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Asymptotic symmetry algebras of conformal gravity in four dimensions

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We consider conformal gravity boundary conditions and outline the highest dimensional non-trivial asymptotic symmetry algebras of conformal gravity. The highest among them is five dimensional and leads to a global geon solution.

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I. INTRODUCTION AND MOTIVATION

In this talk, we consider the asymptotic symmetry algebra (ASA) of conformal gravity (CG). ASA is generally investigated for three main reasons. First one is to learn about the ASA itself, the second one is to investigate the theory of gravity, and the third one is to learn about the charges defined by the ASA which describe the field theory at the boundary. CG is a very interesting theory of gravity that has recently received much attention. Compared to Einstein gravity, it has an advantage that it is two loop renormalizable while Einstein gravity is not, however, it introduces an issue common for higher derivative gravity theories. It contains ghosts.

It was studied from theoretical aspects by Maldacena, who simultaneously showed the importance of the boundary conditions obtaining Einstein gravity solutions from conformal gravity by imposing appropriate boundary conditions [1]. Further motivation to study CG comes from the the number of papers by 't Hooft in which he considers CG and conformal symmetry, and speculates that conformal symmetry could play a crucial role for the physics at the Planck energy scale [2–5]. CG arises from five dimensional Einstein gravity (EG) as a boundary counterterm and from the twistor string theory [6]. The analysis of the holographic [7] and canonical [8] aspect of conformal gravity, showed that consistent set of boundary conditions leads to well defined variational principle and finite charges of the conformal gravity holography in four dimensions, while the charge associated to Weyl transformations vanishes. On phenomenological grounds, it was mostly studied by Manheim, in the explanation of the galactic rotation curves without the addition of dark matter [9–11].

The aspect from which we are considering CG is the AdS/CFT correspondence that has been demonstrated to work on number of examples such as $AdS_3/LCFT_2$ [12, 13], AdS/Ricci flat correspondence and other examples of gauge/gravity correspondence.

Within this classification, beside ASAs of CG, we also obtain the asymptotic solutions that can be uplifted to the global solutions. These solutions are pp waves or geons [14]. The classification also contains the known CG solutions, such as Manheim–Kazanas–Riegert solution and rotating black hole solutions [15].

II. CONFORMAL GRAVITY

Given a manifold $\mathcal{M}$, conformal gravity action is described by the

$$S = \alpha \int d^4x C^{\mu}{}_{\nu\sigma\rho} C_{\mu}{}^{\nu\sigma\rho} \tag{1}$$

living on that manifold. In (1), α is dimensionless coupling constant responsible for the power counting renormalizability of the action and $C^{\mu}{}_{\nu\sigma\rho}$ is Weyl tensor. The action is invariant under Weyl rescalings of the metric

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu} \tag{2}$$

for ω Weyl factor. Variation of the action leads to the CG equations of motion which require vanishing of the Bach tensor

$$\left(\nabla^{\rho}\nabla_{\sigma} + \frac{1}{2}R^{\rho}{}_{\sigma}\right) C^{\sigma}{}_{\mu\rho\nu} = 0. \tag{3}$$

III. BOUNDARY CONDITIONS

The asymptotic $(0 < \rho << \ell)$ line element is described with

$$ds^2 = \frac{\ell^2}{\rho^2}\left(-\sigma d\rho^2 + \gamma_{ij} dx^i dx^j\right) \tag{4}$$

for ℓ AdS radius which we set to 1 for simplicity, ρ holographic component using which we approach to the boundary $\partial\mathcal{M}$, and $\sigma = \pm 1$ for (A)dS space. The metric on the boundary γ_{ij} defines the first part of the boundary conditions in terms of the generalised Fefferman-Graham expansion

$$\gamma_{ij} = \gamma_{ij}^{(0)} + \rho\gamma_{ij}^{(1)} + \frac{1}{2}\rho^2\gamma_{ij}^{(2)} + ... \tag{5}$$

In addition, we take that variations of the first two terms in the expansion of the boundary metric (5) are

$$\delta\gamma_{ij}^{(0)} = \lambda\gamma_{ij}^{(0)}, \delta\gamma_{ij}^{(1)} = 2\lambda\gamma_{ij}^{(1)}. \tag{6}$$

Since the metric $g_{\mu\nu}$ is invariant under small diffeomorphisms $x^{\mu} \to x^{\mu} + \xi^{\mu}$, its variation

$$\delta g_{\mu\nu} = \left(e^{2\omega} - 1\right) g_{\mu\nu} + \pounds_{\xi} g_{\mu\nu} \tag{7}$$

consists of the part that appears due to invariance under these diffeomorphisms and the part that appears due to invariance under Weyl rescalings. We expand the Weyl factor ω and the vector ξ in the holographic coordinate ρ analogously to the expansion of the boundary metric γ_{ij} (5) and insert them in the Killing equation (7). This leads to equations that define the conditions on the components in the metric expansion (5). Taking $\delta g_{\rho\rho} = \delta g_{\rho i} = 0$, we obtain that $\omega^{(0)}$ is zero and ij component of (7) defines the leading and the subleading order of the Killing equation. The leading order Killing equation

$$\mathcal{D}_i\xi_j^{(0)} + \mathcal{D}_j\xi_i^{(0)} = \frac{2}{3}\gamma_{ij}^{(0)}\mathcal{D}_k\xi^{(0)k} \tag{8}$$

defines the leading term in the expansion of the boundary metric, $\gamma^{(0)}_{ij}$, in the dependence on the leading order term in the expansion of Killing vectors (KV) ξ. The choice of the Minkowski metric for $\gamma^{(0)}_{ij} = \eta_{ij} = diag(-1,1,1)$ leads to vectors ξ which define the conformal algebra $so(3,2)$ at the boundary. Conformal algebra is consisted of KVs of translations

$$\xi^{(0)} = \partial_t, \qquad \xi^{(1)} = \partial_x, \qquad \xi^{(2)} = \partial_y, \tag{9}$$

Lorentz rotations

$$L_{ij} = x_i\partial_j - x_j\partial_i \tag{10}$$

which define $\xi^{(3)}, \xi^{(4)}$ and $\xi^{(5)}$ for $i,j = t,x,y$, the dilatation KV

$$\xi^{(6)} = t\partial_t + x\partial_x + y\partial_y \tag{11}$$

and special conformal transformations (SCTs)

$$\xi^{(7)} = tx\partial_t + \frac{t^2+x^2-y^2}{2}\partial_x + xy\partial_y \tag{12}$$

$$\xi^{(8)} = ty\partial_t + xy\partial_x + \frac{t^2+y^2-x^2}{2}\partial_y \tag{13}$$

$$\xi^{(9)} = \frac{t^2+x^2+y^2}{2}\partial_t + tx\partial_x + ty\partial_y. \tag{14}$$

We denote $\xi^t = \{\xi^{(0)}, \xi^{(1)}, \xi^{(2)}\}$ as translations, $\xi^{(6)} = \xi^d$ as dilatation, and $\xi^{sct} = \{\xi^{(7)}, \xi^{(8)}, \xi^{(9)}\}$ as SCT Killing vectors to define the $so(3,2)$

$$[\xi^d, \xi^t_i] = -\xi^t_i \qquad [\xi^d, \xi^{sct}_i] = \xi^{sct}_i \tag{15}$$

$$[\xi^t_k, L_{ij}] = (\eta_{ki}\xi^t_j - \eta_{kj}\xi^t_i) \qquad [\xi^{sct}_k, L_{ij}] = -(\eta_{ki}\xi^{sct}_j - \eta_{kj}\xi^{sct}_i) \tag{16}$$

$$[\xi^{sct}_i, \xi^t_j] = -(\eta_{ij}\xi^d - L_{ij}) \tag{17}$$

$$[L_{ij}, L_{kj}] = -L_{ik}. \tag{18}$$

Due to the boundary conditions the subleading order Killing equation

$$\pounds_{\xi^{(0)}}\gamma^{(1)}_{ij} = \frac{1}{3}\mathcal{D}_k\xi^k_{(0)}\gamma^{(1)}_{ij} \tag{19}$$

consists of the Killing vectors $\xi^{(0)}_i$, leading term in the expansion of the boundary metric $\gamma^{(0)}_{ij}$, and the subleading term in the expansion of the boundary metric $\gamma^{(1)}_{ij}$. From this equation we can proceed in the two possible directions. Since both $\xi_i^{(0)}$ and $\gamma_{ij}^{(1)}$ are undefined, we can choose the condition on the $\gamma_{ij}^{(1)}$, and inserting it in (19) determine the set of KVs $\xi_i^{(0)}$ which are conserved by that $\gamma_{ij}^{(1)}$. That set of KVs should also form a closed algebra. From the other side, we can choose the subalgebra of $so(3,2)$ defined by a set of $\xi_i^{(0)}$s and find the corresponding $\gamma_{ij}^{(1)}$.

In this talk we focus on the approach in which, using (19) we determine $\gamma_{ij}^{(1)}$ for the chosen subalgebra of $so(3,2)$. The $\gamma_{ij}^{(1)}$ can depend on all the coordinates on the boundary, while the simplest cases are of course those in which the components of the $\gamma_{ij}^{(1)}$ are constant.

To be able to describe the required subalgebras, we will have to define the new KVs formed from the linear combinations of the existing ones. The most general linear combination of the KVs is

$$\xi^{linear\ combination} = a_0\xi^{(0)} + a_1\xi^{(1)} + a_2\xi^{(2)} + a_3\xi^{(3)} + a_4\xi^{(4)} + a_5\xi^{(5)} + a_6\xi^{(6)} + a_7\xi^{(7)} + a_8\xi^{(8)} + a_9\xi^{(9)}. \tag{20}$$

In the following chapter we consider the largest subalgebras.

IV. ASYMPTOTIC SYMMETRY ALGEBRA OF CONFORMAL GRAVITY

The most interesting subalgebras of $so(3,2)$ for which we find $\gamma_{ij}^{(1)}$ are the highest dimensional subalgebras. They consist of five and four KVs. Here, we list first all the subalgebras of $so(3,2)$ [16]

- similitude algebra, *sim(2,1)*,
- optical algebra *opt(2,1)*,
- maximal compact algebra $o(3) \otimes o(2)$
- $o(2) \otimes o(2,1)$
- *o(2,2)*
- Lorentz algebra $o(3,1)$
- irreducible subalgbra $o(2,1)$,

while below we demonstrate how to define these subalgebras in terms of our KVs.

1. Similitude algebra sim(2,1)

The number of KVs in this algebra is 7, and we can identify them with the KVs

$$P_0 = \xi^{(0)}, \qquad P_1 = \xi^{(1)}, \qquad P_2 = \xi^{(2)} \qquad F = \xi^{(6)} \tag{21}$$

$$K_1 = \xi^{(3)} \qquad K_2 = \xi^{(4)} \qquad L_3 = \xi^{(5)}. \tag{22}$$

They close into

$$\left[\xi^d, \xi^t_j\right] = -\xi^t_j \tag{23}$$

$$[\xi^t_l, L_{ij}] = -\left(\eta_{li}\xi^t_j - \eta_{lj}\xi^t_i\right) \tag{24}$$

$$[L_{ij}, L_{mj}] = L_{im}. \tag{25}$$

If we insert the corresponding KVs in the subleading order Killing equation (19), we do not get a solution for $\gamma^{(1)}_{ij}$ unless it is trivial. The subalgebra of $sim(2,1)$ which contains five KVs is the highest dimensional one that leads to non-trivial $\gamma^{(1)}_{ij}$.

2. Optical algebra opt(2,1)

as well consists of 7 KVs which do not lead to non-trivial $\gamma^{(1)}_{ij}$. The realised subalgebra of $opt(2,1)$ is 5 dimensional. When we write its KVs in a form

$$W = -\frac{\xi^{(6)} + \xi^{(4)}}{2} \qquad K_1 = \frac{\xi^{(6)} - \xi^{(4)}}{2} \qquad K_2 = \frac{1}{2}\left[\xi^{(0)} - \xi^{(2)} + \frac{(\xi^{(8)} - \xi^{(9)})}{2}\right] \tag{26}$$

$$Q = \frac{\xi^{(5)} - \xi^{(3)}}{2\sqrt{2}} \qquad M = -\sqrt{2}\xi^{(1)} \qquad L_3 = \frac{1}{2}\left[\xi^{(0)} - \xi^{(2)} - \frac{(\xi^{(8)} - \xi^{(9)})}{2}\right] \tag{27}$$

$$N = -(\xi^{(0)} + \xi^{(2)}) \tag{28}$$

their algebra is defined with

$$[K_1, K_2] = -L_3, \quad [L_3, K_1] = K_2, \quad [L_3, K_2] = -K_1, \quad [M, Q] = -N, \quad [K_1, M] = -\frac{1}{2}M, \tag{29}$$

$$[K_1, Q] = \frac{1}{2}Q, \quad [K_1, N] = 0, \quad [K_2, M] = \frac{1}{2}Q, \quad [K_2, Q] = \frac{1}{2}M, \quad [K_2, N] = 0 \tag{30}$$

$$[L_3, M] = -\frac{1}{2}Q, \quad [L_3, Q] = \frac{1}{2}M, \quad [L_3, N] = 0 \quad [W, M] = \frac{1}{2}M, \quad [W, Q] = \frac{1}{2}Q, \tag{31}$$

$$[W, N] = \frac{1}{2}N. \tag{32}$$

3. $o(2,2)$

The algebra $o(2,2)$ is 6 dimensional while its highest realised subalgebra consists of 4 KVs. Rewriting the KVs in the form,

$$A_1 = -\frac{1}{2}\left[\frac{\xi^{(9)}+\xi^{(8)}}{2} - \left(\xi^{(0)}+\right)\right], \qquad A_2 = \frac{1}{2}\left(\xi^{(6)}+\xi^{(4)}\right), \tag{33}$$

$$A_3 = \frac{1}{2}\left[-\frac{\xi^{(9)}+\xi^{(8)}}{2} - \left(\xi^{(0)}+\xi^{(2)}\right)\right] \qquad B_1 = -\frac{1}{2}\left[\frac{-\xi^{(9)}-\xi^{(8)}}{2} + \left(\xi^{(0)}-\xi^{(2)}\right)\right] \tag{34}$$

$$B_2 = \frac{1}{2}\left(\xi^{(6)}-\xi^{(4)}\right) \qquad B_3 = \frac{1}{2}\left[\frac{\xi^{(9)}-\xi^{(8)}}{2} + \left(\xi^{(0)}-\xi^{(2)}\right)\right] \tag{35}$$

we can identify the algebra

$$[A_1, A_2] = -A_3 \qquad [A_3, A_1] = A_2 \qquad [A_2, A_3] = A_1 \tag{36}$$

$$[B_1, B_2] = -B_3 \qquad [B_3, B_1] = B_2 \qquad [B_2, B_3] = B_1 \tag{37}$$

$$\tag{38}$$

which can be summarised into $[A_i, B_k] = 0$ for $(i,k) = 1,2,3$.

4. $o(3,1)$

This algebra consists of 6 KVs, while the highest dimensional realised subalgebra is 4 dimensional. If we define the KVs as

$$L_1 = \xi^{(7)} + \frac{\xi^{(2)}}{2} \qquad L_2 = \xi^{(5)} \qquad L_3 = \xi^{(8)} + \frac{1}{2}\xi^{(1)} \tag{39}$$

$$K_1 = \xi^{(8)} - \frac{1}{2}\xi^{(1)} \qquad K_2 = \xi^{(6)} \qquad K_3 = -\xi^{(7)} + \frac{1}{2}\xi(2) \tag{40}$$

the algebra is

$$[L_i, L_j] = \epsilon_{ijk} L_k, \tag{41}$$

$$[L_i, K_j] = \epsilon_{ijk} K_k, \tag{42}$$

$$[K_i, K_j] = -\epsilon_{ijk} L_k. \tag{43}$$

The subalgebras which have for the highest dimensional realised subalgebra, algebra with 4 KVs are $o(3)\otimes o(2)$, $o(2)\otimes o(2,1)$ and $o(2,1)$ and we will not consider their algebras and KVs explicitly here, for more information see [14]. The explicit solutions defined by some of the algebras we mentioned above are written in the table in the Appendix: "Highest realised subalgebras of $sim(2,1)$", while now we focus on the realised subalgbras that can be extended to global solutions.

V. GLOBAL SOLUTION

To obtain the five dimensional subalgebra we consider the KVs of the similitude algebra which form the subalgebra $\alpha_{5,4}$ (adopting the notation from [16]) and insert them in the equation (19). The equation (19) is solvable when $\gamma_{ij}^{(1)}$ is

$$\gamma_{ij}^{(1)} = \begin{pmatrix} c & c & 0 \\ c & c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{44}$$

and the KVs that define $\alpha_{5,4}$ are KVs of translation and two new linearly combined KVs

$$\chi_{new}^{(1)} = \xi^{(6)} - \frac{1}{2}\xi^{(3)} \qquad \chi_{new}^{(2)} = \xi^{(5)} - \xi^{(4)}. \tag{45}$$

The solution (44) helps us to find the global soultion, because we can use it to write the ansatz metric

$$ds^2 = dr^2 + (-1 + cf(r))dx_i^2 + 2cf(r)dx_i dx_j + (1 + cf(r))dx_j^2 + dx_x^2 \tag{46}$$

for $x_i = \{t, x, y\}$. The line element (46) gives global solution and solves Bach equation (3) for $f(r) = c_1 + c_2 + c_3 r^2 + c_4 r^3$.

Interestingly, we can notice form the solution for $f(r)$ that choosing the coefficients c_i ($i = 1, 2, 3, 4$) we can decide whether we will have the corresponding charges. The AdS/CFT correspondence tells us how to define the stress energy tensors at the boundary, and for conformal gravity and this particular metric, stress energy tensors are partially massless response P_{ij} in the sense of Deser, Nepomechie, and Waldron [17, 18], and standard Brown-York stress energy tensor τ_{ij}. We obtain

$$\tau_{ij} = \begin{pmatrix} -cc_4 & -cc_4 & 0 \\ -cc_4 & -cc_4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{47}$$

and

$$P_{ij} = \begin{pmatrix} cc_3 & cc_3 & 0 \\ cc_3 & cc_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{48}$$

Using the analogous ansatz metric and $\gamma_{ij}^{(1)}$ of the form

$$\gamma_{ij} = \begin{pmatrix} -cb(t-y) & 0 & cb(t-y) \\ 0 & 0 & 0 \\ cb(t-y) & 0 & -cb(t-y) \end{pmatrix}. \tag{49}$$

which conserve the KVs

$$\xi^{(n1)} = -P_0 + P_2 \qquad \xi_2^{(n2)} = P_1 \qquad \xi^{(n3)} = P_1 \qquad \xi^{(n3)} = L_3 - K_1 \tag{50}$$

we as well obtain the global solution of the Bach equation (3).

The particularity of this metric is that $b(t-y)$ is a function, which allows us to solve (19) for the further conserved KV. We can obtain the forms of $\gamma_{ij}^{(1)}$ for the corresponding KV written in the table

4th KV	b(t-y)	4th KV	$b(t-y)$
F	$\frac{b}{t-y}$	$F - K_2$	$\frac{b}{(t-y)^{3/2}}$
$F + K_2 + \epsilon(-P_0 - P_2)$	$b \cdot e^{\frac{t-1}{2\epsilon}}$	K_2	$\frac{b}{(t-y)^2}$
$P_0 - P_2$	b(t-1)	$F + cK_2$	$b \cdot (t-y)^{\frac{1-2c}{-1+c}}$.

VI. CONCLUSION AND OUTLOOK

Depending on the linear term in the FG expansion, we can impose a number of boundary conditions in conformal gravity. These boundary conditions are classified according to subalgebras of $so(3,2)$, and with the clever choice of an ansatz metric, using the realised $\gamma_{ij}^{(1)}$ matrices, we can obtain a global solution. Global solutions can therefore be classified according to the realised subalgebras.

Largest asymptotic symmetry algebra we find is 5 dimensional, belongs to subalgbra of $sim(2,1)$ and $opt(2,1)$ and defines pp wave or geon solution. $o(2,2)$ and $o(3,1)$ algebras define ASAs with maximally 4 KVs. Each of these can also give pp wave global solution.

However, there are more global solutions. To find them using this approach, one has to carefully choose the ansatz metric and depending on the requirement of the global solutions, impose additional conditions which may lead to easy or demanding task. Further research in this direction include studying the black hole solutions, black branes and black strings.

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VIII. APPENDIX: HIGHEST REALISED SUBALGEBRAS OF $sim(2,1)$

Here we write an example of the asymptotic solutions for $\gamma_{ij}^{(1)}$ and their corresponding algebra. The subalgebras of $sim(2,1)$ we denote with "α" and adopt the notation of [16], while the generators are identified with with $P_0 = -\xi^{(0)}, P_1 = \xi^{(1)}, P_2 = \xi^{(2)}, F = \xi^{(6)}, K_1 = \xi^{(3)}, K_2 = \xi^{(4)}, L_3 = \xi^{(5)}$.

Realized subalgebras			
Name	Patera name	generators	$\gamma_{ij}^{(1)}$
	$\alpha^{a}_{5,4}$	$F+\frac{1}{2}K_2, -K_1+L_3,$ P_0, P_1, P_2	$\gamma_{ij}^{(1)} = \begin{pmatrix} -b & 0 & b \\ 0 & 0 & 0 \\ b & 0 & -b \end{pmatrix}$
	$\alpha \neq 0, \pm 1$	$\alpha = \frac{1}{2}$	
$\mathcal{R} \oplus o(3)$	$\alpha_{4,1}$	$P_1 \oplus \{K_2, P_0, P_2\}$	$\begin{pmatrix} \frac{b}{2} & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -\frac{b}{2} \end{pmatrix}$
	$\alpha_{4,2}$	$P_0 - P_2 \oplus$ $\{F - K_2; P_0 + P_2, P_1\}$	$\begin{pmatrix} -b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -2b \end{pmatrix}$
MKR $\mathcal{R} \oplus o(3)$	$\alpha_{4,3}$	$P_0 \oplus \{L_3, P_1, P_2\}$	$\begin{pmatrix} 2b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$
	$\alpha_{4,4}$	$F \oplus \{K_1, K_2, L_3\}$	$\gamma_{ij}^{(1)} = \begin{pmatrix} 2f(t) & 0 & 0 \\ 0 & f(t) & 0 \\ 0 & 0 & f(t) \end{pmatrix}$
	$\alpha_{4,5}$	$F\{K_2; P_0 - P_2\} \oplus$ $\{F - K_2, P_1\}$	$\begin{pmatrix} 0 & \frac{b}{t-y} & 0 \\ \frac{b}{t-y} & 0 & \frac{b}{y-t} \\ 0 & \frac{b}{y-t} & 0 \end{pmatrix}$
	$\alpha_{4,6}$	$\{F + K_2, P_0 - P_2\} \oplus$ $\{F - K_2, P_0 + P_2\}$	$\begin{pmatrix} \frac{b}{x} & 0 & 0 \\ 0 & \frac{2b}{x} & 0 \\ 0 & 0 & -\frac{b}{x} \end{pmatrix}$
	$\alpha_{4,7}$	$L_3 - K_1, P_0 + P_2;$ $P_0 - P_2, P_1$	gives 5 KV subalgebra for constant components of $\gamma_{ij}^{(1)}$

[1] J. Maldacena (2011), 1105.5632.

[2] G. 't Hooft, pp. 251–266 (2009), 0909.3426.

[3] G. 't Hooft (2010), 1009.0669.

[4] G. t. Hooft (2010), 1011.0061.

[5] G. T. Hooft (2014), 1410.6675.

[6] N. Berkovits and E. Witten, JHEP **0408**, 009 (2004), hep-th/0406051.

[7] D. Grumiller, M. Irakleidou, I. Lovrekovic, and R. McNees, Phys.Rev.Lett. **112**, 111102 (2014), 1310.0819.

[8] M. Irakleidou, I. Lovrekovic, and F. Preis (2014), 1412.7508.

[9] P. D. Mannheim and J. G. O'Brien, Phys.Rev. **D85**, 124020 (2012), 1011.3495.

[10] P. D. Mannheim, Found.Phys. **42**, 388 (2012), 1101.2186.

[11] P. D. Mannheim and J. G. O'Brien, J.Phys.Conf.Ser. **437**, 012002 (2013), 1211.0188.

[12] D. Grumiller and I. Sachs, JHEP **1003**, 012 (2010), 0910.5241.

[13] D. Grumiller and O. Hohm, Phys.Lett. **B686**, 264 (2010), 0911.4274.

[14] M. Irakleidou and I. Lovrekovic (2016), 1611.01810.

[15] P. D. Mannheim and D. Kazanas, Astrophys.J. **342**, 635 (1989).

[16] J. Patera, R. Sharp, P. Winternitz, and H. Zassenhaus, J.Math.Phys. **18**, 2259 (1977).

[17] S. Deser, E. Joung, and A. Waldron, J.Phys. **A46**, 214019 (2013), 1208.1307.

[18] S. Deser, M. Sandora, and A. Waldron, Phys.Rev. **D87**, 101501 (2013), 1301.5621.