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SUPERSYMMETRIC STRING VACUA WITH TORSION AND GEOMETRIC FLOWS

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In memoriam of Ioannis Bakas

In 1986, a system of equations for compactifications of the heterotic string which preserve supersymmetry was proposed independently by C. Hull and A. Strominger. They are more complicated than the Calabi-Yau compactifications proposed earlier by P. Candelas, G. Horowitz, A. Strominger, and E. Witten, because they allow non-vanishing torsion and they incorporate terms which are quadratic in the curvature tensor. As such they are also particularly interesting from the point of view of both non-Kaehler geometry and the theory of non-linear partial differential equations. While the complete solution of such partial differential equations seems out of reach at the present time, we describe progress in developing a new general approach based on geometric flows which shares some features with the Ricci flow. In particular, this approach can recover the non-perturbative solutions found in 2006 by J.X. Fu and S.T. Yau.

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1. Supersymmetric vacua of the heterotic string

In 1986, the following system of equations was proposed independently by C. Hull [22, 23] and A. Strominger [36] for compactifications of the heterotic string which preserve supersymmetry. Let Y be a 3-dimensional compact complex manifold, equipped with a nowhere vanishing holomorphic 3-form Ω, and let $E \to Y$ be a holomorphic vector bundle over Y. We look then for a Hermitian metric $g_{\bar{k}j}$ on Y (identified with the corresponding symplectic form $\omega = ig_{\bar{k}j}dz^j \wedge d\bar{z}^k$) , and for a Hermitian metric $H_{\bar{\alpha}\beta}$ on E, satisfying the following system

$$F^{2,0} = F^{0,2} = 0, \qquad \omega^2 \wedge F^{1,1} = 0 \tag{1}$$

$$i\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\mathrm{Tr}(Rm \wedge Rm) - \mathrm{Tr}(F \wedge F)) = 0 \tag{2}$$

$$d^\dagger \omega = i(\bar{\partial} - \partial)\log \|\Omega\|_\omega \tag{3}$$

Here α' is the slope parameter. The expressions Rm and F are the curvatures of the metrics ω and $H_{\bar{\alpha}\beta}$, viewed as a $(1,1)$-forms valued in $End(T^{1,0}(Y))$ and in $End(E)$ respectively. The expressions $F^{p,q}$ denote the (p,q)-components of the curvature form F. The norm $\|\Omega\|_\omega$ is defined by

$$\|\Omega\|^2_\omega = i\Omega \wedge \overline{\Omega}\,\omega^{-3}.$$

This system is an extension of a well-known set of conditions for compactifications of the heterotic string with unbroken supersymmetry proposed earlier by P. Candelas, G. Horowitz, A. Strominger, and E. Witten [4]. The first equation is just the usual Hermitian-Einstein equation, which ensures the invariance of the gluino under supersymmetry. If we identify the de Kalb-Ramond field strength with the 3-form $T + \overline{T}$, where $T = i\partial\omega$ is the torsion of the Hermitian metric ω, then the second equation can be recognized as the seminal anomaly cancellation mechanism due to M. Green and J. Schwarz (1984). The distinctive feature of the system proposed by Hull and Strominger is the third equation, which is actually a torsion constraint less restrictive than the Kähler condition $T = 0$. In components, if we express the torsion as $T = \frac{1}{2}T_{\bar{k}jm}dz^m \wedge dz^j \wedge d\bar{z}^k$, it can be written more explicitly as

$$g^{j\bar{k}}T_{\bar{k}jm} = \partial_m \log \|\Omega\|_\omega.$$

The Calabi-Yau compactifications found by Candelas, Horowitz, Strominger, and Witten [4] can be recovered from the above system in the following manner. We take (Y,ω) to be Kähler, and set $E = T^{1,0}(Y)$, $H_{\bar{\alpha}\beta} = \omega$. Then $Rm = F$ and the second equation is trivially satisfied. Next $\omega^2 \wedge Rm = 3\,Ric(\omega)$ (viewed as an endomorphism of $T^{1,0}(Y)$), and thus the first equation reduces to the condition of vanishing Ricci curvature

$$Ric(\omega) = 0.$$

As conjectured by Calabi, and proved by Yau [39], manifolds admitting such metrics are exactly the ones with vanishing first Chern class $c_1(Y) = 0$. Taking Ω a non-trivial holomorphic, covariantly constant 3-form, the third equation follows from the Kähler condition, and we obtain a solution of the Hull-Strominger system.

Because ω is not necessarily Kähler, there are many natural unitary connections which preserve the complex structure. As shown by C. Hull [24], the anomaly cancellation mechanism does not require a specific unitary connection for ω. In this work, we restrict ourselves to the choice of the Chern unitary connection, characterized by

$$\nabla_{\bar{j}} V^k = \partial_{\bar{j}} V^k, \qquad \nabla_j V^k = g^{k\bar{p}} \partial_j (g_{\bar{p}m} V^m).$$

In this case, the Riemann curvature tensor is given by,

$$Rm = R_{\bar{k}j}{}^p{}_q dz^j \wedge d\bar{z}^k, \qquad R_{\bar{k}j}{}^p{}_q = -\partial_{\bar{k}} (g^{p\bar{m}} \partial_j g_{\bar{m}q}),$$

with a similar expression for the curvature F of $H_{\bar{\alpha}\beta}$, $F = F_{\bar{k}j}{}^\alpha{}_\beta dz^j \wedge d\bar{z}^k$, $F_{\bar{k}j}{}^\alpha{}_\beta = -\partial_{\bar{k}} (H^{\alpha\bar{\gamma}} \partial_j H_{\bar{\gamma}\beta})$.

2. Non-Kähler geometry and non-linear partial differential equations

While the system (1-3) originally arose from string theory, it is potentially of considerable interest in mathematics as well for several reasons.

First, it can be interpreted as providing a notion of canonical metric in a particular non-Kähler setting. In Kähler geometry, a canonical metric is usually defined by a cohomological condition (e.g. $d\omega = 0$), and by a curvature condition (e.g. ω has constant scalar curvature, see e.g. [34] for a survey). As pointed out by J. Li and S.T. Yau [25], the third equation (3) in the Hull-Strominger system is equivalent to the following "conformally balanced" condition

$$d(\|\Omega\|_\omega \omega^2) = 0. \tag{4}$$

The notion of balanced metric, i.e. $d(\omega^2) = 0$, was introduced in mathematics by Michelsohn [26] (1981). It is a natural notion, as the existence of a balanced metric is a property invariant under modifications (see Alessandrini-Bassanelli [1]). The first two equations in the Hull-Strominger system can then be viewed as the analogue of the curvature condition in the setting of conformally balanced metrics.

Second, the expression $\mathrm{Tr}(Rm \wedge Rm)$, which appears in the equation (2) and is fundamental to the Green-Schwarz anomaly cancellation in string theory, does not seem to have been studied before as a curvature condition in complex differential geometry. What sets it apart from much studied conditions such as constant scalar or constant Ricci curvature is that it is quadratic in the curvature tensor. In particular, it leads to a class of fully non-linear equations which is new in the theory of partial differential equations, and whose geometric meaning is yet to be explored. In this context, it is natural to consider generalizations of these equations to dimensions different from 3. This has already led to some remarkable equations of complex Hessian type [17, 18, 28, 30, 31].

Finally, we shall see shortly that there are compelling reasons for studying the Hull-Strominger system as the fixed point of a geometric flow. Remarkably, this flow will turn out to have some strong resemblance with the Ricci flow [21] (or renormalization group flow for sigma models [15]), although it will of course be more complicated. As such, it should provide a good laboratory for the development of new techniques for the study of geometric flows.

2.1 Some special solutions of the Hull-Strominger system

By now many special solutions have been found in the physics literature (see e.g. Strominger [36], Dasgupta, Rajesh, and Sethi [7], Becker, Becker, Fu, Tseng, and Yau [3], Carlevaro and Israel [5], Andreas and Garcia-Fernandez [2], and others).

Other special solutions have been found using some specific geometric constructions. They include invariant solutions on Lie groups and their quotients (see e.g. Grantcharov [20], Fernandez, Ivanov, Ugarte and Villacampa [14], Otal, Ugarte and Villacampa [27], Fei and Yau [12], and references therein) using connections which are not always Chern connections. They also include local models, such as torus bundles over an ALE space (Fu, Tseng, and Yau [16]), torus bundles over conformally $\mathbb{T}^4$ manifolds (Fernandez, Ivanov, Ugarte, and Vassilev [13]) and a local model based on the twistor space of a hyperkähler manifold (Fei [10]). Recently, compact non-Kähler solutions were constructed by Fei, Huang, and Picard [11] on hyperkähler fibrations over a Riemann surface, building on previous work by Fei [9].

But the first non-perturbative, non-Kähler solution was found by Fu and Yau [17] on certain toric fibrations $\pi : Y \to X$ over K3 surfaces constructed by Goldstein and Prokushkin [19], building on earlier ideas of Calabi and Eckmann [6]. We shall say more about this geometric set-up later, but for the moment, we just discuss the analytic features of the Fu-Yau solution. It turns out that, in this case, the Hull-Strominger system can be reduced to a single non-linear PDE of complex Monge-Ampère type on the two-dimensional base X,

$$i\partial\bar\partial(e^u\omega - \alpha' e^{-u}\rho) + \frac{\alpha'}{2} i\partial\bar\partial u \wedge i\partial\bar\partial u + \mu = 0. \tag{5}$$

Here ρ and μ are given smooth $(1,1)$ and $(2,2)$ forms respectively, with μ satisfying the integrability condition

$$\int_X \mu = 0.$$

The existence of solutions to equations of this type was shown by Fu and Yau [17, 18] using the method of continuity in two separate papers, for $\alpha' > 0$ and $\alpha' < 0$ respectively. While the geometric set-up is the same in both cases, the equations are analytically quite different, and the key a priori estimates for their solutions are also quite different.

3. The Anomaly flow

We begin by discussing some of the key difficulties which have to be addressed when trying to solve the Hull-Strominger system. For given ω, the first equation (1) is the equation for F to be the curvature of an integrable Hermitian-Einstein connection. For given ω, the classical theorem of Donaldson-Uhlenbeck-Yau [8, 38] gives a necessary and sufficient condition for the existence of Hermitian-Einstein connections in terms of the Mumford stability of the bundle $E \to (Y,\omega)$. While the Hull-Strominger system is a system for the pair $(\omega, H_{\bar\alpha\beta})$, at this preliminary stage of our considerations, it is then not unreasonable to assume that $H_{\bar\alpha\beta}$ is known and to focus on the equations (2-3) for ω (this is for example what will happen in the case of the Fu-Yau solution to be discussed later).

If we take the case of canonical metrics in Kähler geometry as a guideline, we run immediately into a new difficulty: a Kähler metric can be characterized by a potential which is unique up to a harmless constant, while there is no such known characterization for balanced or conformally balanced metrics. Various ansatze for balanced metrics have been constructed by many authors, e.g. Tosatti-Weinkove [37], Popovici [35], Fei [9], et al. For example, if ω_0 is balanced ($d\omega_0^2 = 0$), then any metric of the form

$$\omega^2 = \omega_0^2 + i\partial\bar\partial(u\tilde\omega)$$

is balanced (for any scalar function u and $\tilde\omega$ any $(1,1)$-form which keep ω^2 positive). The drawback is that no particular ansatz seems more compelling than the others, and the resulting equations all seem very complicated and unnatural.

In this talk, we describe a series of papers [29],[31],[32] whose goal is to bypass this problem of any particular Ansatz for balanced or conformally balanced metrics by viewing the solutions of the Hull-Strominger systems as the stationary points of the following flow of metrics $(\omega, H_{\bar\alpha\beta})$, where the balanced or conformally balanced condition is automatically preserved,

$$\partial_t(||\Omega||_\omega \omega^2) = i\partial\bar\partial\omega - \frac{\alpha'}{4}(\mathrm{Tr}(Rm\wedge Rm) - \mathrm{Tr}(F\wedge F))$$

$$H^{-1}\partial_t H = -3\frac{\omega^2\wedge F}{\omega^3} \tag{6}$$

with $\omega = \omega_0$ when $t = 0$, where ω_0 is a balanced metric. For fixed ω, the flow of the metric H in the second line above is just the Donaldson heat flow [8].

We can also consider the flow of ω alone, as given by the first line in (6), for a given $(2,2)$-form $\mathrm{Tr}(F\wedge F)$. We call all these flows “Anomaly flows", in reference to the Green-Schwarz anomaly cancellation mechanism. To lighten the discussion, we don't indicate which Anomaly flow we discuss in each instance, as it should be clear from the context, and also how to adapt the discussion from one flow to the other.

Theorem 1 [29] *The above flow of positive $(2,2)$-forms defines a vector field on the space of positive $(1,1)$-forms.*

(a) *The corresponding flow preserves the balanced property of the metric $\omega(t)$.*

(b) *Clearly its stationary points are solutions of the Hull-Strominger system.*

(c) *The flow exists at least for a short time, assuming that $|\alpha' Rm(\omega)|$ is small enough.*

The proof of the first statement in the theorem makes essential use of an early work of Michelsohn [26], who showed that, given a positive $(n-1,n-1)$-form Ψ, there is a unique positive $(1,1)$-form ω so that $\omega^{n-1} = \Psi$. It turns out that ω can be expressed algebraically in Ψ. In fact, $(n-1)! \star\omega = \Psi$, if $\star$ is the Hodge operator defined by ω itself. The statements (a-b) are obvious. The only non-trivial statement left is (c), which is proved by establishing the weak parabolicity of the flow and applying the Nash implicit function theorem.

Even though Michelsohn's theorem can be used to show that the Anomaly flow does define a smooth vector field on the space of metrics, it does not give a practical formula for this vector field. In particular, it is hard to deduce from it the flows of the curvature and torsion tensors. This difficulty was recently overcome in [31]:

Theorem 2 [31] *Consider the Anomaly flow with a conformally balanced initial metric. Then the flow is given by*

$$\partial_t g_{\bar{k}j} = \frac{1}{2\|\Omega\|_\omega}\Big\{ -\tilde{R}_{\bar{k}j} + g^{s\bar{r}} g^{p\bar{q}} T_{\bar{q}sj} \bar{T}_{p\bar{r}\bar{k}} - \frac{\alpha'}{4} g^{s\bar{r}} (R_{[\bar{k}s}{}^{\alpha}{}_{\beta} R_{\bar{r}j]}{}^{\beta}{}_{\alpha} - \Phi_{\bar{k}s\bar{r}j}) \Big\}$$

Here $\tilde{R}_{\bar{k}j} = g^{p\bar{q}} R_{\bar{q}p\bar{k}j}$ *is the Ricci tensor for general Hermitian metrics,* $i\partial\omega = \frac{1}{2} T_{\bar{k}jm} dz^m \wedge dz^j \wedge d\bar{z}^k$ *is the torsion tensor, and we have set* $\Phi = \mathrm{Tr}(F \wedge F)$. *The bracket* $[,]$ *denote anti-symmetrization in each of the two sets of barred and unbarred indices.*

Once a description of the flow as in Theorem 2 is available, it is easy to derive the flows of the curvature tensor and of the torsion. The complete formulas are provided in [31]. Here for illustrative purposes, we quote only the leading terms. We find for the full curvature tensor

$$\partial_t R_{\bar{k}j}{}^{\rho}{}_{\lambda} = \frac{1}{2\|\Omega\|_\omega} (\Delta R_{\bar{k}j}{}^{\rho}{}_{\lambda} + \frac{\alpha'}{2} g^{\rho\bar{\mu}} g^{s\bar{r}} R_{[\bar{r}\lambda}{}^{\beta}{}_{\alpha} \nabla_s \nabla_{\bar{\mu}]} R_{\bar{k}j}{}^{\alpha}{}_{\beta}) + \cdots$$

for the Ricci curvature,

$$\partial_t R_{\bar{k}j} = \frac{1}{2\|\Omega\|_\omega} (\Delta R_{\bar{k}j} + \frac{\alpha'}{2} g^{\lambda\bar{\mu}} g^{s\bar{r}} R_{[\bar{r}\lambda}{}^{\beta}{}_{\alpha} \nabla_s \nabla_{\bar{\mu}]} R_{\bar{k}j}{}^{\alpha}{}_{\beta}) + \cdots$$

for the scalar curvature $R = g^{j\bar{k}} R_{\bar{k}j} = g^{j\bar{k}} \tilde{R}_{\bar{k}j}$,

$$\partial_t R = \frac{1}{2\|\Omega\|_\omega} (\Delta R + \frac{\alpha'}{2} g^{\lambda\bar{\mu}} g^{s\bar{r}} R_{[\bar{r}\lambda}{}^{\beta}{}_{\alpha} \nabla_s \nabla_{\bar{\mu}]} R^{\alpha}{}_{\beta}) + \cdots$$

and for the torsion $T_{\bar{p}jq}$,

$$\partial_t T_{\bar{p}jq} = \frac{1}{2\|\Omega\|_\omega} \big[\Delta T_{\bar{p}jq} - \frac{\alpha'}{4} g^{s\bar{r}} (\nabla_j (R_{[\bar{p}s}{}^{\alpha}{}_{\beta} R_{\bar{r}q]}{}^{\beta}{}_{\alpha}) - \nabla_q (R_{[\bar{p}s}{}^{\alpha}{}_{\beta} R_{\bar{r}j]}{}^{\beta}{}_{\alpha}))\big] + \cdots$$

Here $\Delta = g^{j\bar{k}} \nabla_j \nabla_{\bar{k}}$ is the Laplacian. By definition, the diffusion operator is the leading linearized differential operator on the right hand side. Thus, for the Riemann curvature tensor, it is given by

$$\delta R_{\bar{k}j}{}^{\rho}{}_{\lambda} \to \frac{1}{2\|\Omega\|_\omega} \big(\Delta(\delta R_{\bar{k}j}{}^{\rho}{}_{\lambda}) + \frac{\alpha'}{2} g^{\rho\bar{\mu}} g^{s\bar{r}} R_{[\bar{r}\lambda}{}^{\beta}{}_{\alpha} \nabla_s \nabla_{\bar{\mu}]} \delta R_{\bar{k}j}{}^{\alpha}{}_{\beta}\big)$$

with similar expressions for the diffusion operators for the Ricci tensor, as well as the scalar curvature. Naively, the diffusion operator for the torsion tensor is $\frac{1}{2\|\Omega\|_\omega}\Delta$, but in this case, there are additional terms in the curvature which are of the same order.

It is instructive to compare the Anomaly flow with the well-known Ricci flow

$$\partial_t g_{\bar{k}j} = -R_{\bar{k}j}$$

on Kähler metrics. In the case of the Ricci flow, the curvatures evolve as follows [21]

$$\partial_t R = \Delta R + R_{\bar{k}j} R^{j\bar{k}}, \qquad \partial_t R_{\bar{k}j} = \Delta R_{\bar{k}j} + R_{\bar{k}m\bar{p}q} R^{q\bar{p}m}{}_j$$

Thus the diffusion operator for the curvatures in the Ricci flow is Δ. The relation between the diffusion operators for the Ricci flow and for the Anomaly flow can clearly be traced to the similarity in the flows, as shown by Theorem 2. However, there is no clear analogy for the diffusion of the torsion terms, since the torsion is identically 0 in the case of the Ricci flow. Also, the Anomaly flow is clearly more complicated due to the factors of $(2\|\Omega\|_\omega)^{-1}$, and especially the terms which are quadratic in the curvature tensor.

4. Ellipticity vs Parabolicity in the theory of Partial Differential Equations

The Anomaly flow provides one particular parabolic approach to finding solutions of the Hull-Strominger system. It may be appropriate to pause here to discuss briefly the issue of selecting a parabolic approach to a particular elliptic partial differential equation.

For a given elliptic equation, say $F(D^2u) = e^\psi$, there are an infinite number of possible parabolic equations with the same equation as stationary point, for example

$$\partial_t u = F(D^2u) - e^\psi \qquad \text{or} \qquad \partial_t u = \log F(D^2u) - \psi.$$

However, they can behave quite differently. A well-known example is the Monge-Ampère equation, with $F(D^2u) = \det D^2u$, where the parabolic equation with $\log F(D^2u)$ is much better behaved, because of the concavity of the function $\log F(D^2u)$ in the second derivatives of u.

In the present case of Hull-Strominger systems, our choice of parabolic equation is dictated by the need to preserve the conformally balanced condition. There is no further flexibility, and thus it is a particularly important issue to determine whether the parabolic flow which is the Anomaly flow is well-behaved.

Since for a given manifold Y with $c_1(Y) = 0$ and a holomorphic vector bundle $E \to Y$, there may not be any solution to the Hull-Strominger system, no parabolic flow with the Hull-Strominger system as stationary point can always converge. So what can be good criteria for the well-behavior of a given parabolic flow ? Certainly weak-parabolicity, which ensures at least the existence of the flow for a short time, is a minimum requirement. Beyond that, we can hope for the long-time existence of the flow, or cogent geometric conditions for when singularities may appear and when the flow may fail to converge. The general difficulty is that, if the flow exists on an interval $[0,T)$, the metric $g_{\bar{k}j}(t)$ may become either degenerate or infinite, as $t \to T$. Similarly, the curvature Rm or the torsion T may blow up as $t \to T$. This would prevent the continuation of the flow beyond time T, let alone convergence. Whether this happens or not will require suitable a priori estimates. Another very important criterion for the well-behavior of a flow is that it should converge whenever there is a stationary point, at least for a large basin of initial data. A good example is provided by the Kähler-Ricci flow on manifolds Y with $c_1(Y) > 0$: while stationary points, which are Kähler-Einstein metrics in this case, do not always exist, the flow will always converge when they do.

The answer to all these questions for the Anomaly flow in full generality appears out of reach at the present time. But we shall see that the Anomaly flow passes all the tests in the following two important special cases, which will be discussed in detail in the next two sections:

• The case $\alpha'=0$: The most difficult quadratic terms in the curvature tensor won't occur. But the flow still presents new difficulties due to the factor $\|\Omega\|_\omega$ and the non-vanishing torsion, and it appears still at least as complicated as the Ricci flow.

• The case of Calabi-Eckmann-Goldstein-Prokushkin fibrations: this is the case where the elliptic equation was solved by Fu and Yau. So it is important to find out whether the Anomaly flow can at least recapture this case. We shall see that it can, and even though it requires a different set of techniques, it will prove to be even more powerful than the Monge-Ampère techniques used by Fu and Yau.

5. The case of $\alpha'=0$

In this case, the Anomaly flow reduces to the following flow,

$$\partial_t(\|\Omega\|_\omega \omega^2) = i\partial\bar\partial\omega,$$

or equivalently, in view of Theorem 2,

$$\partial_t g_{\bar k j} = \frac{1}{2\|\Omega\|_\omega}(-\tilde R_{\bar k j} + g^{s\bar r} g^{q\bar p} T_{\bar p s j}\overline{T_{q\bar r k}}).$$

Even though the terms quadratic in the curvature are absent in this case, and the stationary point is only a truncation of the Green-Schwarz anomaly cancellation mechanism, the flow is still quite interesting from the geometric view point. Its stationary points would satisfy $i\partial\bar\partial\omega=0$. When combined with the conformally balanced condition, this would imply that ω is Kähler, so the flow would provide a way of answering the basic and long-standing question of when a balanced or conformally balanced manifold is actually Kähler.

In this case, we can establish an essential property of well-behaved flows, which is that it suffices to control a *finite* number of geometric quantities in order to control all derivatives of the metric:

Theorem 3 [31] *Assume that the flow exists for $t\in[0,\frac{1}{A}]$ and that*

$$|Rm|_\omega + |DT|_\omega + |T|^2_\omega \le A, \ \ z\in X.$$

Then for any $k\in\mathbf{N}$, there exists a constant C_k depending on a uniform lower bound for $\|\Omega\|_\omega$ so that

$$|D^k Rm|_\omega \le C_k A t^{-\frac{k}{2}}, \qquad |D^{k+1}T|_\omega \le C_k A t^{-\frac{k}{2}}.$$

This leads immediately to the following simple criteria for the long-time existence of the flow:

Theorem 4 [31] *The flow exists for all time $t\ge 0$, unless there is a finite time T and a sequence (z_j,t_j) with $t_j\to T$, and either $\|\Omega(z_j,t_j)\|_{\omega_j}\to 0$, or*

$$(|Rm|_\omega + |DT|_\omega + |T|^2_\omega)(z_j,t_j)\to\infty.$$

6. The Anomaly flow and the Fu-Yau equation

Next, we consider the case of the Calabi-Eckmann-Goldstein-Prokushkin fibrations, which is the case solved by Fu and Yau [17] using elliptic methods.

We begin by recalling the geometric set-up for these fibrations. Let $(X,\hat{\omega})$ be a Calabi-Yau surface, with Ricci-flat metric $\hat{\omega}$, and holomorphic form Ω normalized so that $\|\Omega\|^2_{\hat{\omega}} = 1$. Given any two forms $\omega_1, \omega_2 \in 2\pi H^2(X,\mathbf{Z})$ with $\omega_1 \wedge \hat{\omega} = \omega_2 \wedge \hat{\omega} = 0$, building on earlier ideas of Calabi and Eckmann [6], Goldstein and Prokushkin [19] construct a toric fibration $\pi : Y \to X$, equipped with a $(1,0)$-form θ on Y satisfying $\partial\theta = 0$, $\bar{\partial}\theta = \pi^*(\omega_1 + i\omega_2)$. Furthermore, the form

$$\Omega_Y = \sqrt{3}\,\Omega \wedge \theta$$

is a holomorphic nowhere vanishing $(3,0)$-form on Y, and for any scalar function u on X, the $(1,1)$-form

$$\omega_u = \pi^*(e^u\hat{\omega}) + i\theta \wedge \bar{\theta} \tag{7}$$

is a conformally balanced metric on Y.

Next, look for a solution of the Hull-Strominger system on $Y, \pi^*(E)$ under the form $(\omega_u, \pi^*(H))$, where H is a Hermitian-Einstein metric on a stable vector bundle $E \to (X,\hat{\omega})$. Then the only equation left to solve is the Green-Schwarz anomaly equation (3),

$$i\partial\bar{\partial}\omega_u - \frac{\alpha'}{4}\mathrm{Tr}(Rm(\omega_u) \wedge Rm(\omega_u) - F \wedge F) = 0.$$

In a key calculation, Fu and Yau [17] showed that this equation descends to an equation for the scalar function u on X,

$$i\partial\bar{\partial}(e^u\hat{\omega} - \alpha' e^{-u}\rho) + \frac{\alpha'}{2} i\partial\bar{\partial}u \wedge i\partial\bar{\partial}u + \mu = 0$$

where ρ and μ are given $(1,1)$ and $(2,2)$-forms. They then showed that the existence of solutions to this equation is equivalent to the integrability condition $\int_X \mu = 0$.

In our case, we consider the Anomaly flow on a Calabi-Eckmann-Goldstein-Prokushkin fibration, with an initial data ω_0 of the form (7). Then we have

Theorem 5 [33] *Consider the Anomaly flow*

$$\partial_t(\|\Omega\|_\chi \chi^2) = i\partial\bar{\partial}\chi - \frac{\alpha'}{4}\mathrm{Tr}(Rm(\chi) \wedge Rm(\chi) - F \wedge F)$$

on a Calabi-Eckmann-Goldstein-Prokushkin fibration $\pi : Y \to X$, with initial data $\chi(0) = \pi^*(M\hat{\omega}) + i\theta \wedge \bar{\theta}$, where M is a positive constant. Assume the integrability condition on μ (which depends only on the Goldstein-Prokushkin data). Then there exists $M_0 > 0$, so that for all $M \geq M_0$, the flow exists for all time, and converges to a metric ω_∞ with $(\omega_\infty, \pi^*(H))$ satisfying the Hull-Strominger system.

This theorem holds for $\alpha' > 0$ and $\alpha' < 0$. We formulated it in terms of flows on the 3-fold Y. But of course the advantage of Calabi-Eckmann-Goldstein-Prokushkin fibrations is that it descends to a flow on the surface X, and the theorem which is equivalent to Theorem 5 and which we shall actually prove is the following:

Theorem 5' *Let $(X,\hat{\omega})$ be a Calabi-Yau surface, with a Ricci-flat metric $\hat{\omega}$ and a holomorphic $(2,0)$-form Ω normalized to $\|\Omega\|_{\hat{\omega}} = 1$. Consider the flow*

$$\partial_t \omega = -\frac{1}{2\|\Omega\|_\omega}\Big(\frac{R}{2} - |T|^2 - \frac{\alpha'}{4}\sigma_2(iRic_\omega) + 2\alpha'\frac{i\partial\bar{\partial}(\|\Omega\|_\omega \rho)}{\omega^2} - 2\frac{\mu}{\omega^2}\Big)\omega \tag{8}$$

with an initial metric of the form $\omega(0) = M\hat{\omega}$. Here $\sigma_2(iRic_\omega)$ is the second symmetric polynomial in the eigenvalues of $iRic_\omega$. Assume the integrability condition on μ. Then there exists a constant M_0 so that, for all $M \geq M_0$, the flow exists for all time and converges exponentially fast to a metric ω_∞ satisfying the Fu-Yau equation

$$i\partial\bar{\partial}(\omega_\infty - \alpha'\|\Omega\|_{\omega_\infty}\rho) - \frac{\alpha'}{8}Ric_{\omega_\infty}\wedge Ric_{\omega_\infty} + \mu = 0.$$

Because for Calabi-Eckmann-Goldstein-Prokushkin fibrations, the relevant metrics ω_u are determined by a single conformal factor u, the Anomaly flow also provides an interesting example of a flow in conformal geometry. It may be instructive to examine it in this light. For this, we write it as a parabolic equation for the conformal factor u ($\omega = e^u\hat{\omega}$),

$$\partial_t u = \frac{e^{-u}}{2}\Big(\Delta_{\hat{\omega}} e^u - \alpha'\frac{i\partial\bar{\partial}(e^{-u}\rho)}{\det \hat{g}_{\bar{k}j}} + \alpha'\frac{\det u_{\bar{k}j}}{\det \hat{g}_{\bar{k}j}} + \frac{\mu}{\det \hat{g}_{\bar{k}j}}\Big) \tag{9}$$

with stationary points given by the Fu-Yau equation,

$$\Delta_{\hat{\omega}} e^u - \alpha'\frac{i\partial\bar{\partial}(e^{-u}\rho)}{\det \hat{g}_{\bar{k}j}} + \alpha'\frac{\det u_{\bar{k}j}}{\det \hat{g}_{\bar{k}j}} + \frac{\mu}{\det \hat{g}_{\bar{k}j}} = 0.$$

This parabolic version of the Fu-Yau equation does not have any desirable concavity property. Due to that, none of the techniques, except for Moser iteration, used to solve the elliptic equation can be adapted to this parabolic version.

Another big difference between the elliptic and the parabolic versions can be seen from the dependence of their behavior on the slope parameter α'. While in string theory, the parameter α' is > 0, from the point of view of geometry, it makes sense to consider the Hull-Strominger system for both $\alpha' > 0$ and $\alpha' < 0$. Then the behavior of the elliptic version changes drastically with the sign of α'. Indeed, if we rewrite the equation as (setting $\rho = 0$ for notational simplicity)

$$\frac{\det(e^u\hat{\omega} + \alpha' i\partial\bar{\partial}u)}{\det\hat{\omega}} = (e^{2u} - \alpha' e^u|Du|^2) - \alpha'\frac{\mu}{\det\hat{\omega}}$$

and impose the ellipticity condition that $\omega' = e^u\hat{\omega} + \alpha' i\partial\bar{\partial}u > 0$, then for $\alpha' > 0$, the estimate for $\|Du\|_{C^0}$ is easy, but the determinant of ω' may slide to 0. On the other hand, for $\alpha' < 0$, a lower bound for the determinant of ω' is easy, but the estimate for $\|Du\|_{C^0}$ is hard. Thus the two cases $\alpha' > 0$ and $\alpha' < 0$ require different methods in the elliptic case [17, 18] (and also [28, 31] for generalizations to higher dimensions). On the other hand, as we shall see below, the behavior of the Anomaly flow is insensitive to the sign of α'. This indicates a greater robustness and capacity for generalization for this method.

7. Estimates for the Anomaly flow

It turns out that the simplicity that we seem to gain by writing the Anomaly flow as a parabolic equation in a scalar unknown function u is only apparent. Rather, it is important not to lose sight of the geometric significance of the flow, and to work directly with the evolving metric ω, without having to specify the sign of α'. Even though we have to deal then with a system, the geometric insight more than compensates for it.

We make the key assumption that $|\alpha' Ric_\omega| << 1$ and $e^{-u} << 1$, which implies the parabolicity condition (c) of the flow in Theorem 1, so that the diffusion operator

$$\Delta_F = F^{p\bar q}\nabla_p\nabla_{\bar q}, \qquad F^{p\bar q} = g^{p\bar q} + \alpha'\|\Omega\|_\omega^3\tilde\rho^{p\bar q} - \frac{\alpha'}{2}(Rg^{p\bar q} - R^{p\bar q})$$

is elliptic. Of course a crucial and difficult step will be to prove that this condition is preserved along the flow. Here $\tilde\rho$ is defined such that $i\partial\bar\partial f \wedge \rho = \tilde\rho^{j\bar k} f_{\bar k j}\frac{\hat\omega^2}{2!}$ for any function f.

We also exploit some simplifications which occur in the case of Calabi-Eckmann-Goldstein-Prokushkin fibrations, by opposition to the general case. The first is that the full curvature Rm of the metric $e^u\hat\omega$ is determined by its Ricci curvature

$$Rm = -\partial\bar\partial u \otimes I + Rm(\hat\omega), \qquad Ric_\omega = -2\partial\bar\partial u$$

and the second is that the full torsion tensor

$$T = i\partial\omega = i\partial u \wedge \omega$$

is also completely determined by the components

$$T_j = g^{p\bar q}T_{\bar q p j} = -\partial_j u.$$

Upon descending to the base, the metric $e^u\hat\omega$ happens to satisfy the same useful relations between curvature and torsion as the original metric $e^u\hat\omega + i\theta\wedge\bar\theta$,

$$R_{\bar k j} = 2\nabla_{\bar k}T_j, \qquad T_j = \partial_j \log\|\Omega\|_\omega.$$

Third, and most important, the leading diffusion operators for the curvature and the torsion are given by the same operator $\frac{1}{2\|\Omega\|_\omega}\Delta_F$.

7.1 Uniform equivalence of the metrics $\omega(t)$

This is equivalent to a uniform estimate for the conformal factor u in $\omega = e^u\hat\omega$, and is established by Moser iteration, exploiting the fact that the quantity $\int_X \|\Omega\|_\omega \omega^2$ is conserved along the flow:

Proposition 1 Assume that the flow exists for $t \in [0,T)$ and starts with $\omega(0) = M\hat\omega$. Then there exists M_0 so that, for $M \geq M_0$, we have

$$\sup_{X\times[0,T)} e^u \leq C_1 M, \qquad \sup_{X\times[0,T)} e^{-u} \leq \frac{C_2}{M}$$

where C_1, C_2 depend only on $(X,\hat\omega)$, μ, ρ, and α'.

7.2 Estimates for the torsion

These are equivalent to estimates for $\|Du\|_{C^0}$ if we work with the scalar parabolic equation (9). However, they appear inaccessible from (9). Working instead with the geometric formulation (8) of the flow, we can establish the following:

Proposition 2 There exists M_0 with the following property. If the flow is started with $\omega(0) = M\hat{\omega}$ and $M \geq M_0$, and if

$$|\alpha' Ric_\omega| \leq 10^{-6}$$

along the flow, then there exists a constant C_3 depending only on $(X,\hat{\omega})$, μ, ρ, and α' so that

$$|T|^2 \leq \frac{C_3}{M^{4/3}} << 1.$$

7.3 Estimates for the Ricci curvature

These are equivalent to estimates for $\|i\partial\bar{\partial}u\|_{C^0}$ for the scalar equation (9). In the geometric formulation, we have

Proposition 3 Start the flow with $\omega(0) = M\hat{\omega}$. There exists $M_0 >> 1$ such that, for every $M \geq M_0$, if

$$\|\Omega\|^2 \leq \frac{C_2}{M^2}, \qquad |T|^2 \leq \frac{C_3}{M^{4/3}}$$

along the flow, then

$$|\alpha' Ric_\omega| \leq \frac{1}{M^{1/2}}$$

Note that these estimates imply in particular that the ellipticity condition $|\alpha' Ric_\omega| << 1$ is preserved along the flow.

7.4 Estimates for higher order derivatives

The next step is to obtain estimates for all higher order derivatives of the curvature and torsion. From general PDE theory, we need only to obtain them for the first derivatives (since they result in all derivatives up to third order for the metric), and this is done in the following proposition:

Proposition 4 There exists $0 < \delta_1, \delta_2$ with the following property. If

$$-\frac{1}{8}g^{p\bar{q}} < \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} < \frac{1}{8}g^{p\bar{q}}, \qquad \|\Omega\| \leq 1$$

and

$$|\alpha' Ric_\omega| \leq \delta_1, \qquad |T|^2 \leq \delta_2$$

then

$$|\nabla Ric_\omega| \leq C, \qquad |\nabla T| \leq C$$

for a constant C depending only on $\delta_1, \delta_2, \alpha', \rho, \mu$ and $(X,\hat{\omega})$.

It may be worth noting here the use in the maximum principle of a seemingly new type of test function

$$G = (|\alpha' Ric_\omega| + \tau_1)|\nabla Ric_\omega|^2 + (|T|^2 + \tau_2)|\nabla T|^2.$$

7.5 Long-time existence of the flow

Proposition 5 There exists $M_0 >> 1$ such that, for all $M \geq M_0$, if the flow exists on $[0, t_0)$ and initially starts from $\omega_0 = M\hat{\omega}$, then along the flow

$$\frac{1}{C_1 M} \leq e^{-u} \leq \frac{C_2}{M}, \qquad |T|^2 \leq \frac{C_3}{M^{4/3}}, \qquad |\alpha' Ric_\omega| \leq \frac{1}{M^{1/2}}$$

and

$$|D^k u|_{\hat{\omega}} \leq \tilde{C}_k, \qquad \frac{1}{2}\hat{g}^{j\bar{k}} \leq e^u F^{j\bar{k}} \leq 2\hat{g}^{j\bar{k}}.$$

This readily implies

Proposition 6 There exists M_0 so that, for all $M \geq M_0$, if the flow starts from $\omega(0) = M\hat{\omega}$, then it will exist on $[0, \infty)$.

7.6 Convergence of the flow

Even though it does not appear that the Anomaly flow is a gradient flow, once a priori estimates for all the derivatives are available, we can establish the convergence of the flow:

Proposition 7 There exists $M_0 >> 1$ so that for all $M \geq M_0$, if the flow starts initially with $\omega(0) = M\hat{\omega}$, then it exists for all time, and converges in C^∞ to a metric ω_∞ satisfying

$$0 = i\partial\bar{\partial}(\omega_\infty - \alpha' \|\Omega\|_{\omega_\infty}\rho) + \frac{\alpha'}{8} iRic_{\omega_\infty} \wedge iRic_{\omega_\infty} + \mu, \qquad \int_X \|\Omega\|_\infty \omega_\infty^2 = M.$$

For the proof, recall $\omega = e^u\hat{\omega}$ and introduce

$$v = \partial_t e^u, \qquad J(t) = \int_X v^2 \frac{\hat{\omega}^2}{2}$$

Then we can show that

$$\int_X v = 0, \qquad \frac{dJ}{dt} \leq -\eta J$$

for some constant $\eta > 0$. It follows that $J(t) \leq Ce^{-\eta t}$, and $v \to 0$ in L^2. With the a priori estimates for all derivatives available, it is not difficult to deduce that e^u converges in C^∞ to e^{u_∞}.

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