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Calogero, spherically reduced and PT-deformed

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The rational Calogero model based on the A_{n-1} root system is spherically reduced to a superintegrable angular model of a particle moving on S^{n-2} subject to a very particular potential singular at the Weyl chamber walls. We review the computation of its energy spectrum (including the eigenstates), conserved charges and intertwining operators shifting the coupling constant by one. These models are deformed in a $\mathscr{P}\mathscr{T}$-symmetric manner by judicious complex coordinate transformations, which render the potential less singular. The $\mathscr{P}\mathscr{T}$ deformation does not change the energy levels but in some cases adds a previously unphysical tower of states. For integral couplings this roughly doubles the previous degeneracy and allows for a conserved nonlinear supersymmetry-type charge. We illustrate the general constructions by presenting the details for the cases of A_2 and A_3 and point out open questions.

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1. Some history

The rational Calogero model (for a review, see [1]) generalizes to any root system of a (finite-dimensional) Lie algebra or, better, to any Coxeter root system. Given such a system of rank n, it describes a conformal particle moving in $\mathbb{R}^n$ under the influence of a very special potential. Since this potential has a universal inverse-square radial dependence and otherwise depends only on the angular coordinates (of S^{n-1}), a spherical reduction to its angular subsystem, the angular Calogero model, is natural. Like the full model on $\mathbb{R}^n$, the reduced dynamics on S^{n-1} is superintegrable, so that it enjoys $2n-3$ integrals of motion, which are however not in involution. Recently, the angular models have been analyzed in some detail, both classically and quantum mechanically [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

It has been known for a long time that hermiticity is not an essential feature of a Hamiltonian for its spectrum to be real. For instance, it suffices that the Hamiltonian commutes with an antilinear involution (one example is provided by the $\mathscr{P}\mathscr{T}$ operator where $\mathscr{P}$ correspond to the parity operator and $\mathscr{T}$ the time reversal operator) which also leaves the eigenfunctions invariant ("unbroken $\mathscr{P}\mathscr{T}$ symmetry") [13]. Such a non-hermitian Hamiltonian is related to a hermitian one by a (non-unitary) similarity transformation, which may be impossibly complicated. Often, however, there exists a family H_ε of non-hermitian $\mathscr{P}\mathscr{T}$-invariant Hamiltonians representing a smooth deformation of a hermitian H_0. In this case we speak of a "$\mathscr{P}\mathscr{T}$ deformation", with the parameter ε measuring the deviation from hermiticity. For rational Calogero models, a particularly nice set of $\mathscr{P}\mathscr{T}$ deformations can be generated by a specific *complex* orthogonal deformation of the coordinates in the expression for the Hamiltonian. If such a $\mathscr{P}\mathscr{T}$ deformation is in accordance with the Coxeter reflection symmetry of the system, integrability will be preserved. This kind of $\mathscr{P}\mathscr{T}$ deformation has been applied to the full rational Calogero model about ten years ago by Fring and Znojil [14], and corresponding complex root systems were constructed by Fring and Smith thereafter [15, 16, 17]. For a review of $\mathscr{P}\mathscr{T}$ deformations of integrable models, see [18].

It is worth recalling the relevant part (for this talk) of the Calogero model's long history:

- 1971 Calogero [19]:
  Solution of the one-dim'l N-body problem with ... inversely quadratic pair potentials
- 1981 Olshanetsky & Perelomov [20, 21]:
  Classical integrable finite-dimensional systems related to Lie algebras (1983: quantum)
- 1983 Wojciechowski [22]:
  Superintegrability of the Calogero–Moser system
- 1989 Dunkl [23]:
  Differential-difference operators associated to reflection groups
- 1990 Chalykh & Veselov [24]:
  Commutative rings of partial differential operators and Lie algebras, supercompleteness
- 1991 Heckman [25]:
  Elementary construction for commuting charges and intertwiners (shift operators)
- 2003 M. Feigin [2]:
  Intertwining relations for the spherical parts of generalized Calogero operators

- 2008 A. Fring, M. Znojil [14]:
  $\mathscr{P}\mathscr{T}$-symmetric deformations of Calogero models
- 2008 Hakobyan, Nersessian, Yeghikyan [3]:
  The cuboctahedric Higgs oscillator from the rational Calogero model (classical)
- 2010 A. Fring, M. Smith [15, 16, 17]:
  Complex root systems in the Calogero model
- 2013 M. Feigin, Lechtenfeld, Polychronakos [9]:
  The quantum angular Calogero–Moser model (spectra, eigenstates)
- 2013 Correa, Lechtenfeld, Plyushchay [10]:
  Nonlinear supersymmetry in the quantum Calogero model
- 2014 M. Feigin, Hakobyan [11]:
  On the algebra of Dunkl angular momentum operators
- 2015 Correa, Lechtenfeld [12]:
  The tetrahexahedric angular Calogero model

2. The angular (relative) Calogero model

The A_{n-1} Calogero Hamiltonian (without the center of mass) reads

$$H \;=\; \sum_{\mu<\nu}^{n}\Big\{\tfrac{1}{2n}(p_\mu-p_\nu)^2 \;+\; \frac{g(g-1)}{(x^\mu-x^\nu)^2}\Big\} \tag{2.1}$$

for $\mathbb{R}^n$ coordinates and momenta x^μ and p_ν, respectively, subject to canonical quantization

$$[x^\mu\,,\,p_\nu] \;=\; \mathrm{i}\,\delta^\mu_\nu \qquad\qquad \text{with} \quad \mu,\nu=1,\ldots,n\,. \tag{2.2}$$

This Hamiltonian acts on states in the 'relative' space orthogonal to the center of mass. In this reduced space we define a radial coordinate and momentum,

$$\tfrac{1}{n}\sum_{\mu<\nu}(x^\mu-x^\nu)^2 \;=\; r^2 \qquad\text{and}\qquad \tfrac{1}{n}\sum_{\mu<\nu}(p_\mu-p_\nu)^2 \;=\; p_r^2+\tfrac{1}{r^2}L^2+\tfrac{(n-2)(n-4)}{4\,r^2}\;, \tag{2.3}$$

revealing also an angular-momentum contribution to H. We introduce $n-1$ suitable relative coordinates y^i and momenta p_j,

$$r^2 \;=\; \sum_{i=1}^{n-1}(y^i)^2 \quad,\quad p_i\equiv p_{y^i} \quad,\quad L_{ij} \;=\; -\mathrm{i}(y^ip_j-y^jp_i) \quad,\quad L^2 \;=\; -\sum_{i<j}L_{ij}^2\,. \tag{2.4}$$

This Hamiltonian is part of an SL(2,$\mathbb{R}$) conformal algebra generated by

$$H \;=\; \tfrac12 p_r^2 \;+\; \tfrac{(n-2)(n-4)}{8\,r^2} \;+\; \tfrac{1}{r^2}H_\Omega \quad,\quad D \;=\; \tfrac12(r\,p_r+p_r r) \quad,\quad K \;=\; \tfrac12 r^2\,, \tag{2.5}$$

where all dependence on the non-radial (i.e. angular) coordinates and momenta is subsumed in the angular Calogero Hamiltonian

$$H_\Omega \;=\; \tfrac12 L^2+U(\vec\theta) \;=\; C-\tfrac18(n-1)(n-5) \qquad\text{with}\qquad C \;=\; K\,H+H\,K-\tfrac12 D^2\,, \tag{2.6}$$

where C denotes the SL(2,$\mathbb{R}$) Casimir and

$$U(\vec{\theta}) \;=\; r^2 \sum_{\mu<\nu} \frac{g(g-1)}{(x^\mu - x^\nu)^2} \;=\; r^2 \sum_{\alpha\in\mathscr{R}_+} \frac{g(g-1)}{(\alpha\cdot y)^2} \;=\; \tfrac{g(g-1)}{2} \sum_{\alpha\in\mathscr{R}_+} \cos^{-2}\theta_\alpha\;, \tag{2.7}$$

with $\mathscr{R}_+$ denoting the set of positive roots of A_{n-1}. Our H_Ω describes a single particle on the $(n-2)$-sphere at $r{=}1$ in relative space, trapped in a spherical $(n-2)$-simplex bounded by Weyl-chamber walls $\alpha\cdot y = 0$, where the particular potential U blows up. We remark that the coupling $g(g-1)$ is invariant under $g \mapsto 1-g$.

Let us pass to the position representation and deal with differential operators,

$$p_i \;\mapsto\; -\mathrm{i}\partial_i \qquad\Longrightarrow\qquad p_r \;\mapsto\; -\mathrm{i}\big(\partial_r + \tfrac{n-2}{2r}\big)\;, \tag{2.8}$$

$$\begin{aligned} H \;&\mapsto\; -\tfrac12\big(\partial_r^2 + \tfrac{n-2}{r}\partial_r\big) \;+\; \tfrac{1}{r^2}H_\Omega \;=\; w^{-1}\Big[-\tfrac12\big(\partial_r^2 - \tfrac{(n-2)(n-4)}{4r^2}\big) \;+\; \tfrac{1}{r^2}H_\Omega\Big]\,w \\ H_\Omega \;&\mapsto\; -\tfrac12\sum_{i<j}\big(y^i\partial_j - y^j\partial_i\big)^2 \;+\; r^2\sum_{\alpha\in\mathscr{R}_+}\frac{g(g-1)}{(\alpha\cdot y)^2} \qquad\qquad \text{with}\quad w = r^{\frac{n-2}{2}} \end{aligned} \tag{2.9}$$

The Schrödinger equation is solved via a standard radial-angular separation ansatz. Employing a free-particle (on S^{n-2}) parametrization of the energy, $\varepsilon = \frac12 q\,(q+n-3)$ with generalized angular momentum q, we have

$$H\Psi \;=\; E\Psi \qquad\text{with}\qquad \Psi \;=\; R(r)\,v(\vec{\theta}) \qquad\text{and}\qquad H_\Omega\, v \;=\; \varepsilon\, v\;, \tag{2.10}$$

$$\begin{aligned} wHw^{-1}\big|_\varepsilon \;&\mapsto\; -\tfrac12\partial_r^2 + \tfrac{1}{2r^2}\big[(\tfrac{n}{2}-1)(\tfrac{n}{2}-2) + q\,(q{+}n{-}3)\big] \\ &=\; -\tfrac12\partial_r^2 + \tfrac{1}{2r^2}\big(q{+}\tfrac{n}{2}{-}1\big)\big(q{+}\tfrac{n}{2}{-}2\big)\;. \end{aligned} \tag{2.11}$$

To 'discretize' the continuous E spectrum, it is convenient to add a harmonic confining potential $\frac12\omega^2 r^2$. Then, on the one hand, the radial problem for a given q-value is a textbook one, with the solution

$$E\big|_\varepsilon \;=\; \omega\,\big(2\ell_2 + q + \tfrac{n-1}{2}\big) \quad\text{with}\quad \ell_2\in\mathbb{N}_0 \tag{2.12}$$

involving a 'radial' quantum number ℓ_2. On the other hand, the harmonic Calogero spectrum is well known otherwise,

$$E \;=\; \omega\,\big(2\ell_2 + 3\ell_3 + \ldots + n\ell_n + \tfrac12 n(n-1)\,g + \tfrac{n-1}{2}\big) \quad\text{with}\quad \ell_\mu\in\mathbb{N}_0\;. \tag{2.13}$$

Comparing (2.12) to (2.13), we learn that

$$q \;=\; \tfrac12 n(n-1)\,g + \ell \qquad\text{where}\qquad \ell \;=\; 3\ell_3 + 4\ell_4 + \ldots + n\ell_n \;\in\mathbb{N}_0\;. \tag{2.14}$$

Reassuringly, ℓ_1 (counting translational excitation) is absent, and ℓ_2 (measuring radial excitation) is missing as well on S^{n-2}. So indeed it is appropriate to express the angular spectrum as

$$\varepsilon_q \;=\; \tfrac12 q\,(q+n-3)\;. \tag{2.15}$$

The degeneracy of these energy levels is, however, much lower than for a free particle, because the potential U breaks the SO$(n-1)$ symmetry of S^{n-2} to the Weyl group S_n of A_{n-1}. It is essentially given by the number $p_n(\ell)$ of partitions of the integer ℓ into at most n summands (or, equivalently, intoparts not larger than n),

$$\deg_n(\varepsilon_q) \;=\; p_n(\ell) - p_n(\ell{-}1) - p_n(\ell{-}2) + p_n(\ell{-}3)\;, \tag{2.16}$$

accounting for the absence of 1's and 2's. Even though its generating function is very simple,

$$p_n(t) \;:=\; \sum_{\ell=0}^{\infty} p_n(\ell)\,t^\ell \;=\; \prod_{m=1}^{n}\bigl(1-t^m\bigr)^{-1}\;, \tag{2.17}$$

explicit formulæ exist only for small values of n. We shall need

$$\begin{aligned}
\deg_3(\ell) \;&=\; \begin{cases} 0 & \text{for}\quad \ell = 1,2 \bmod 3 \\ 1 & \text{for}\quad \ell = 0 \bmod 3 \end{cases}\;, \\
\deg_4(\ell) \;&=\; \Bigl\lfloor \frac{\ell}{12} \Bigr\rfloor \;+\; \begin{cases} 0 & \text{for}\quad \ell = 1,2,5 \bmod 12 \\ 1 & \text{for}\quad \ell = \text{else} \bmod 12 \end{cases}\;.
\end{aligned} \tag{2.18}$$

The limits $g \to 0$ and $g \to 1$ keep the memory of the infinite Weyl-chamber walls and thus the above degeneracy, although the particle becomes free otherwise.

What about the energy eigenstates? Their radial part, $R_{\ell_2\ell}(r)$, is given by a Bessel function. The angular part, $v_{\{\ell\}}(\vec\theta)$, is a bit harder to obtain. In this talk, we demand the Weyl invariance of the potential to be unbroken, i.e. we impose that $v^{(g)}_{\{\ell\}}(s_\alpha\vec\theta) = \mathrm{e}^{\mathrm{i}\pi g}\,v^{(g)}_{\{\ell\}}(\vec\theta)$ under the reflection s_α pertaining to a root α. As ingredients to construct $v^{(g)}_{\{\ell\}}(\vec\theta)$, one needs first the elementary Weyl-symmetric polynomials $\sigma_\mu(y)$ of degrees $\mu = 3,\dots,n$ ($\sigma_2 = r^2$ is trivial), second and third the Vandermonde and the Dunkl operators,

$$\Delta \;=\; \prod_{\alpha\in\mathscr{R}_+} \alpha\cdot y \qquad\text{and}\qquad \mathscr{D}_i \;=\; \partial_i \;-\; g\sum_{\alpha\in\mathscr{R}_+} \frac{\alpha_i}{\alpha\cdot y}\,s_\alpha\;, \tag{2.19}$$

respectively. With these,

$$v_q(\vec\theta) \;\equiv\; v^{(g)}_{\{\ell\}}(\vec\theta) \;\sim\; r^{n-3+q}\Bigl(\prod_{\mu=3}^{n} \sigma_\mu\bigl(\{\mathscr{D}_i\}\bigr)^{\ell_\mu}\Bigr)\,\Delta^g\, r^{3-n-n(n-1)g}\;. \tag{2.20}$$

Pulling out the factor of Δ^g, the angular wave functions are essentially determined by Weyl-symmetric Dunkl-deformed harmonic polynomials of degree ℓ,

$$v^{(g)}_{\{\ell\}}(\vec\theta) \;=\; r^{-q}\,\Delta^g\,h^{(g)}_\ell \qquad\Longrightarrow\qquad H\bigl(\Delta^g\,h^{(g)}_\ell\bigr) \;=\; 0\;. \tag{2.21}$$

In fact, one may understand the Hamiltonians H and H_Ω as Dunkl deformations of the corresponding Laplacians. To see this, we 'Dunklize' the angular momenta,

$$L_{ij} \;\mapsto\; -(y^i\partial_j - y^j\partial_i) \qquad\Longrightarrow\qquad \mathscr{L}_{ij} \;=\; -(y^i\mathscr{D}_j - y^j\mathscr{D}_i)\;, \tag{2.22}$$

and compute their squares,

$$-\tfrac12\sum_i \mathscr{D}_i^2 \;=:\; \mathscr{H} \qquad\text{and}\qquad -\tfrac12\sum_{i<j}\mathscr{L}_{ij}^2 \;=:\; \mathscr{H}_\Omega \;-\; \tfrac12\,g\textstyle\sum_\alpha s_\alpha\,\bigl(g\sum_\alpha s_\alpha + n{-}3\bigr)\;. \tag{2.23}$$

The so-defined differential-reflection operators $\mathscr{H}$ and $\mathscr{H}_\Omega$ reduce to our Hamiltonians when restricted to act on Weyl-symmetric functions,

$$\mathrm{res}(\mathscr{H}) \;=\; H \qquad \text{and} \qquad \mathrm{res}(\mathscr{H}_\Omega) \;=\; \tfrac{1}{2}\mathrm{res}\big(\mathscr{L}^2\big) \;+\; \varepsilon_q(\ell{=}0) \;=\; H_\Omega\ . \tag{2.24}$$

More generally, conserved charges of order t are found from degree-t polynomials in $\mathscr{L}_{ij}$,

$$\mathscr{C}_t(\mathscr{L}_{ij}) \quad \text{Weyl-invariant} \qquad \Longrightarrow \qquad C_t = \mathrm{res}(\mathscr{C}_t) \quad \text{commutes with } H_\Omega\ . \tag{2.25}$$

The angular Calogero model is maximally superintegrable, i.e. there exist $2n{-}5$ algebraically independent charges of this kind ($C_2 = H_\Omega$), but they are not in involution!

Other (degree-s) polynomials in $\mathscr{L}_{ij}$ produce intertwining operators,

$$\mathscr{M}_s(\mathscr{L}_{ij}) \quad \text{Weyl-}\underline{\text{anti}}\text{invariant} \qquad \Longrightarrow \qquad M_s \;=\; \mathrm{res}(\mathscr{M}_s) \quad \text{intertwines with } H_\Omega\ . \tag{2.26}$$

By this we mean the following. Since $[\mathscr{L}_{ij},\mathscr{H}]=0$ and (from (2.19)) $s_\alpha \to -s_\alpha$ equals $g\to -g$, we have:

$$\begin{aligned} [\mathscr{M}_s,\mathscr{H}]=0 \qquad &\Longrightarrow \qquad M_s^{(g)}H^{(g)} \;=\; H^{(-g)}M_s^{(g)} \;=\; H^{(g+1)}M_s^{(g)} \\ &\text{and} \qquad M_s^{(g)}\,:\,\big\{\Psi^{(g)}_{E,q}\big\} \;\to\; \big\{\Psi^{(g+1)}_{E,q}\big\}\ , \end{aligned} \tag{2.27}$$

$$\begin{aligned} [\mathscr{M}_s,\mathscr{H}_\Omega]=0 \qquad &\Longrightarrow \qquad M_s^{(g)}H_\Omega^{(g)} \;=\; H_\Omega^{(-g)}M_s^{(g)} \;=\; H_\Omega^{(g+1)}M_s^{(g)} \\ &\text{and} \qquad M_s^{(g)}\,:\,\big\{v^{(g)}_\ell\big\} \;\to\; \big\{v^{(g+1)}_{\ell-n(n-1)/2}\big\}\ . \end{aligned} \tag{2.28}$$

The operators $\mathscr{L}_{ij}$ and the Weyl reflections generate an interesting algebra,

$$[\mathscr{L}_{ij},\mathscr{L}_{k\ell}] \;=\; \mathscr{L}_{i\ell}\mathscr{S}_{jk}-\mathscr{L}_{ik}\mathscr{S}_{j\ell}-\mathscr{L}_{j\ell}\mathscr{S}_{ik}+\mathscr{L}_{jk}\mathscr{S}_{i\ell} \tag{2.29}$$

$$\text{with} \qquad \mathscr{S}_{ij} \;=\; \begin{cases} -g\,s_{ij} & \text{for} \quad i\neq j \\ 1+g\sum_{k(\neq i)}s_{ik} & \text{for} \quad i=j \end{cases}\ , \tag{2.30}$$

$$[\mathscr{S}_{ij},\mathscr{L}_{k\ell}]=0 \quad , \qquad \{\mathscr{S}_{ij},\mathscr{L}_{ij}\}=0 \quad , \qquad \mathscr{S}_{ij}\mathscr{L}_{ik} \;=\; \mathscr{L}_{jk}\mathscr{S}_{ij}\ . \tag{2.31}$$

It is a 'Dunkl deformation' of $so(n{-}1)$, with H_Ω being its Casimir invariant.

3. Complex $\mathscr{P}\mathscr{T}$ deformation

Quantum mechanics achieves $E\in\mathbb{R}$ by $H^\dagger=H$, but actually $H^\dagger=\rho\,H\,\rho^{-1}$ suffices to guarantee a real spectrum. Such a non-hermitian H is related to a hermitian H_0 by a similarity transformation. The spectrum of $H\neq H^\dagger$ is still real if the system (and ground state) is invariant under a combined involution $\mathscr{P}\mathscr{T}$, where $\mathscr{P}$ is linear and $\mathscr{T}$ is <u>anti</u>linear (usually $\mathscr{P}$ = parity and $\mathscr{T}$ = complex conjugation $\mathrm{i}\mapsto-\mathrm{i}$). A $\mathscr{P}\mathscr{T}$ deformation is a non-hermitian $\mathscr{P}\mathscr{T}$-invariant family H_ε smoothly deforming $H_0{=}H_0^\dagger$. For the A_{n-1} Calogero system, we will induce $H{=}H_0\mapsto H_\varepsilon$ from a complex coordinate deformation $\Gamma(\varepsilon):\mathbb{R}^{n-1}\to\mathbb{C}^{n-1}$. We shall see that superintegrability will be preserved as long as $\Gamma(\varepsilon)$ is chosen to be compatible with Weyl invariance. The simplest possibility for $\mathscr{P}$ is an order-2 element s from the Weyl group (e.g. a reflection s_α), while $\mathscr{T}$ remains complex conjugation.

Reasonable (but debatable) onditions on a complex angular coordinate deformation $\Gamma(\varepsilon)$ are:

- it should be linear
- it should not change the kinetic term $L^2 \quad\Rightarrow\quad \Gamma(\varepsilon)\in \mathrm{SO}(n{-}1,\mathbb{C})/\mathrm{SO}(n{-}1,\mathbb{R})$
- it should render $U_\varepsilon(y) := U\big(\Gamma(\varepsilon)\,y\big)$ $\mathscr{P}\mathscr{T}$-invariant $\quad\Rightarrow\quad \mathscr{P}\mathscr{T}\,\Gamma(\varepsilon)=\Gamma(\varepsilon)$

As a consequence,

$$\Gamma(\varepsilon) \;=\; \exp\Big\{\sum_{i<j}\varepsilon_{ij}G_{ij}\Big\} \qquad\text{with}\qquad G_{ij}: y^k \;\mapsto\; \mathrm{i}\big(\delta^{kj}y^i-\delta^{ki}y^j\big) \tag{3.1}$$

$$\text{and}\qquad \mathscr{P}\,\Gamma(\varepsilon) \;=\; s\,\Gamma(\varepsilon)\,s \;\stackrel{!}{=}\; \Gamma(-\varepsilon) \;=\; \Gamma(\varepsilon)^* \;=\; \mathscr{T}\,\Gamma(\varepsilon)\;. \tag{3.2}$$

We infer that $\varepsilon{:}G \equiv \sum_{i<j}\varepsilon_{ij}G_{ij}$ intertwines between the $+1$ and -1 eigenspaces of s. When $\mathscr{P}$ is simply a root reflection s_γ,

$$\varepsilon{:}G \;\sim\; \varepsilon\,\gamma\wedge G\gamma \;\in su(1,1)\;, \tag{3.3}$$

and we may adapt our coordinates to the 2-plane $\gamma\wedge G\gamma$ so that

$$\Gamma(\varepsilon) \;=\; \mathrm{e}^{\varepsilon:G} \;=\; \begin{pmatrix} \cosh(\varepsilon) & -\mathrm{i}\sinh(\varepsilon) & 0\,\cdots\,0 \\ \mathrm{i}\sinh(\varepsilon) & \cosh(\varepsilon) & 0\,\cdots\,0 \\ 0 & 0 & \\ \vdots & \vdots & \mathbb{1}_{n-3} \\ 0 & 0 & \end{pmatrix}\;. \tag{3.4}$$

Hence, the deformation $y\mapsto\Gamma(\varepsilon)\,y$ just complexifies the angle $\phi \;\mapsto\; \phi+\varepsilon$ in the 2-plane $\gamma\wedge G\gamma$. Much more general $\mathscr{P}\mathscr{T}$ deformations are possible, their classification is open.

The benefit of such a deformation is a partial de-singularization of the potential: Its singular loci obey

$$\alpha\cdot y=0 \quad\mapsto\quad \alpha\cdot\Gamma(\varepsilon)\,y=0 \qquad\Rightarrow\qquad \underline{\text{two}}\text{ real conditions for each root }\alpha\;, \tag{3.5}$$

so that the singularities are (generically) reduced from codimension one to codimension two. In other words, the Calogero particle is liberated from its Weyl-chamber trap and can move everywhere on S^{n-2}! Explicitly,

$$U_\varepsilon(\vec\theta) \;=\; r^2\sum_{\alpha\in\mathscr{R}_+}\frac{g(g-1)}{(\alpha\cdot\Gamma(\varepsilon)\,y)^2} \;=\; \tfrac{g(g-1)}{2}\sum_{\alpha\in\mathscr{R}_+}\cos^{-2}\theta_\alpha(\varepsilon) \tag{3.6}$$

with $\theta_\alpha(\varepsilon)=\theta_\alpha+\mathrm{i}\eta_\alpha(\vec\theta,\varepsilon))$ is less singular due to

$$\frac{1}{\cos^2(\theta_\alpha+\mathrm{i}\eta_\alpha)} \;=\; \frac{\cosh^2\eta_\alpha\cos^2\theta_\alpha-\sinh^2\eta_\alpha\sin^2\theta_\alpha \;+\; \frac{\mathrm{i}}{2}\sinh 2\eta_\alpha\sin 2\theta_\alpha}{(\cosh^2\eta_\alpha\cos^2\theta_\alpha+\sinh^2\eta_\alpha\sin^2\theta_\alpha)^2}\;. \tag{3.7}$$

However, the wave functions $v_q(\vec\theta)$ carry the factor

$$\Delta^g \;=\; \prod_{\alpha\in\mathscr{R}_+}(\alpha\cdot y)^g \qquad\mapsto\qquad \Delta^g_\varepsilon \;=\; \prod_{\alpha\in\mathscr{R}_+}\big(\alpha\cdot\Gamma(\varepsilon)\,y\big)^g\;, \tag{3.8}$$

which still prevents their normalizability for $g<0$, except at $n{=}3$ (see below).

There do exist nonlinear $\mathscr{P}\mathscr{T}$ deformations which totally de-singularize Δ and U at $n{>}3$. For them, states with negative values of the coupling $g'=1-g<0$ become physical and have to be added to the spectrum for $g>1$. For $g\in\mathbb{Z}$ such $\mathscr{P}\mathscr{T}$ deformations then roughly double the degeneracy of the energy levels. Furthermore, the intertwiners $Q^{(g)}$ linking the states at g' and at g become bona fide conserved charges realizing a nonlinear type of supersymmetry,

$$Q^{(g)}\,H^{(1-g)}_{\Omega\varepsilon} \;=\; H^{(g)}_{\Omega\varepsilon}\,Q^{(g)} \qquad\text{and}\qquad \big(Q^{(g)}\big)^2 \;=\; \text{a polynomial in the }\mathscr{C}^{(g)}_t\;. \tag{3.9}$$

4. Warmup: the hexagonal or Pöschl-Teller model

For illustration of the previous two sections, we present the details for the simplest nontrivial case of $n{=}3$. The two-dimensional root system is of hexagonal shape, and the angular submodel describes a particle on a circle subject to a Pöschl-Teller Potential. The Jacobi relative coordinates on $\mathbb{R}^2$ orthogonal to the center-of-mass coordinate X read

$$\begin{aligned} x^1 &= X+\tfrac{1}{\sqrt{2}}y^1+\tfrac{1}{\sqrt{6}}y^2 \quad , \qquad & \partial_{x^1} &= \tfrac13\partial_X+\tfrac{1}{\sqrt{2}}\partial_{y^1}+\tfrac{1}{\sqrt{6}}\partial_{y^2} \ , \\ x^2 &= X-\tfrac{1}{\sqrt{2}}y^1+\tfrac{1}{\sqrt{6}}y^2 \quad , \qquad & \partial_{x^2} &= \tfrac13\partial_X-\tfrac{1}{\sqrt{2}}\partial_{y^1}+\tfrac{1}{\sqrt{6}}\partial_{y^2} \ , \\ x^3 &= X-\tfrac{2}{\sqrt{6}}y^2 \quad , \qquad & \partial_{x^3} &= \tfrac13\partial_X-\tfrac{2}{\sqrt{6}}\partial_{y^2} \ , \end{aligned} \tag{4.1}$$

from which we define polar and complex coordinates as well,

$$y^1 \ = \ r\cos\phi \quad \text{and} \quad y^2 \ = \ r\sin\phi \qquad \Longrightarrow \qquad w \ := \ y^1+\mathrm{i}y^2 \ = \ r\,\mathrm{e}^{\mathrm{i}\phi} \ . \tag{4.2}$$

The angular Hamiltonian takes the form

$$H_\Omega \ = \ \tfrac12\big(w\partial_w-\bar{w}\partial_{\bar{w}}\big)^2 \ + \ g(g{-}1)\frac{18\,(w\bar{w})^3}{(w^3+\bar{w}^3)^2} \ , \tag{4.3}$$

where the Pöschl-Teller potential emerges from

$$U(\phi) \ = \ \tfrac{g(g-1)}{2}\sum_{k=0,1,2}\cos^{-2}(\phi{+}k\tfrac{2\pi}{3}) \ = \ \tfrac92 g(g{-}1)\,\cos^{-2}(3\phi) \ . \tag{4.4}$$

It is singular at the Weyl-chamber walls $\phi=\pm\frac{\pi}{6},\pm\frac{\pi}{2},\pm\frac{5\pi}{6}$ produced by the Weyl group S_3.

Specializing the general formulæ of Section 2 we obtain the spectrum and eigenfunctions:

$$\varepsilon_q \ = \ \tfrac12 q^2 \qquad \text{with} \qquad q \ = \ 3g+\ell \ = \ 3(g+\ell_3)\ , \quad \ell=3\ell_3 \qquad \text{and} \qquad \deg(\varepsilon_q)=1\ , \tag{4.5}$$

$$\sigma_3 \ = \ 3(y^1)^2y^2-(y^2)^3 \ \sim \ w^3-\bar{w}^3 \ \sim \ r^3\sin(3\phi)\ , \tag{4.6}$$

$$\Delta \ \sim \ (y^1)^3-3y^1(y^2)^2 \ \sim \ w^3+\bar{w}^3 \ \sim \ r^3\cos(3\phi)\ , \tag{4.7}$$

$$\begin{aligned} \mathscr{D}_w \ &= \ \partial_w \ - \ g\Big\{\frac{1}{w+\bar{w}}\,s_0+\frac{\rho}{\rho w+\bar{\rho}\bar{w}}\,s_+ + \frac{\bar{\rho}}{\bar{\rho}w+\rho\bar{w}}\,s_-\Big\} \quad \text{with} \quad \rho=\mathrm{e}^{2\pi\mathrm{i}/3} \\ &\text{and} \qquad s_0:\ w\mapsto-\bar{w}\ , \quad s_+:\ w\mapsto-\rho\bar{w}\ , \quad s_-:\ w\mapsto-\bar{\rho}\bar{w}\ , \end{aligned} \tag{4.8}$$

$$v_q(\phi) \ \equiv \ v_\ell^{(g)}(\phi) \ \sim \ r^q\big(\mathscr{D}_w^3-\mathscr{D}_{\bar{w}}^3\big)^{\ell_3}\Delta^g\,r^{-6g} \ = \ r^{-q}\Delta^g\,h_\ell^{(g)}(w^3,\bar{w}^3)\ . \tag{4.9}$$

The homogeneous polynomials are given by Jacobi polynomials,

$$h_\ell^{(g)}(w^3,\bar{w}^3) \ = \ \sum_{k=0}^{\ell_3}(-1)^k\,\tfrac{\Gamma(1+\ell_3)\,\Gamma(g+k)\,\Gamma(g+\ell_3-k)}{\Gamma(2g+\ell_3)\,\Gamma(g)\,\Gamma(1+k)\,\Gamma(1+\ell_3-k)}\ w^{\ell-3k}\bar{w}^{3k}\ , \tag{4.10}$$

of which the lowest few are listed here for $g{=}0,1,2$, with the notation $(m\bar{m}):=w^{3m}\bar{w}^{3\bar{m}}$:

q	$h_\ell^{(0)}$	$h_\ell^{(1)}$	$h_\ell^{(2)}$	$\cdots$
0	(00)			
3	(10) − (01)	(00)		
6	(20) + (02)	(10) − (01)	(00)	
9	(30) − (03)	(20) − (11) + (02)	(10) − (01)	$\ddots$
12	(40) + (04)	(30) − (21) + (12) − (03)	3(20) − 4(11) + 3(02)	$\cdots$
15	(50) − (05)	(40) − (31) + (22) − (13) + (04)	4(30) − 6(21) + 6(12) − 4(03)	$\cdots$
18	(60) + (06)	(50) − (41) + (32) − (23) + (14) − (05)	5(40) − 8(31) + 9(22) − 8(13) + 5(04)	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

. (4.11)

We note that $\sigma_3 = (10) - (01)$, $\Delta = (10) + (01)$ and $r^6 = (11)$. The normalization is arbitrary.

Since the single angular momentum L_{12} is already Weyl-antiinvariant (it is O(2)-antiinvariant), its Dunkl deformation yields a degree-one angular intertwiner,

$$\mathscr{M}_1 \;\sim\; \mathrm{i}\big(w\mathscr{D}_w - \bar{w}\mathscr{D}_{\bar{w}}\big) \sim\; \mathrm{i}\big(w\partial_w - \bar{w}\partial_{\bar{w}}\big) \;-\; \mathrm{i}\,g\left\{\frac{w-\bar{w}}{w+\bar{w}}\,s_0 + \frac{\rho w - \bar{\rho}\bar{w}}{\rho w + \bar{\rho}\bar{w}}\,s_+ + \frac{\bar{\rho} w - \rho\bar{w}}{\bar{\rho} w + \rho\bar{w}}\,s_-\right\} \tag{4.12}$$

$$\Rightarrow \quad M_1 \;\sim\; \mathrm{i}\big(w\partial_w - \bar{w}\partial_{\bar{w}}\big) \;-\; 3\mathrm{i}\,g\,\frac{w^3-\bar{w}^3}{w^3+\bar{w}^3} \;=\; \mathrm{i}\Delta^g\big(w\partial_w - \bar{w}\partial_{\bar{w}}\big)\Delta^{-g} \;=\; \partial_\phi + 3\,g\,\tan 3\phi\ , \tag{4.13}$$

It allows for a simple recursion for the homogeneous polynomials,

$$v_\ell^{(g+1)} \;\sim\; M_1\, v_{\ell+3}^{(g)} \qquad \Rightarrow \qquad h_\ell^{(g+1)} \;\sim\; \mathrm{i}\Delta^{-1}\big(w\partial_w - \bar{w}\partial_{\bar{w}}\big)\, h_{\ell+3}^{(g)} \;\sim\; \Delta^{-1}\partial_\phi h_{\ell+3}^{(g)}\ . \tag{4.14}$$

There are no further conserved charges, because $M_1^\dagger M_1$ just yields H_Ω again.

What is the effect of an $\mathscr{P}\mathscr{T}$ deformation? Any $\mathrm{SO}(2,\mathbb{C})/\mathrm{SO}(2,\mathbb{R})$ transformation can be written as

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \;\longmapsto\; \frac{1}{3}\begin{pmatrix} 1{+}2\cosh\varepsilon & 1{-}\cosh\varepsilon{-}\mathrm{i}\sqrt{3}\sinh\varepsilon & 1{-}\cosh\varepsilon{+}\mathrm{i}\sqrt{3}\sinh\varepsilon \\ 1{-}\cosh\varepsilon{+}\mathrm{i}\sqrt{3}\sinh\varepsilon & 1{+}2\cosh\varepsilon & 1{-}\cosh\varepsilon{-}\mathrm{i}\sqrt{3}\sinh\varepsilon \\ 1{-}\cosh\varepsilon{-}\mathrm{i}\sqrt{3}\sinh\varepsilon & 1{-}\cosh\varepsilon{+}\mathrm{i}\sqrt{3}\sinh\varepsilon & 1{+}2\cosh\varepsilon \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{4.15}$$

or

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \;\longmapsto\; \begin{pmatrix} \cosh\varepsilon & -\mathrm{i}\sinh\varepsilon \\ \mathrm{i}\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \;=\; r\begin{pmatrix} \cos(\phi{+}\mathrm{i}\varepsilon) \\ \sin(\phi{+}\mathrm{i}\varepsilon) \end{pmatrix}\ , \tag{4.16}$$

which is nothing but

$$\phi \;\mapsto\; \phi + \mathrm{i}\varepsilon \qquad \Longleftrightarrow \qquad (w,\bar{w}) \;\mapsto\; (\mathrm{e}^{-\varepsilon}w, \mathrm{e}^{\varepsilon}\bar{w})\ . \tag{4.17}$$

The complex deformed potential reads

$$U_\varepsilon(\phi) \;=\; 9\,g(g{-}1)\,\frac{(1+\cosh 6\varepsilon\cos 6\phi) + 2\mathrm{i}\sinh 6\varepsilon\sin 6\phi}{(\cosh 6\varepsilon + \cos 6\phi)^2}\ , \tag{4.18}$$

and we know that its spectrum is real and independent on ε. Moreover, it is nonsingular, and thus the previously singular states for $g<0$ are transformed into physical states! The formulæ (4.5) still apply, but states with $\ell_3 = -g{-}q$ are proportional to states at $\ell_3 = -g{+}q$, so effectively $\ell_3 \geq \min(-g,0)$. With

$$\Delta_\varepsilon \;\sim\; \mathrm{e}^{-3\varepsilon}w^3 + \mathrm{e}^{3\varepsilon}\bar{w}^3 \;\sim\; r^3\left(\cosh(3\varepsilon)\cos(3\phi) - \mathrm{i}\sinh(3\varepsilon)\sin(3\phi)\right) \;\neq 0 \tag{4.19}$$

and

$$h_\ell^{\varepsilon(g)}(w^3,\bar{w}^3) \;=\; \sum_{k=0}^{\ell_3}(-1)^k\,\frac{\Gamma(1{+}\ell_3)\,\Gamma(g{+}k)\,\Gamma(g{+}\ell_3{-}k)}{\Gamma(2g{+}\ell_3)\,\Gamma(g)\,\Gamma(1{+}k)\,\Gamma(1{+}\ell_3{-}k)}\;(\mathrm{e}^{-\varepsilon}w)^{\ell-3k}(\mathrm{e}^{\varepsilon}\bar{w})^{3k} \tag{4.20}$$

extending to $g<0$ with proper $\frac{\infty}{\infty}$ regularization, the deformed angular wave functions $v_\ell^{\varepsilon(g)} = r^{-\ell-3g}\Delta_\varepsilon^{\;g}\,h_\ell^{\varepsilon(g)}$ are normalizable for any value of g, and we must extend the above table to

q	$\cdots$	$h_\ell^{\varepsilon(-1)}$	$h_\ell^{\varepsilon(0)}$	$h_\ell^{\varepsilon(1)}$	$h_\ell^{\varepsilon(2)}$	$\cdots$
0	$\cdots$	$(10)-(01)$	(00)			
3	$\cdots$	(11)	$(10)-(01)$	(00)		
6	$\cdots$	$(30)+3(21)-3(12)-(03)$	$(20)+(02)$	$(10)-(01)$	(00)	
9	$\cdots$	$2(40)+4(31)+4(13)+2(04)$	$(30)-(03)$	$(20)-(11)+(02)$	$(10)-(01)$	$\ddots$
12	$\cdots$	$3(50)+5(41)-5(14)-3(05)$	$(40)+(04)$	$(30)-(21)+(12)-(03)$	$3(20)-4(11)+3(02)$	$\cdots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	

(4.21)

where of course now $(m\bar{m}) := \mathrm{e}^{-3(m-\bar{m})\varepsilon}\, w^{3m}\bar{w}^{3\bar{m}}$. Restoring the factor $r^{-q}\Delta_\varepsilon^{\;g}$, we also list the corresponding angular wave functions:

q	$\cdots$	$r^q\, v_\ell^{\varepsilon(-1)}$	$r^q\, v_\ell^{\varepsilon(0)}$	$r^q\, v_\ell^{\varepsilon(1)}$	$r^q\, v_\ell^{\varepsilon(2)}$	$\cdots$
0	$\cdots$	$\frac{(10)-(01)}{(10)+(01)}$	(00)			
3	$\cdots$	$\frac{(11)}{(10)+(01)}$	$(10)-(01)$	$(10)+(01)$		
6	$\cdots$	$\frac{(30)+3(21)-3(12)-(03)}{(10)+(01)}$	$(20)+(02)$	$(20)-(02)$	$(20)+2(11)+(02)$	
9	$\cdots$	$\frac{2(40)+4(31)+4(13)+2(04)}{(10)+(01)}$	$(30)-(03)$	$(30)+(03)$	$(30)+(21)-(12)-(03)$	$\ddots$
12	$\cdots$	$\frac{3(50)+5(41)-5(14)-3(05)}{(10)+(01)}$	$(40)+(04)$	$(40)-(04)$	$3(40)+2(31)-2(22)+2(13)+3(04)$	$\cdots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	

. (4.22)

Finally, we have to join the state spaces at couplings g and $1-g$, and thus the degeneracy (for $g>0$) increases to $\deg(\varepsilon_q) = 2$ for $q \geq 3g$.

5. Tetrahexahedric model: the spectrum

Now we turn to the novel case of $n=4$, which is no longer fully separable. Exploiting $A_3 \simeq D_3$ to introduce Walsh-Hadamard coordinates $(y^1, y^2, y^3) \equiv (x, y, z)$,

$$\begin{aligned} x^1 &= X + \tfrac12(+x+y+z) \quad , \qquad \partial_{x^1} = \tfrac14\partial_X + \tfrac12(+\partial_x+\partial_y+\partial_z)\ , \\ x^2 &= X + \tfrac12(+x-y-z) \quad , \qquad \partial_{x^2} = \tfrac14\partial_X + \tfrac12(+\partial_x-\partial_y-\partial_z)\ , \\ x^3 &= X + \tfrac12(-x+y-z) \quad , \qquad \partial_{x^3} = \tfrac14\partial_X + \tfrac12(-\partial_x+\partial_y-\partial_z)\ , \\ x^4 &= X + \tfrac12(-x-y+z) \quad , \qquad \partial_{x^4} = \tfrac14\partial_X + \tfrac12(-\partial_x-\partial_y+\partial_z)\ , \end{aligned} \tag{5.1}$$

$$x = r\sin\theta\cos\phi \quad , \qquad y = r\sin\theta\sin\phi \quad , \qquad z = r\cos\theta\ , \tag{5.2}$$

the 2-sphere in relative $\mathbb{R}^3$ is parametrized by (θ,ϕ). The angular momenta

$$L_x = -(y\partial_z - z\partial_y)\ , \qquad L_y = -(z\partial_x - x\partial_z)\ , \qquad L_z = -(x\partial_y - y\partial_x) \tag{5.3}$$

square to the S^2 Laplacian

$$L^2 \;=\; -(L_x^2+L_y^2+L_z^2) \;=\; -\tfrac{1}{\sin\theta}\partial_\theta\sin\theta\,\partial_\theta - \tfrac{1}{\sin^2\theta}\partial_\phi^2\ , \tag{5.4}$$

and the six positive roots $\{e_x \pm e_y, e_x \pm e_z, e_y \pm e_z\}$ of A_3 yield the rational A_3 Calogero Hamiltonian

$$H \;=\; -\tfrac12(\partial_x^2+\partial_y^2+\partial_z^2) \;+\; 2g(g-1)\Big(\frac{x^2+y^2}{(x^2-y^2)^2} + \frac{y^2+z^2}{(y^2-z^2)^2} + \frac{z^2+x^2}{(z^2-x^2)^2}\Big)\ . \tag{5.5}$$

In the angular coordinates, the potential takes the less transparent form

$$U(\theta,\phi) \;=\; 2g(g-1)\Big\{\frac{1}{\sin^2\theta\cos^2 2\phi} + \frac{\cos^2\theta+\sin^2\theta\cos^2\phi}{(\cos^2\theta-\sin^2\theta\cos^2\phi)^2} + \frac{\cos^2\theta+\sin^2\theta\sin^2\phi}{(\cos^2\theta-\sin^2\theta\sin^2\phi)^2}\Big\}\ , \tag{5.6}$$

and it is invariant under the elementary S_4 Weyl reflections

$$\begin{aligned} s_{x+y}:\ (x,y,z) &\mapsto (-y,-x,+z) \quad , \qquad s_{x-y}:\ (x,y,z) \mapsto (+y,+x,+z) \quad , \\ s_{y+z}:\ (x,y,z) &\mapsto (+x,-z,-y) \quad , \qquad s_{y-z}:\ (x,y,z) \mapsto (+x,+z,+y) \quad , \\ s_{z+x}:\ (x,y,z) &\mapsto (-z,+y,-x) \quad , \qquad s_{z-x}:\ (x,y,z) \mapsto (+z,+y,+x) \quad . \end{aligned} \tag{5.7}$$

Each of these leaves fixed a plane which intersects the 2-sphere $r{=}1$ in a great circle. The six great circles form the edges of a spherical tetrahexahedron (or tetrakis hexahedron), tessalating the sphere in 24 identical right isosceles triangles, in which the particle is trapped by infinite potential walls.

Specializing Section 2 to $n{=}4$, we find

$$\varepsilon_q \;=\; \tfrac12 q\,(q{+}1) \qquad \text{with} \qquad q \;=\; 6g+\ell \;=\; 6g+3\ell_3{+}4\ell_4 \qquad \text{and} \qquad \ell \;=\; 3\ell_3{+}4\ell_4\;, \tag{5.8}$$

$$\sigma_3 \;=\; xyz \qquad \text{and} \qquad \sigma_4 \;=\; x^4+y^4+z^4\;, \tag{5.9}$$

$$\Delta \;\sim\; (x^2-y^2)(y^2-z^2)(x^2-z^2)\;, \tag{5.10}$$

$$\begin{aligned}
\mathscr{D}_x &= \partial_x - \frac{g}{x{+}y}\,s_{x+y} - \frac{g}{x{-}y}\,s_{x-y} - \frac{g}{z{+}x}\,s_{x+z} - \frac{g}{x{-}z}\,s_{z-x}\;,\\
\mathscr{D}_y &= \partial_y - \frac{g}{y{+}x}\,s_{x+y} - \frac{g}{y{-}x}\,s_{x-y} - \frac{g}{y{+}z}\,s_{y+z} - \frac{g}{y{-}z}\,s_{y-z}\;,\\
\mathscr{D}_z &= \partial_z - \frac{g}{z{+}x}\,s_{z+x} - \frac{g}{z{-}x}\,s_{z-x} - \frac{g}{z{+}y}\,s_{y+z} - \frac{g}{z{-}y}\,s_{y-z}\;,
\end{aligned} \tag{5.11}$$

so that

$$v_\ell^{(g)}(\theta,\phi) \;\sim\; r^{q+1}\big(\mathscr{D}_x\mathscr{D}_y\mathscr{D}_z\big)^{\ell_3}\big(\mathscr{D}_x^4{+}\mathscr{D}_y^4{+}\mathscr{D}_z^4\big)^{\ell_4}\Delta^g\,r^{1-12g} \;=\; r^{-q}\Delta^g\,h_\ell^{(g)}(x,y,z)\;. \tag{5.12}$$

Unlike for $n{=}3$, we have no analytic expression for $h_\ell^{(g)}$, but display these polynomials for $\ell{\le}12$:

ℓ	ℓ_3	ℓ_4	$h_\ell^{(g)}$
0	0	0	$\{000\}$
3	1	0	$\{111\}$
4	0	1	$\{400\}+\{220\}$
6	2	0	$\{600\}+\{420\}+\{222\}$
7	1	1	$\{511\}+\{331\}$
8	0	2	$\{800\}+\{620\}+\{440\}+\{422\}$
9	3	0	$\{711\}+\{531\}+\{333\}$
10	2	1	$\{1000\}+\{820\}+\{640\}+\{622\}+\{442\}$
11	1	2	$\{911\}+\{731\}+\{551\}$
12	4	0	$\{1200\}_1+\{1020\}_1+\{840\}_1+\{822\}_1+\{660\}_1+\{642\}_1+\{444\}_1$
12	0	3	$\{1200\}_2+\{1020\}_2+\{840\}_2+\{822\}_2+\{660\}_2+\{642\}_2+\{444\}_2$

(5.13)

with the notation $\{rst\} := c_{rst}(g)\,\big(x^ry^sz^t+x^ry^tz^s+x^sy^tz^r+x^sy^rz^t+x^ty^rz^s+x^ty^sz^r\big)$ and $c_{rst}(g)\in\mathbb{Z}$. Their degeneracy (for $g{\ge}0$) is given by

$$\deg(E_\ell) \;=\; \Big\lfloor\frac{\ell}{12}\Big\rfloor \;+\; \begin{cases} 0 & \text{for} \quad \ell=1,2,5 \mod 12 \\ 1 & \text{for} \quad \ell=\text{else} \mod 12 \end{cases}\;. \tag{5.14}$$

Due to the factor Δ^g, the formal eigenstates at $g{<}0$ are non-normalizable. Can we again employ a $\mathscr{P}\mathscr{T}$ deformation to sufficiently desingularize the potential? Up to an SO(3) rotation, the simplest linear $\mathscr{P}\mathscr{T}$ deformation reads

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \;\longmapsto\; \begin{pmatrix} 1{+}\cosh\varepsilon & \mathrm{i}\sinh\varepsilon & -\mathrm{i}\sinh\varepsilon & 1-\cosh\varepsilon \\ -\mathrm{i}\sinh\varepsilon & 1{+}\cosh\varepsilon & 1-\cosh\varepsilon & \mathrm{i}\sinh\varepsilon \\ \mathrm{i}\sinh\varepsilon & 1-\cosh\varepsilon & 1{+}\cosh\varepsilon & -\mathrm{i}\sinh\varepsilon \\ 1-\cosh\varepsilon & -\mathrm{i}\sinh\varepsilon & \mathrm{i}\sinh\varepsilon & 1{+}\cosh\varepsilon \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \tag{5.15}$$

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or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \longmapsto \quad \begin{pmatrix} \cosh\varepsilon & -\mathrm{i}\sinh\varepsilon & 0 \\ \mathrm{i}\sinh\varepsilon & \cosh\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \sin\theta\cos(\phi{+}\mathrm{i}\varepsilon) \\ \sin\theta\sin(\phi{+}\mathrm{i}\varepsilon) \\ \cos\theta \end{pmatrix} , \tag{5.16}$$

in short:

$$\phi \;\mapsto\; \phi + \mathrm{i}\varepsilon \qquad \Longleftrightarrow \qquad \big(x \pm \mathrm{i}y, z\big) \;\mapsto\; \big(\mathrm{e}^{\mp\varepsilon}(x \pm \mathrm{i}y), z\big) \; . \tag{5.17}$$

This deformation smoothens the potential to

$$\frac{U_\varepsilon(\theta,\phi)}{2g(g{-}1)} = \frac{1}{\sin^2\theta\,\cos^2 2(\phi{+}\mathrm{i}\varepsilon)} + \frac{\cos^2\theta + \sin^2\theta\cos^2(\phi{+}\mathrm{i}\varepsilon)}{(\cos^2\theta - \sin^2\theta\cos^2(\phi{+}\mathrm{i}\varepsilon))^2} + \frac{\cos^2\theta + \sin^2\theta\sin^2(\phi{+}\mathrm{i}\varepsilon)}{(\cos^2\theta - \sin^2\theta\sin^2(\phi{+}\mathrm{i}\varepsilon))^2} \, . \tag{5.18}$$

However, there remain five antipodal pairs of singular points, which prevents the normalizability of the $g{<}0$ states.

Still, all singularities may be removed with the nonlinear complex deformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto r \begin{pmatrix} \sin(\theta{+}\mathrm{i}\varepsilon_1)\cos(\phi{+}\mathrm{i}\varepsilon_2) \\ \sin(\theta{+}\mathrm{i}\varepsilon_1)\sin(\phi{+}\mathrm{i}\varepsilon_2) \\ \cos(\theta{+}\mathrm{i}\varepsilon_1) \end{pmatrix} = r \begin{pmatrix} c_1c_2\,x - \mathrm{i}c_1s_2\,y + s_1s_2\,\frac{zy}{\rho} \;+\; \mathrm{i}s_1c_2\,\frac{zx}{\rho} \\ c_1c_2\,y + \mathrm{i}c_1s_2\,x - s_1s_2\,\frac{zx}{\rho} \;+\; \mathrm{i}s_1c_2\,\frac{zy}{\rho} \\ c_1\,z \;-\; \mathrm{i}s_1\,\rho \end{pmatrix} \tag{5.19}$$

$$\text{with} \qquad c_i = \cosh(\varepsilon_i) \quad , \qquad s_i = \sinh(\varepsilon_i) \quad , \qquad \rho = \sqrt{x^2+y^2} \, ,$$

which modifies both L^2 and U. If both ε_1 and ε_2 are nonzero then the Vandermonde

$$\begin{aligned} \Delta_\varepsilon \;&\sim r^6 \sin^2(\theta{+}\mathrm{i}\varepsilon_1)\cos^4(\theta{+}\mathrm{i}\varepsilon_1)\cos^2(2\phi{+}2\mathrm{i}\varepsilon_2) \\ &\times \big(\tan^2(\theta{+}\mathrm{i}\varepsilon_1)\cos^2(\phi{+}\mathrm{i}\varepsilon_2) - 1\big)\big(\tan^2(\theta{+}\mathrm{i}\varepsilon_1)\sin^2(\phi{+}\mathrm{i}\varepsilon_2) - 1\big) \end{aligned} \tag{5.20}$$

is nowhere vanishing, which makes the $g < 0$ wave functions nonsingular and thus physical. The linear involution

$$\mathscr{P}: \; (\theta,\phi) \mapsto (-\theta,-\phi) \qquad \Longleftrightarrow \qquad (x,y,z) \mapsto (-x,y,z) \tag{5.21}$$

together with the complex conjugation $\mathscr{T}$ leaves the so-deformed Hamiltonian $H_{\Omega\varepsilon}$ invariant. Therefore, this is a viable $\mathscr{P}\mathscr{T}$ deformation, and we get a real ε-independent spectrum including the previously singular $g{<}0$ states! The energy formula

$$\varepsilon_q \;=\; \tfrac12 q\,(q{+}1) \qquad \text{with} \qquad q \;=\; 6g + \ell \;=\; 6g + 3\ell_3{+}4\ell_4 \tag{5.22}$$

develops a second branch for $g' = 1{-}g < 0$, so joining the state spaces at g and g' yields (for $g{>}0$)

$$\begin{aligned} \deg(\varepsilon_q) \;&=\; \deg_4(q{-}6g) + \deg_4(q{+}6g{-}6) + \deg_4(-q{+}6g{-}7) \\ &= \left\{ \begin{array}{ll} g{-}1 \;+\; \begin{Bmatrix} 0 & \text{for} & q+6g = 0,3,4,7,8,11 \bmod 12 \\ 1 & \text{for} & q+6g = 1,2,5,6,9,10 \bmod 12 \end{Bmatrix} & \text{if} \quad q < 6g{-}6 \\ \left\lfloor \frac{q}{6} \right\rfloor \;+\; \begin{Bmatrix} 0 & \text{for} & q = 1,2,5 \bmod 6 \\ 1 & \text{for} & q = 0,3,4 \bmod 6 \end{Bmatrix} & \text{if} \quad q \geq 6g{-}6 \end{array} \right. . \end{aligned} \tag{5.23}$$

One sees that the high-energy growth is g-independent.

6. Tetrahexahedric model: intertwiner & integrability

Let us finally hunt for conserved charges and intertwinders in the angular A_3 model. With the angular Dunkl operators

$$\begin{aligned} \mathscr{L}_x &= L_x + g\big\{\tfrac{z}{x-y}s_{x-y} - \tfrac{z}{x+y}s_{x+y} - \tfrac{y}{x-z}s_{z-x} + \tfrac{y}{z+x}s_{z+x} - \tfrac{y+z}{y-z}s_{y-z} + \tfrac{y-z}{y+z}s_{y+z}\big\}\ , \\ \mathscr{L}_y &= L_y + g\big\{\tfrac{x}{y-z}s_{y-z} - \tfrac{x}{y+z}s_{y+z} - \tfrac{z}{y-x}s_{x-y} + \tfrac{z}{y+x}s_{x+y} - \tfrac{z+x}{z-x}s_{z-x} + \tfrac{z-x}{z+x}s_{z+x}\big\}\ , \\ \mathscr{L}_z &= L_z + g\big\{\tfrac{y}{z-x}s_{z-x} - \tfrac{y}{z+x}s_{z+x} - \tfrac{x}{z-y}s_{y-z} + \tfrac{x}{z+y}s_{y+z} - \tfrac{x+y}{x-y}s_{x-y} + \tfrac{x-y}{x+y}s_{x+y}\big\} \end{aligned} \tag{6.1}$$

we find the obviously conserved charges

$$J_k := \operatorname{res}\big(\mathscr{L}_x^k + \mathscr{L}_y^k + \mathscr{L}_z^k\big) \qquad \text{for} \quad k \in 2\mathbb{N}\ . \tag{6.2}$$

In particular,

$$J_0 = C_0 = 1 \qquad \text{and} \qquad J_2 = -C_2 = -2H_\Omega + 6g(6g+1)\ . \tag{6.3}$$

Only J_2, J_4 and J_6 are algebraically independent. Thus, $J_{k>6}$ may be written in terms of $J_{k\le 6}$, e.g.

$$\begin{aligned} 6J_8 = {}& 8J_6J_2 + 3J_4J_4 - 6J_4J_2J_2 + J_2J_2J_2J_2 \\ & - 12(8+5g+12g^2)J_6 + 4(34+23g+30g^2)J_4J_2 - 8(5+3g+3g^2)J_2J_2J_2 \\ & + 24(13+15g-102g^2-72g^3)J_4 - 4(43+70g-252g^2-144g^3)J_2J_2 \\ & - 48(1+3g)(1+4g)(1-12g)J_2\ . \end{aligned} \tag{6.4}$$

The generators $\{J_2, J_4, J_6\}$ form a free algebra modulo

$$[J_2, J_k] = 0 \qquad \Longleftrightarrow \qquad \text{Center} = \langle\langle J_0, J_2 \rangle\rangle\ . \tag{6.5}$$

There is no relation between J_4 and J_6, and any word in the three generators is conserved.

A first angular intertwiner derives from a cubic Weyl antiinvariant polynomial,

$$\mathscr{M}_3 \sim \tfrac{1}{6}\big(\mathscr{L}_x\mathscr{L}_y\mathscr{L}_z + \mathscr{L}_x\mathscr{L}_z\mathscr{L}_y + \mathscr{L}_y\mathscr{L}_z\mathscr{L}_x + \mathscr{L}_y\mathscr{L}_x\mathscr{L}_z + \mathscr{L}_z\mathscr{L}_x\mathscr{L}_y + \mathscr{L}_z\mathscr{L}_y\mathscr{L}_x\big)\ , \tag{6.6}$$

$$\begin{aligned} M_3 \sim {}& y^2z\partial_{zxx} - yz^2\partial_{xxy} + \tfrac{1}{2}(y^2-z^2)\,\partial_{xx} + 4g\,\tfrac{yz}{y^2-z^2}\big(yz\partial_{xx} + x^2\partial_{yz} - zx\partial_{xy}\big) \\ & + g\Big[2g\,y^2z^2\big(\tfrac{8g}{(x^2-y^2)(z^2-x^2)} + \tfrac{16g}{(z^2-x^2)(y^2-z^2)} - \tfrac{2g-1}{(x^2-y^2)^2} + \tfrac{2g-1}{(z^2-x^2)^2}\big) \\ & \qquad - \tfrac{2x^2y^2}{(z^2-x^2)^2} + \tfrac{2x^2z^2}{(x^2-y^2)^2} - \tfrac{2y^2}{x^2-y^2} - \tfrac{2z^2}{z^2-x^2} - 2\tfrac{y^2+z^2}{y^2-z^2}\Big]\,x\partial_x \\ & + 2g(g-1)(g+2)\,x^2\Big[\tfrac{y^2+z^2}{(y^2-z^2)^2} + z\big(\tfrac{1}{(y-z)^3} - \tfrac{1}{(y+z)^3}\big)\Big] + g\,\big(2g^2+8g-1\big)\tfrac{y^2+z^2}{y^2-z^2} \\ & + 2g^2(8+9g)\tfrac{x^2y^2z^2}{(x^2-y^2)(x^2-z^2)(y^2-z^2)} - \tfrac{2}{3}g^3\,\tfrac{x^6+y^6+z^6}{(x^2-y^2)(x^2-z^2)(y^2-z^2)} + \text{ cyclic permutations} \end{aligned} \tag{6.7}$$

or

$$\begin{aligned} \Delta^{-g}M_3\,\Delta^g \sim {}& y^2z\partial_{zxx} - yz^2\partial_{xxy} + \tfrac{1}{2}(y^2-z^2)\,\partial_{xx} + 2g\,\tfrac{y^2z^2(y^2-z^2)}{(x^2-y^2)(x^2-z^2)}\,\partial_{xx} \\ & + 4g\,\tfrac{xy^2z}{x^2-z^2}\,\partial_{xz} + 2g\,x\Big[\tfrac{y^2(x^2+3z^2)}{(x^2-z^2)^2} - \tfrac{z^2(x^2+3y^2)}{(x^2-y^2)^2}\Big]\,\partial_x + \text{ cyclic permutations}\ . \end{aligned} \tag{6.8}$$

There exists a second, sextic Weyl antiinvariant polynomial,

$$\mathscr{M}_6 \sim \{\mathscr{L}_x^4,\mathscr{L}_y^2\} - \{\mathscr{L}_y^4,\mathscr{L}_x^2\} + \{\mathscr{L}_y^4,\mathscr{L}_z^2\} - \{\mathscr{L}_z^4,\mathscr{L}_y^2\} + \{\mathscr{L}_z^4,\mathscr{L}_x^2\} - \{\mathscr{L}_x^4,\mathscr{L}_z^2\}\ , \tag{6.9}$$

which is independent of $\mathscr{M}_1$ and thus gives rise to another intertwiner, $M_6 = \mathrm{res}(\mathscr{M}_6)$, whose lengthy expression we do not display here. Its potential-free variant $\Delta^{-g}M_6\Delta^g$ should take a somewhat simpler form, however. Higher angular intertwiners are reduced to M_3 and M_6.

The intertwining relations between the M_s and the J_k involve some mixing, except for the Hamiltonian H_Ω. For M_3 the relations read

$$\begin{aligned}
M_3^{(g)}J_2^{(g)} &= \big(J_2^{(g+1)} - 6(7{+}12g)\big)\,M_3^{(g)}\ ,\\
M_3^{(g)}J_4^{(g)} &= \big(J_4^{(g+1)} - 4(11{+}12g)J_2^{(g+1)} + 48(26{+}73g{+}48g^2)\big)\,M_3^{(g)}\\
&\quad +\ 2M_6^{(g)}\ ,\\
M_3^{(g)}J_6^{(g)} &= \big(J_6^{(g+1)} - (35{+}36g)J_4^{(g+1)} - 3(7{+}4g)J_2^{(g+1)}J_2^{(g+1)}\\
&\quad +\ 2(1111{+}2668g{+}1392g^2)J_2^{(g+1)}\\
&\quad +\ 96(457{+}1933g{+}2717g^2{+}1368g^3{+}144g^4)\big)M_3^{(g)}\\
&\quad +\ \big(3J_2^{(g+1)} - (115{+}200g{+}48g^2)\big)M_6^{(g)}\ ,
\end{aligned} \tag{6.10}$$

By iterating the interwiners we may create a ladder

$$1{-}g\ \to\ 2{-}g\ \to\ \ldots\ \to\ g{-}2\ \to\ g{-}1\ \to\ g\ , \tag{6.11}$$

which closes to a loop due to the identification of state spaces at $1{-}g$ and g. However, this becomes relevant only after applying the nonlinear $\mathscr{P}\mathscr{T}$ deformation (5.19), which brings the $g{<}0$ states to life. In this situation, the composite intertwiners ($* = 3$ or 6)

$$Q^{(g)}\ =\ M_*^{(g-1)}M_*^{(g-2)}\cdots M_*^{(1-g)} \tag{6.12}$$

have a well-defined action which commutes with the Hamiltonian, which turns them into additional ‘odd’ conserved charges. Since some $Q^{(g)}$ are odd-order differential operators, they are functionally independent of the J_k, but their squares can be expressed through them. Hence, $\{Q,J_2,J_4,J_6\}$ form a $\mathbb{Z}_2$ graded nonlinear supersymmetry-type algebra.

7. Summary and outlook

Let us summarize the key points of this talk and mention some future prospects.

- Geometric picture of potential on S^{n-2}, superintegrable but not separable
- Characterization of the full set of conserved charges: Weyl invariants from $\mathscr{L}_{ij}$
- Characterization of the algebra generated by the conserved charges
- Are there more than two charges in involution? (need $n>4$ to test)
- Characterization of the independent intertwiners: Weyl antiinvariants from $\mathscr{L}_{ij}$
- Intertwining relations of the conserved charges
- $\mathscr{P}\mathscr{T}$ deformation: regularized potential, $g<0$ states, degeneracy doubling
- Additional ‘odd’ conserved charges for integer coupling
- Generalization to trigonometric, hyperbolic, elliptic Calogero systems?

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