldots),$$

$$A_q(\alpha_s, n_f=4) = 0.42441 \alpha_s (1 + 0.63815 \alpha_s + 0.50998 a_s^2 + 0.3168(2) a_s^3 + \ldots),$$

$$A_q(\alpha_s, n_f=5) = 0.42441 \alpha_s (1 + 0.54973 \alpha_s + 0.28403 a_s^2 + 0.0133(2) a_s^3 + \ldots). \quad (6.2)$$
The corresponding results for the virtual anomalous dimension, i.e., the coefficient of \( \delta(1-x) \) show a similarly benign expansion with

\[
B_4 = 23393(10) - 5551(1)n_f + 193.8554n_f^2 + 3.014982n_f^3
\]

and

\[
B_q(\alpha_s, n_f = 3) = 0.31831\alpha_s(1 + 0.99712\alpha_s + 1.24116\alpha_s^2 + 1.0791(13)\alpha_s^3 + \ldots),
\]

\[
B_q(\alpha_s, n_f = 4) = 0.31831\alpha_s(1 + 0.87192\alpha_s + 0.97833\alpha_s^2 + 0.5649(13)\alpha_s^3 + \ldots),
\]

\[
B_q(\alpha_s, n_f = 5) = 0.31831\alpha_s(1 + 1.74672\alpha_s + 0.71907\alpha_s^2 + 0.1085(13)\alpha_s^3 + \ldots).
\]

Due to constraints by large-\( N \) moments, the errors of \( A_4 \) and \( B_4 \) are fully correlated. The accuracy in eqs. (6.2) and (6.4) should be amply sufficient for phenomenological applications.

By repeating the approximation procedure in section 5 for individual colour factors, it is possible to obtain corresponding approximate coefficients for \( A_4 \) and \( B_4 \) which can be summarized as (for a table of the relevant group invariants see, e.g., appendix C of ref. [44])

<table>
<thead>
<tr>
<th>( A_4 )</th>
<th>( B_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_F^4 )</td>
<td>0</td>
</tr>
<tr>
<td>( C_F^2C_A )</td>
<td>0</td>
</tr>
<tr>
<td>( C_F^2C_A^2 )</td>
<td>0</td>
</tr>
<tr>
<td>( C_FC_A^3 )</td>
<td>610.3 ± 0.3</td>
</tr>
<tr>
<td>( d^a_{abcd} / N_R )</td>
<td>507.5 ± 6.0</td>
</tr>
<tr>
<td>( n_fC_F^3 )</td>
<td>−31.00 ± 0.4</td>
</tr>
<tr>
<td>( n_fC_F^2C_A )</td>
<td>38.75 ± 0.2</td>
</tr>
<tr>
<td>( n_fC_FC_A^2 )</td>
<td>−440.65 ± 0.2</td>
</tr>
<tr>
<td>( n_fd^abcd / N_R )</td>
<td>−123.90 ± 0.2</td>
</tr>
<tr>
<td>( n_f^2C_F^2 )</td>
<td>−21.31439</td>
</tr>
<tr>
<td>( n_f^2C_FC_A )</td>
<td>58.36737</td>
</tr>
<tr>
<td>( n_f^2C_F )</td>
<td>2.454258</td>
</tr>
</tbody>
</table>

where the exactly known \( n_f^2 \) and \( n_f^3 \) coefficients have been included for completeness. Due to the constraint provided by the exact large-\( n_c \) limit, the errors in this table are highly correlated; for numerical applications in QCD eqs. (6.2) and (6.4) should be used instead. The above results show that both quartic group invariants definitely contribute to the four-loop cusp anomalous dimension, for this issue see also refs. [45–48] and references therein. This implies that the so-called Casimir scaling between the quark and gluon cases, \( A_q = C_F/C_A/\alpha_s \), does not hold beyond three loops.

### 7. \( N^3\text{LO} \) corrections to the evolution of non-singlet PDFs

The effect of the fourth-order contributions on the evolution of the non-singlet PDFs can be illustrated by considering the logarithmic derivatives of the respective combinations of quark PDFs with respect to the factorization scale, \( \tilde{q}_{\text{ins}}^a = d\ln q_{\text{ins}}^a / d\ln \mu_f \), at a suitably chosen reference point.
As in ref. [1], we choose the schematic, order-independent initial conditions

$$xq_{\text{ns}}^\pm(x, \mu_0^2) = x^{0.5}(1-x)^3 \quad \text{and} \quad \alpha_s(\mu_0^2) = 0.2.$$  \hspace{1cm} (7.1)

For $\alpha_s(M_Z^2) = 0.114 \ldots 0.120$ this value for $\alpha_s$ corresponds to $\mu_0^2 \simeq 25 \ldots 50 \text{GeV}^2$ beyond the leading order, a scale range typical for DIS at fixed-target experiments and at the ep collider HERA.

The new N$^3$LO corrections to $\dot{q}_{\text{ns}}^i$ are generally small, hence they are illustrated in fig. 3 by comparing their relative effect to that of the N$^2$LO contributions for the standard identification $\mu_r = \mu_f \equiv \mu$ of the renormalization scale with the factorization scale. Except close to the sign change of the scaling violations at $x \simeq 0.07$, the relative N$^3$LO effects are (well) below 1% for the flavour-differences $\dot{q}_{\text{ns}}^+ \text{and} \dot{q}_{\text{ns}}^-$ (left and middle panel). The N$^2$LO and N$^3$LO corrections are larger for the valence distribution $\dot{q}_{\text{ns}}^v$ at $x < 0.07$ due to the effect of the $d_{abc}d_{abc}$ ‘sea’ contribution $P_{\text{ns}}^s(x)$, note the different scale of the right panel in fig. 3. Also in this case the N$^3$LO evolution represents a clear improvement, and the relative four-loop corrections are below 2%.

The remaining uncertainty due to the approximate character of the four-loop splitting functions beyond the large-$n_c$ limit is indicated by the difference between the solid and dotted (red) curves in fig. 3 and fig. 4 below. Due to the small size of the four-loop contributions and the ‘$x$-averaging’ effect of the Mellin convolution,

$$[P_{\text{ns}} \otimes q_{\text{ns}}](x) = \int_x^1 \frac{dy}{y} P_{\text{ns}}(y) q_{\text{ns}}\left(\frac{x}{y}\right),$$  \hspace{1cm} (7.2)

the results of section 4 are safely applicable to lower values of $x$ than one might expect from fig. 2.

The stability of the NLO, N$^2$LO and N$^3$LO results under variation of the renormalization scale over the range $\frac{1}{2} \mu_f^2 \leq \mu_r^2 \leq 8 \mu_f^2$ is illustrated in fig. 4 at typical values of $x$. Except close to the sign change of $\dot{q}_{\text{ns}}^i$, the variation is well below 1% for the conventional interval $\frac{1}{2} \mu_f \leq \mu_r \leq 2 \mu_f$.  

**Figure 3:** The relative N$^2$LO and N$^3$LO corrections to the logarithmic scale derivative of the non-singlet combinations $q_{\text{ns}}^i$ of quark PDFs for the schematic order-independent input (7.1) for $n_f = 4$ at $\mu_r = \mu_f$. 

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Figure 4: The dependence of the NLO, N^2LO and N^3LO results for \( \hat{q}_{ns}^+ \equiv d \ln q_{ns}^+ / d \ln \mu_f^2 \) on the renormalization scale \( \mu_r \) at six typical values of \( x \) for the initial conditions (7.1) and \( n_f = 4 \) flavours. The remaining uncertainty of the four-loop splitting function \( P_n^{(3)}(x) \) leads to the difference of the solid and dotted curves.

8. Summary and Outlook

The splitting functions for the non-singlet combinations of quark PDFs have been addressed at the fourth-order (N^3LO) of perturbative QCD. The quantities \( P_n^{\pm(3)} \) are now known exactly in the limit of a large number of colours \( n_c \). Present results for the large-\( n_c \) suppressed contributions with \( n_f^0 \) and \( n_1^f \) are still approximate, but sufficiently accurate for phenomenological applications in deep-inelastic scattering and collider physics. FORM and FORTRAN files of these results can be obtained by downloading the source of ref. [8] from arXiv.org.

It would be desirable, mostly for theoretical purposes, to obtain also the analytic forms \( n_f^0 \) and \( n_1^f \) parts of \( P_n^{\pm(3)} \). So far, only their contributions proportional to the values \( \zeta_4 \) and \( \zeta_5 \) of the Riemann \( \zeta \)-function have been completely determined, together with the (unpublished) \( \zeta_3 \) part of the \( n_1^f \) contributions. The \( \zeta_4 \) parts are particularly simple; in fact, it turns out that they (and other \( \pi^2 \) terms) can be predicted via physical evolution kernels from lower-order quantities, see refs. [49,50].

The \( \zeta_5 \) part of \( P_n^{\pm(3)} \), presented in appendix D of ref. [8], includes a (non large-\( n_c \)) contribution

\[
- \frac{128}{3} \left\{ 3 C_F^2 C_A^2 - 2 C_F C_A^3 + 12 d^{abcd} d_A^{abcd} / N_R \right\} 5 \zeta_5 [S_1(N)]^2 .
\]

The resulting \( \ln^2 N \) large-\( N \) behaviour needs to be compensated by non-\( \zeta_5 \) terms. Eq. (8.1) looks exactly like the \( \zeta_5 \)-‘tail’ of the so-called wrapping correction in the anomalous dimensions in \( \mathcal{N} = 4 \) maximally supersymmetric Yang-Mills theory, see refs. [51,52].
Phenomenologically, of course, one rather needs corresponding results for the flavour-singlet splitting functions \( H^{(3)}_{ij}(x) \), \( i, j = q, g \). At present, it appears computationally too hard to obtain moments of all four functions beyond \( N = 6 \) using the method of refs. [9–12]. Therefore one will need to resort to the OPE, which offers additional theoretical challenges in the massless flavour-singlet case, see refs. [53–55]. We hope to address this issue in a future publication.

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References

Four-loop results on anomalous dimensions and splitting functions

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