

Rapidity evolution of gluon TMDs from low to moderate x

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I discuss how the rapidity evolution of gluon transverse momentum dependent distribution changes from nonlinear evolution at small $x \ll 1$ to linear evolution at moderate $x \sim 1$.

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A TMD factorization [1, 2, 3] generalizes the usual concept of parton density by allowing PDFs to depend on intrinsic transverse momenta in addition to the usual longitudinal momentum fraction variable. These transverse-momentum dependent parton distributions (also called unintegrated parton distributions) are widely used in the analysis of semi-inclusive processes like semi-inclusive deep inelastic scattering (SIDIS) or dijet production in hadron-hadron collisions (for a review, see Ref. [3]). However, the analysis of TMD evolution in these cases is mostly restricted to the evolution of quark TMDs, whereas at high collider energies the majority of produced particles will be small- x gluons. In this case one has to understand the transition between non-linear dynamics at small x and presumably linear evolution of gluon TMDs at intermediate x .

In this presentation I discuss the connection between rapidity evolution of gluon TMD at low x_B and at moderate $x_B \sim 1$. (The discussion is based on papers [4, 5]). We assume $k_\perp^2 \geq \text{few GeV}^2$ so that we can use perturbative QCD, but otherwise k_\perp is arbitrary and can be of order of s as in the DGLAP evolution. In this kinematic region we will vary Bjorken x_B and look how non-linear evolution at small x transforms into linear evolution at moderate x_B . It should be noted that at least at moderate x_B gluon TMDs mix with the quark ones. Here I disregard this mixing leaving the discussion of full matrix for future publications.

It is convenient to define the field-strength operator with attached light-like Wilson line:

$$\mathcal{F}_i^{a\eta}(x_B, z_\perp) \equiv \frac{2}{s} \int dz_* e^{ix_B z_*} ([\infty, z_*]_z^{am} g F_{\bullet i}^m(z_*, z_\perp))^\eta \quad (1)$$

where the index η denotes the rapidity cutoff (2) for all gluon fields in this operator:

$$A_\mu^\eta(x) = \int \frac{d^4 k}{16\pi^4} \theta(e^\eta - |\alpha|) e^{-ik \cdot x} A_\mu(k) \quad (2)$$

The Sudakov variable α is defined as usual, $k = \alpha p_1 + \beta p_2 + k_\perp$. We define the light-like vectors p_1 and p_2 such that $p_1 = n$ and $p_2 = p - \frac{m^2}{s} n$, where p is the momentum of the target particle of mass m . We use metric $g^{\mu\nu} = (1, -1, -1, -1)$ so $p \cdot q = (\alpha_p \beta_q + \alpha_q \beta_p) \frac{s}{2} - (p, q)_\perp$. For the coordinates we use the notations $x_\bullet \equiv x_\mu p_1^\mu$ and $x_* \equiv x_\mu p_2^\mu$ related to the light-cone coordinates by $x_* = \sqrt{\frac{s}{2}} x_+$ and $x_\bullet = \sqrt{\frac{s}{2}} x_-$.

Hereafter we use the notation $[\infty, z_*]_z \equiv [\infty_* p_1 + z_\perp, \frac{2}{s} z_* p_1 + z_\perp]$ where $[x, y]$ stands for the straight-line gauge link connecting points x and y . Our convention is that the Latin Lorentz indices always correspond to transverse coordinates while Greek Lorentz indices are four-dimensional.

Similarly, we define

$$\tilde{\mathcal{F}}_i^{a\eta}(x_B, z_\perp) \equiv \frac{2}{s} \int dz_* e^{-ix_B z_*} g(\tilde{F}_{\bullet i}^m(z_*, z_\perp)[z_*, \infty]^{ma})^\eta \quad (3)$$

in the complex-conjugate part of the amplitude.

In this notations the unintegrated gluon TMD $\mathcal{D}(x_B, z_\perp, \eta)$ can be represented as

$$\begin{aligned} \langle p | \tilde{\mathcal{F}}_i^{a\eta}(x_B, z_\perp) \mathcal{F}_i^{a\eta}(x_B, 0_\perp) | p + \xi p_2 \rangle &\equiv \sum_X \langle p | \tilde{\mathcal{F}}_i^{a\eta}(x_B, z_\perp) | X \rangle \langle X | \mathcal{F}_i^{a\eta}(x_B, 0_\perp) | p + \xi p_2 \rangle \\ &= -4\pi^2 \delta(\xi) x_B g^2 \mathcal{D}(x_B, z_\perp, \eta) \end{aligned} \quad (4)$$

Hereafter we use a short-hand notation

$$\langle p | \tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_m \mathcal{O}_1 \dots \mathcal{O}_n | p' \rangle \equiv \sum_X \langle p | \tilde{T} \{ \tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_m \} | X \rangle \langle X | T \{ \mathcal{O}_1 \dots \mathcal{O}_n \} | p' \rangle \quad (5)$$

where tilde on the operators in the l.h.s. of this formula stands as a reminder that they should be inverse time ordered as indicated by inverse-time ordering \tilde{T} in the r.h.s. of the above equation.

As discussed e.g. in Ref. [6], such matrix element can be represented by a double functional integral

$$\langle \tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_m \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \int D\tilde{A} D\tilde{\psi} D\tilde{\bar{\psi}} e^{-iS_{\text{QCD}}(\tilde{A}, \tilde{\psi})} \int DAD\bar{\psi} D\psi e^{iS_{\text{QCD}}(A, \psi)} \tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_m \mathcal{O}_1 \dots \mathcal{O}_n \quad (6)$$

with the boundary condition $\tilde{A}(\vec{x}, t = \infty) = A(\vec{x}, t = \infty)$ (and similarly for quark fields) reflecting the sum over all intermediate states X .

We will study the rapidity evolution of the operator $\tilde{\mathcal{F}}_i^{a\eta}(x_B, x_\perp) \mathcal{F}_j^{a\eta}(x_B, y_\perp)$. In the spirit of rapidity factorization, in order to find the evolution of this operator with respect to rapidity cutoff η (see Eq. (2)) one should integrate in the matrix element (4) over gluons and quarks with rapidities $\eta > Y > \eta'$ and temporarily “freeze” fields with $Y < \eta'$ to be integrated over later. (For a review, see Refs. [7, 8].) In this case, we obtain functional integral of Eq. (6) type over fields with $\eta > Y > \eta'$ in the “external” fields with $Y < \eta'$. In terms of Sudakov variables we integrate over gluons with α between $\sigma = e^\eta$ and $\sigma' = e^{\eta'}$ and, in the leading order, only the diagrams with gluon emissions are relevant - the quark diagrams will enter as loops at the next-to-leading (NLO) level.

To make connections with parton model we will have in mind the frame where target’s velocity is large and call the small α fields by the name “fast fields” and large α fields by “slow” fields. As discussed in Ref. [9], the interaction of “slow” gluons of large α with “fast” fields of small α is described by eikonal gauge factors and the integration over slow fields results in Feynman diagrams in the background of fast fields which form a thin shock wave due to Lorentz contraction. However, in Ref. [9] (as well as in all small- x literature) it was assumed that the characteristic transverse momenta of fast and slow fields are of the same order of magnitude. For our present purposes we need to relax this condition and consider cases where the transverse momenta of fast and slow fields do differ. In this case, we need to rethink the shock-wave approach.

Let us figure out how the relative longitudinal size of fast and slow fields depends on their transverse momenta. The typical longitudinal size of fast fields is $\sigma_* \sim \frac{\sigma_*^s}{l_\perp^2}$ where l_\perp is the characteristic scale of transverse momenta of fast fields. The typical distances traveled by slow gluons are $\sim \frac{\sigma_*^s}{k_\perp^2}$ where k_\perp is the characteristic scale of transverse momenta of slow fields. Effectively, the large- α gluons propagate in the external field of the small- α shock wave, except the case $l_\perp^2 \ll k_\perp^2$ which should be treated separately since the “shock wave” is not necessarily thin in this case. Fortunately, when $l_\perp^2 \ll k_\perp^2$ one can use the light-cone expansion of slow fields and leave at the leading order only the light-ray operators of the leading twist. We use the combination of shock-wave and light-cone expansions and write the interpolating formulas which describe the leading-order contributions in both cases.

Here we present only the final equation for rapidity evolution of gluon TMDs (the details can be found in Refs. [4, 5])

$$\begin{aligned} & \frac{d}{d\eta} \langle p | \tilde{\mathcal{F}}_i^a(x_B, x_\perp) \mathcal{F}_j^a(x_B, y_\perp) | p \rangle^{\eta = \ln \sigma} \\ &= -\alpha_s \langle p | \int \tilde{d}^2 k_\perp \text{Tr} \{ \tilde{L}_i^\mu(k, x_\perp, x_B)^{\text{light-like}} L_{\mu j}(k, y_\perp, x_B)^{\text{light-like}} \} \end{aligned} \quad (7)$$

$$\begin{aligned}
 & + 2\text{Tr} \left\{ \tilde{\mathcal{F}}_i(x_B, x_\perp)(y_\perp| - \frac{p_\perp^m}{p_\perp^2} \mathcal{F}_k(x_B)(i\overleftarrow{\partial}_l + U_l)(2\delta_m^k \delta_j^l - g_{jm} g^{kl}) U \frac{1}{\alpha x_{BS} + p_\perp^2} U^\dagger \right. \\
 & \quad \left. + \mathcal{F}_j(x_B) \frac{\alpha x_{BS}}{p_\perp^2 (\alpha x_{BS} + p_\perp^2)} |y_\perp \right) \\
 & + (x_\perp | \tilde{U} \frac{1}{\alpha x_{BS} + p_\perp^2} \tilde{U}^\dagger (2\delta_i^k \delta_m^l - g_{im} g^{kl})(i\partial_k - \tilde{U}_k) \tilde{\mathcal{F}}_l(x_B) \frac{p_\perp^m}{p_\perp^2} \\
 & \quad \left. + \tilde{\mathcal{F}}_i(x_B) \frac{\alpha x_{BS}}{p_\perp^2 (\alpha x_{BS} + p_\perp^2)} |x_\perp \right) \mathcal{F}_j(x_B, y_\perp) \Big\} |p\rangle + O(\alpha_s^2)
 \end{aligned}$$

where Tr is a trace in the adjoint representation. Here $\mathcal{F}_j^a(x_B, y_\perp)$ is so-called Lipatov vertex - the amplitude of the emission of a real gluon by the operator \mathcal{F}_i^a

$$\begin{aligned}
 & L_{\mu i}^{ab}(k, y_\perp, x_B)^{\text{light-like}} \tag{8} \\
 & = g(k_\perp | \mathcal{F}_j^a(x_B + \frac{k_\perp}{\alpha s}) \left\{ \frac{\alpha x_{BS} g_{\mu i} - 2k_\mu^\perp k_i}{\alpha x_{BS} + k_\perp^2} (k_j U + U p_j) \frac{1}{\alpha x_{BS} + p_\perp^2} U^\dagger \right. \\
 & \quad \left. - 2k_\mu^\perp U \frac{g_{ij}}{\alpha x_{BS} + p_\perp^2} U^\dagger - 2g_{\mu j} U \frac{p_i}{\alpha x_{BS} + p_\perp^2} U^\dagger + \frac{2k_\mu^\perp}{k_\perp^2} g_{ij} \right\} |y_\perp \rangle^{ab} + O(p_{2\mu})
 \end{aligned}$$

where the operator $\mathcal{F}_i(\beta)$ is defined as usual

$$(k_\perp | \mathcal{F}_i(\beta) |y_\perp \rangle \equiv \frac{2}{s} \int dy_* e^{i\beta y_* - i(k, y)_\perp} \mathcal{F}_i(y_*, y_\perp) \tag{9}$$

It is worth noting that at $x_B = 0$ this vertex agrees with the one obtained in Ref. [11].

It is easy to see that our formula for the evolution kernel (7) smoothly interpolates between the k_T -factorization and TMD-factorization cases. Indeed, in the framework of the usual small- x approximation β_B is neglected so the corresponding ‘‘small- x ’’ gluon TMD looks like

$$\mathcal{F}_i^a(z_\perp, 0) = U_i^a(z_\perp) \equiv -2i \text{tr} \{ t^a U \partial_i U^\dagger \} \tag{10}$$

and Eq. (7) reduces to the non-linear equation

$$\begin{aligned}
 & \frac{d}{d\eta} \tilde{U}_i^a(z_2) U_i^a(z_1) \tag{11} \\
 & = -\frac{g^2}{8\pi^3} \text{Tr}(-i\partial_i^z + \tilde{U}_i^z) \left[\int d^2 z_3 (\tilde{U}_{z_2} \tilde{U}_{z_3}^\dagger - 1) \frac{z_{12}^2}{z_{13}^2 z_{23}^2} (U_{z_3} U_{z_1}^\dagger - 1) \right] (i\overleftarrow{\partial}_i^{z_1} + U_i^{z_1})
 \end{aligned}$$

where all indices are 2-dimensional and Tr stands for the trace in the adjoint representation. It is easy to see that the expression in the square brackets is actually the BK kernel for the double-functional integral for cross sections [7, 10].

On the other hand, if $\beta_B \sim 1$ so that $\alpha\beta_{BS} \gg p_\perp^2$ we get a linear equation

$$\begin{aligned}
 & \langle \tilde{\mathcal{F}}_j^a(z_\perp, \beta_B) \mathcal{F}_i^a(z'_\perp, \beta_B) \rangle \tag{12} \\
 & = -\frac{g^2 N_c}{\pi} \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} \int \frac{d^2 p}{p^2} [1 - e^{i(p, z - z')_\perp}] \langle \tilde{\mathcal{F}}^{ai}(z'_\perp, \beta_B) \mathcal{F}_i^a(z_\perp, \beta_B) \rangle
 \end{aligned}$$

which can be rewritten as a linear equation

$$\frac{d}{d\eta} \mathcal{D}(x_B, z_\perp, \eta) = -\frac{\alpha_s N_c}{\pi^2} \mathcal{D}(x_B, z_\perp, \eta) \int \frac{d^2 p}{p^2} [1 - e^{i(p, z)_\perp}]$$

We see that the IR divergence at $p_{\perp}^2 \rightarrow 0$ cancels while the UV divergence in the virtual correction should be cut from above by the condition $p_{\perp}^2 < \sigma s$ following from Eq. (7). Actually, at $x_B \sim 1$ there will be logarithmical region $e^n m \sqrt{s} \gg p_{\perp}^2 \gg m^2$ so one has to sum up leading logarithms $(\alpha_s \eta)^n$ in the evolution kernel Eq. (12) after which the kernel should reproduce the usual Sudakov double logarithms. From Eq. (7) it is clear that the transition between linear evolution (12) and the non-linear evolution (11) occurs at $x_B = \beta_B \sim \frac{m^2}{s}$.

In addition, at $x_{\perp} = y_{\perp}$ (light-cone DIS limit) one obtains from Eq. (7)

$$\begin{aligned} & \frac{d}{d\eta} \langle p | \tilde{\mathcal{F}}_i^n(\beta_B, x_{\perp}) \mathcal{F}^{in}(\beta_B, x_{\perp}) | p \rangle^{\eta} \\ &= \frac{\alpha_s}{\pi} N_c \int_0^{\infty} d\beta \left\{ \theta(1 - \beta_B - \beta) \left[\frac{1}{\beta} - \frac{2\beta_B}{(\beta_B + \beta)^2} + \frac{\beta_B^2}{(\beta_B + \beta)^3} - \frac{\beta_B^3}{(\beta_B + \beta)^4} \right] \right. \\ & \times \left. \langle p | \tilde{\mathcal{F}}_i^n(\beta_B + \beta, x_{\perp}) \mathcal{F}^{ni}(\beta_B + \beta, x_{\perp}) | p \rangle^{\ln \sigma'} - \frac{\beta_B \beta^{-1}}{\beta_B + \beta} \langle p | \tilde{\mathcal{F}}_i^n(\beta_B, x_{\perp}) \mathcal{F}^{in}(\beta_B, x_{\perp}) | p \rangle^{\ln \sigma'} \right\} \end{aligned}$$

which is equivalent to DGLAP equation in the leading log approximation. Thus, our equation (7) smoothly interpolates between DGLAP, Sudakov and low- x limits.

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