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A New Method for Consistency Correction of Judgment Matrix in AHP

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In order to continue to study the consistency correction of the judgment matrix, we studied the relationship between the inconsistency of the judgment matrix and the perturbation matrix, proposed the concept of the perturbation deviation matrix. Then two new methods for the consistency correction of the judgment matrix are presented based on the convex combination of perturbation deviation matrix. Finally, the new methods were validated by examples and compared with the traditional AHP method.

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1. Introduction

The Analytic Hierarchy Process (AHP) [1] has been widely applied as a qualitative and quantitative decision-making tool. A successful decision on making process requires the judgment matrix to be consistent. If a judgment matrix fails to fulfill the requirements of consistency, then the weight obtained from the judgment matrix can not be utilized as the basis for making decision. In that case, certain adjustments on the matrix will be further required; therefore, the problem of consistency correction becomes an important research content in the AHP. With the presentation of AHP, there have been rich literature focusing on the consistency correction research. Ma and Xu [2] proposed a weighted arithmetic mean correction method and two criteria for correction validity. Xu and Wei [3] proposed a weighted geometric mean method. The weighted arithmetic mean and weighted geometric mean method were analyzed and compared in [4]. Xu [5] proposed a weighted arithmetic mean and weighted geometric mean method by analyzing the maximum deviation in the judgment matrix. Some authors proposed the vector correction method and perturbation matrix correction method[6-9]. The consistency of the judgment matrix, the fuzzy judgment matrix and the intuitionistic fuzzy judgment matrix are rectified by the deviation matrix and the vector method [10]. Based on the accelerated genetic algorithm, two kinds of NLP model correction methods were proposed in [11]. Bayesian correction method, Hadamard product induced bias matrix (HPIBM) method and the graph theory correction method were proposed in [12-13]. We have proposed a new algorithm for the consistency test of judgment matrix based on probabilistic statistics and hypothesis testing [14]. Based on the said literature, two new methods for the consistency correction of the judgment matrix based on the convex combination of perturbation deviation matrix are presented hereof to further verify the new methods and make comparison with the traditional AHP.

2. Preliminaries

Definition 2.1 [14] Let $A=(a_{ij})_{p\times q}$ and $B=(b_{ij})_{p\times q}$ be a matrix, if $A*B=(c_{ij})_{p\times q}$ is satisfied, the product of $A*B$ is called the Hadamard product of matrix, where $c_{ij}=a_{ij}*b_{ij}$ and $*$ refer to the Hadamard product symbol.

Definition 2.2[14] Let $A=(a_{ij})_{n\times n}$ be a judgment matrix and the eigenvector corresponding to the largest eigenvalue $\lambda_{\max}$ is $w=[w_1,w_2,...,w_n]^T$,then the matrix $\overline{A}=(w_{ij})_{n\times n}$ is called the characteristic matrix of $A=(a_{ij})_{n\times n}$,where $w_{ij}=\dfrac{w_i}{w_j}$, $i,j\in\{1,2,...,n\}$.

Definition 2.3 [14] Let $A=\left(a_{ij}\right)_{n\times n}$ be a judgment matrix, if $A=\overline{A}*E$,then E is called perturbation matrix of A ,where $E=(\varepsilon_{ij})_{n\times n}$.

Definition 2.4 Let $E=(\varepsilon_{ij})_{n\times n}$ be a perturbation matrix of A ,then

(i) If an element ε_{ij} in E satisfies $\varepsilon_{ij}-1\neq 0$, then $\{\varepsilon_{ij}\}_{n\times n}$ is called perturbation deviation matrix,denoted as D ;

(ii) If an element ε_{ij} in E satisfied $\varepsilon_{ij}-1=0$,then $\{\varepsilon_{ij}\}_{n\times n}$ is called perturbation zero deviation matrix,denoted as D_0 .

Theorem 2.1 If the judgment matrix A satisfies $A=\overline{A}*E$,then A is a completely consistent matrix if and only if $E=D_0=\begin{bmatrix}1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\\ \cdots & \cdots & \ddots & \vdots\\ 1 & 1 & \cdots & 1\end{bmatrix}$.

Proof. Sufficiency: As $\overline{A}$ is the characteristic matrix of A , $\overline{A}$ satisfies the complete consistency, and the perturbation matrix $E=D_0$,according to the Hadamard product definition, we know A satisfies the complete consistency.

Necessity: If A satisfies the complete consistency and $\overline{A}$ also satisfies the complete consistency, according to the Hadamard product definition,so we know $E=D_0$.

Definition 2.5 Let D and D_0 be the perturbation deviation matrix and the perturbation zero deviation matrix respectively,and the convex combination of D and D_0 can be expressed as $\overline{D}=\lambda D+(1-\lambda)D_0$, then $\overline{\varepsilon}_{ij}=\lambda\varepsilon_{ij}+(1-\lambda)$, $\overline{\varepsilon}_{ji}=1/\overline{\varepsilon}_{ij}$,where, $0\leq\lambda\leq 1$.

Theorem 2.2 [9] Let inconsistency judgment matrix A be rectified as $A'=\overline{A}\circ\overline{D}$,if $\max_i\left\{\sum_{j=1}^{n}\varepsilon_{ij}\right\}\leq\lambda_{\max}\left(D\right)$,then $\lambda_{\max}(A')<\lambda_{\max}(A)$.

Definition 2.6 [14] Let $\chi^2_{1-p}\sigma_0^2/n(n-1)$ be a critical value of the consistency index,which is called the Chi- Square Consistency Index(briefly $CSCI$).

Order	3	4	5	6	7	8	9
p=0.01	0.010	0.036	0.064	0.087	0.106	0.121	0.134
p=0.05	0.029	0.068	0.099	0.121	0.138	0.151	0.162
p=0.10	0.049	0.092	0.122	0.138	0.158	0.169	0.178
Order	10	11	12	13	14	15	16
p=0.01	0.144	0.153	0.160	0.166	0.172	0.177	0.181
p=0.05	0.170	0.177	0.183	0.188	0.192	0.196	0.199
p=0.10	0.185	0.191	0.196	0.200	0.204	0.207	0.210
Order	17	18	19	20	21	22	/
p=0.01	0.185	0.189	0.191	0.194	0.197	0.199	/
p=0.05	0.202	0.205	0.207	0.209	0.211	0.213	/
p=0.10	0.212	0.214	0.216	0.218	0.219	0.221	/

Table 1: Critical Value of $CSCI$

If $(\lambda_{\max}-n)/(n-1)<0.1RI$,then the matrix through the consistency test;otherwise,not through the consistency test [1].Where, $CI=(\lambda_{\max}-n)/(n-1)$,and 0.1RI is a critical value for CI to test the consistency of the judgment matrix.In this paper,we present $CSCI$ as proposed in [14] as a new critical value of $(\lambda_{\max}-n)/(n-1)$ to test the consistency of the judgment matrix.

Order	3	4	5	6	7	8	9
RI	0.58	0.90	1.12	1.24	1.32	1.41	1.45
CI=0.1RI	0.058	0.090	0.112	0.124	0.132	0.141	0.145
Order	10	11	12	13	14	15	/
RI	1.49	1.51	1.48	1.56	1.57	1.59	/
CI=0.1RI	0.149	0.151	0.148	0.156	0.157	0.159	/

Table 2: Critical Value of RI and CI [1]

3. A new algorithm for consistency correction

It is proposed to replace d_{ij} in the perturbation matrix with $\gamma d_{ij}+(1-\gamma)$,and the Hadamard product $(w_i/w_j)*[\gamma d_{ij}+(1-\gamma)]$ is used as the correction result of the element in the original matrix[9].However, considering there are too many elements for each correction in [9],it is difficult to maintain the information in the original matrix. In this paper, we propose a consistency correction method, which is easy to maintain the information in the original matrix.

Let $A=(a_{ij})$ be an $n\times n$ judgment matrix, k be the number of iterative times, and the specific steps are as follow:

Step 1: Let $A^{(0)}=\left(a_{ij}^{0}\right)$, $\theta\in(0,1)$, and k =0.

Step 2: Calculate the maximum eigenvalue $\lambda_{\max}^{(k)}$ of $A^{(k)}$ and the priority vector $w^{(k)} = \left(w_1^{(k)}, w_2^{(k)}, \ldots, w_n^{(k)}\right)^T$.

Step 3: If $\left(\lambda_{\max}^{(k)} - n\right)/\left(n-1\right) < CSCI$,then go to Step 6; otherwise, go to next step.

Step 4:

(i) (Method 1)

Normalize all columns of $A^{(k)}$,then get the normalized matrix $\overline{A}^{(k)} = (a_1^{(k)}, a_2^{(k)}, \cdots, a_n^{(k)})$, in which, refers to the column vector of $\overline{A}^{(k)}$. Calculate the cosine value of the included angle between $w^{(k)}$ and $a_i^{(k)}$, namely $\cos\theta_i^{(k)} = \frac{\left(w^{(k)}, a_i^{(k)}\right)}{\left|w^{(k)}\right|\left|a_i^{(k)}\right|}$.

Then determine t so that $\cos\theta_t^{(k)} = \min_i\{\cos\theta_i^{(k)}\}$, and let $A^{(k+1)} = \left(w_i^{(k)} / w_j^{(k)}\right) * \varepsilon_{ij}^{(k)}$, where

$$\varepsilon_{ij}^{(k)} = \begin{cases} \theta\varepsilon_{it}^{(k)} + (1-\theta), j = t \\ \dfrac{1}{\theta\varepsilon_{tj}^{(k)} + (1-\theta)}, i = t \\ \varepsilon_{ij}^{(k)}, \qquad i, j \neq t \end{cases}$$

(ii) (Method 2)

Let $\varepsilon_{ij}^{(k)} = a_{ij}^{(k)}\left(w_j^{(k)} / w_i^{(k)}\right)$ and determine p, q so that $\varepsilon_{pq}^{(k)} = \max_{i,j}\left\{\varepsilon_{ij}^{(k)}\right\}$. Let $A^{(k+1)} = \left(w_i^{(k)} / w_j^{(k)}\right) * \varepsilon_{ij}^{(k)}$, where

$$\varepsilon_{ij}^{(k)} = \begin{cases} \theta\varepsilon_{pq}^{(k)} + (1-\theta), & (i,j) = (p,q) \\ \dfrac{1}{\theta\varepsilon_{pq}^{(k)} + \left(1-\theta\right)}, & (i,j) = (q,p) \\ \varepsilon_{ij}^{(k)}, & (i,j) \neq (p,q), (q,p) \end{cases}$$

Step 5: Let $k = k+1$, and return to Step 2.

Step 6: Output $A^{(k)}, \lambda_{\max}, CI, w^{(k)}$, in which, refers to the correction matrix and refers to the vector of priorities.

Step 7: End.

The two criteria for measuring the proximity of the original matrix to the correction matrix are given in [2],as follows:

$$\delta^{(k)} = \max_{i,j}\left\{\left|a_{ij}^{(k)} - a_{ij}^{(0)}\right|\right\}, i, j \in N$$

$$\sigma^{(k)} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}^{(k)} - a_{ij}^{(0)})^2 / n}$$

Usually we think that the smaller the value of $\delta^{(k)}, \sigma^{(k)}$ is, the more information will be retained from the original matrix and the better correction will be achieved.

4. Case analysis

This paper chooses the Matrix A in [2] and rectifies it according to the above two methods, as follows:

$$A=\begin{bmatrix} 1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\ 1/5 & 1 & 1/3 & 5 & 3 & 3 & 1/5 & 1/7 \\ 1/3 & 3 & 1 & 6 & 3 & 4 & 6 & 1/5 \\ 1/7 & 1/5 & 1/6 & 1 & 1/3 & 1/4 & 1/7 & 1/8 \\ 1/6 & 1/3 & 1/3 & 3 & 1 & 1/2 & 1/5 & 1/6 \\ 1/6 & 1/3 & 1/4 & 4 & 2 & 1 & 1/5 & 1/6 \\ 3 & 5 & 1/6 & 7 & 5 & 5 & 1 & 1/2 \\ 4 & 7 & 5 & 8 & 6 & 6 & 2 & 1 \end{bmatrix}.$$

The perturbation matrix of the original judgment matrix is obtained as follows:

$$\begin{bmatrix} 1 & 1.5599 & 3.2612 & 0.7076 & 1.0766 & 1.2591 & 0.3213 & 0.4813 \\ 0.6411 & 1 & 1.1614 & 1.6200 & 1.7252 & 2.0178 & 0.6180 & 0.8816 \\ 0.3066 & 0.8610 & 1 & 0.5579 & 0.4951 & 0.7722 & 5.3280 & 0.3542 \\ 1.4132 & 0.6173 & 1.7923 & 1 & 0.5916 & 0.5190 & 1.3624 & 2.3809 \\ 0.9290 & 0.5796 & 2.0197 & 1.6903 & 1 & 0.5848 & 1.0746 & 1.7886 \\ 0.7942 & 0.4956 & 1.2951 & 1.9269 & 1.7100 & 1 & 0.9188 & 1.5292 \\ 3.1120 & 1.6182 & 0.1879 & 0.7340 & 0.9306 & 1.0884 & 1 & 0.9986 \\ 2.0775 & 1.1343 & 2.8230 & 0.4200 & 0.5591 & 0.6539 & 1.0014 & 1 \end{bmatrix},$$

$\lambda_{\max} \approx 9.6689$, $\dfrac{\lambda_{\max}-n}{n-1} = 0.2384 > CSCI$,w=(0.1730,0.0540,0.1881,0.175,0.0310,0.0363, 0.1668, 0.3332) .

When p=0.10, $\theta = 0.5$, the correction result in Method 1 is:

$$A^{(k)}=\begin{bmatrix} 1 & 4.9998 & 2.0811 & 6.9998 & 6.0000 & 6.0000 & 0.6853 & 0.2500 \\ 0.2000 & 1 & 0.3524 & 5.0000 & 3.0000 & 3.0000 & 0.2618 & 0.1429 \\ 0.3333 & 3.0000 & 1 & 5.9995 & 2.9997 & 4.0004 & 3.5638 & 0.2000 \\ 0.1429 & 0.2000 & 0.1404 & 1 & 0.3333 & 0.2500 & 0.1239 & 0.1250 \\ 0.1667 & 0.3333 & 0.2688 & 3.0001 & 1 & 0.5000 & 0.1930 & 0.1667 \\ 0.1667 & 0.3333 & 0.2472 & 4.0000 & 2.0000 & 1 & 0.2088 & 0.1667 \\ 3.0000 & 5.0000 & 0.6617 & 7.0003 & 5.0004 & 4.9999 & 1 & 0.5000 \\ 3.9999 & 6.9998 & 3.5899 & 8.0005 & 5.9997 & 6.0000 & 1.9986 & 1 \end{bmatrix},$$

$\lambda_{\max} \approx 8.7633$, $\dfrac{\lambda_{\max}-n}{n-1} = 0.1090 < 0.121 = CSCI$, $w^{(k)}$ =(0.1832,0.0651,0.1289,0.0182, 0.0326,0.0412,0.1954,0.3354), $\delta^{(k)} = 2.4362$, $\sigma^{(k)} = 0.2240$.

When p=0.10, $\theta = 0.5$, the correction result in Method 2 is:

$$A^{(k)}=\begin{bmatrix} 1 & 5.0000 & 3.0000 & 7.0000 & 6.0000 & 6.0000 & 0.3333 & 0.2500 \\ 0.2000 & 1 & 0.3333 & 5.0000 & 3.0000 & 3.0000 & 0.2000 & 0.1429 \\ 0.3333 & 3.0000 & 1 & 6.0000 & 3.0000 & 4.0000 & 1.5486 & 0.2000 \\ 0.1429 & 0.2000 & 0.1667 & 1 & 0.3333 & 0.2500 & 0.1429 & 0.1250 \\ 0.1667 & 0.3333 & 0.3333 & 3.0000 & 1 & 0.5000 & 0.2000 & 0.1667 \\ 0.1667 & 0.3333 & 0.2500 & 4.0000 & 2.0000 & 1 & 0.2000 & 0.1667 \\ 2.0474 & 5.0000 & 0.1667 & 7.0000 & 5.0000 & 5.0000 & 1 & 0.5000 \\ 4.0000 & 7.0000 & 5.0000 & 8.0000 & 6.0000 & 6.0000 & 2.0000 & 1 \end{bmatrix},$$

$\lambda_{\max} \approx 8.8077$, $\dfrac{\lambda_{\max}-n}{n-1}=0.1154$ < $0.121=CSCI$, $w^{(k)}$ =(0.1829,0.0626,0.1258,0.0194, 0.0342,0.0415,0.1793,0.3543), $\delta^{(k)}=3.4514$, $\sigma^{(k)}=0.4476$.

If 0.1RI in the traditional AHP is used as the critical value of $\dfrac{\lambda_{\max}-n}{n-1}$, then the correction result in Method 1 is:

$$A^{(k)}=\begin{bmatrix} 1 & 5.0000 & 3.0001 & 7.0005 & 5.9998 & 5.9994 & 0.9136 & 0.2500 \\ 0.2000 & 1 & 0.3333 & 5.0001 & 2.9998 & 3.0002 & 0.3061 & 0.1429 \\ 0.3333 & 3.0001 & 1 & 5.9994 & 2.9997 & 4.0000 & 1.4626 & 0.2000 \\ 0.1429 & 0.2000 & 0.1667 & 1 & 0.3333 & 0.2500 & 0.1030 & 0.1250 \\ 0.1667 & 0.3333 & 0.3334 & 3.0001 & 1 & 0.5000 & 0.1776 & 0.1667 \\ 0.1667 & 0.3334 & 0.2500 & 4.0000 & 2.0000 & 1 & 0.2085 & 0.1667 \\ 3.0001 & 5.0000 & 0.1667 & 7.0001 & 5.0007 & 4.9999 & 1 & 0.5000 \\ 3.9999 & 7.0000 & 5.0000 & 7.9997 & 6.0003 & 5.9997 & 1.8752 & 1 \end{bmatrix},$$

$\lambda_{\max}=8.9640$, $\dfrac{\lambda_{\max}-n}{n-1}=0.1377$ <0.141=0.1RI, $w^{(k)}$ =(0.1900,0.0626,0.1227,0.0182,0.0328, 0.0294,0.1971,0.3472), $\delta^{(k)}=4.5374$, $\sigma^{(k)}=2.6497$.

If 0.1RI in the traditional AHP is used as the critical value of $\dfrac{\lambda_{\max}-n}{n-1}$, then the correction result in Method 2 is:

$$A^{(k)}=\begin{bmatrix} 1 & 4.9999 & 3.0001 & 7.0004 & 5.9999 & 6.0000 & 0.3334 & 0.2500 \\ 0.2000 & 1 & 0.3333 & 5.0002 & 3.0001 & 3.0000 & 0.2000 & 0.1429 \\ 0.3333 & 3.0001 & 1 & 5.9994 & 2.9997 & 4.0000 & 1.4755 & 0.1988 \\ 0.1429 & 0.2000 & 0.1667 & 1 & 0.3333 & 0.2500 & 0.1429 & 0.1250 \\ 0.1667 & 0.3333 & 0.3334 & 3.0000 & 1 & 0.5000 & 0.2000 & 0.1667 \\ 0.1667 & 0.3333 & 0.2500 & 4.0000 & 2.0000 & 1 & 0.2000 & 0.1667 \\ 3.0000 & 4.9999 & 0.1667 & 6.9995 & 5.0001 & 5.0000 & 1 & 0.5000 \\ 3.9999 & 7.0001 & 5.0000 & 8.0003 & 6.0006 & 5.9990 & 2.0001 & 1 \end{bmatrix},$$

$\lambda_{\max} \approx 8.9058$, $\dfrac{\lambda_{\max}-n}{n-1}=0.1294$ <0.141= 0.1RI, $w^{(k)}$ =(0.1784,0.0613,0.1239,0.0191, 0.0337,0.0407,0.1950,0.3430), $\delta^{(k)}=4.5245$, $\sigma^{(k)}=0.5656$.

By contrast, it is found that the value of $\delta^{(k)}$ and $\sigma^{(k)}$ in the new correction method is smaller than the corresponding traditional correction method, so the new correction method is better.

5. Conclusion

In this paper, we studied the relationship between the perturbation matrix and the judgment matrix inconsistency, proposed two new methods of consistency correction based on the theory of perturbation deviation matrix. The element to be rectified are determined by the size of the disturbance element in the perturbation matrix and the size of the cosine between the vectors. Two new methods have been verified by the satisfied results.

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