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# Two-loop integrals for $\mu e$ scattering

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We report on the analytic evaluation of the full set of two-loop four-point master integrals required for the determination of next-to-next-to-leading order corrections to $\mu e$-scattering in Quantum Electrodynamics. The results are obtained in the massless electron approximation and they retain full dependence on the muon mass. The considered integrals are also relevant for the crossing related process $e^+e^- \to \mu^+\mu^-$, as well as for QCD corrections to the $t\bar{t}$-production at hadron colliders.

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## 1. Introduction

In this contribution, we report on the study of the next-to-next-to-leading-order (NNLO) virtual corrections to the elastic scattering of muons and electrons in Quantum Electrodynamics (QED), which has been presented in the two companion articles [1, 2]. In particular, we discuss the analytic evaluation of the complete set of –planar and non-planar – two-loop four-point master integrals (MIs) that arise from Feynman diagrams of order $\alpha^3$ [3]. The present results, together with the known expression of two-loop QED three-point functions [4, 5, 6], make the analytic evaluation of the NNLO virtual amplitude for $\mu e$ scattering within reach, for instance, in the framework of the adaptive integrand decomposition [7, 8], as discussed during this conference [9].

The NNLO QED corrections to $\mu e \to \mu e$ will play a crucial role in the interpretation of future high-precision experiments like MUonE, recently proposed at CERN, which aims at measuring the cross section of the elastic scattering of high-energy muons on atomic electrons as a function of the negative squared momentum transfer [10, 11, 12] with statistical and systematic uncertainties of the order of 10ppm. This measurement will provide the running of the effective electromagnetic coupling in the spacelike region and, as a result, an independent determination of the leading hadronic contribution to the muon $g$-2, which is expected to be competitive with the precision of the traditional dispersive calculations (see [13] for a review).

The same set of MIs required for $\mu e$ scattering will allow the determination of the NNLO QED corrections to the crossing-related process $e^+e^- \to \mu^+\mu^-$. The latter will be relevant for some of the high-precision studies planned at upcoming low-energy $e^+e^-$ experiments, like Belle-II and VEPP-2000, which will target the forward-backward asymmetry [14] in muon pair production and the determination of the $R(s)$-ratio [15, 16].

In addition, the hereby computed MIs constitute a subset of those needed for the complete QCD corrections to the $t\bar{t}$-pair production at hadron colliders [17, 18, 19, 20, 21] that extends, together with the recent results of [22, 23], the available analytic results [24, 25, 26, 27] to planar and non-planar subleading color contributions.

Given the hierarchy between the electron mass $m_e$ and the muon mass $m$, $m_e/m \sim 5 \cdot 10^{-3}$, we work in the approximation $m_e = 0$. We use integration-by-parts identities (IBPs) [28, 29, 30] in order to identify a set MIs, which we analytically computed by means of the differential equations (DEQs) method [31, 32, 33]. The solution of the system of DEQs for small values of $\epsilon = (4-d)/2$ in terms of generalised polylogarithms (GPLs) [34, 35, 36, 37] is facilitated by the identification of a canonical basis of MIs, in the sense of [38]. Such basis is determined trough a well consolidated procedure, based on the Magnus exponential [39, 40], which has been successfully applied in the context of multi-loop integrals involving several kinematic scales [39, 40, 41, 42, 1]. We addressed the challenging determination of the boundary conditions of the DEQs by exploting either the regularity conditions at pseudo-thresholds or the expression of auxiliary integrals, which are obtained by solving simpler systems of DEQs.

This presentation is organised as follows: in section 2 we classify the relevant four-point integral families, in section 3 we discuss the solution of the DEQs for the associated set of MIs and, in section 4, we address the numerical evaluation of the result. We draw our conclusions in section 5.



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## 2. Integral families

We study the box-type two-loop corrections to the scattering process $\mu^+(p_1) + e^-(p_2) \to e^-(p_3) + \mu^+(p_4)$, with kinematics specified by

$$
\begin{gathered}
p_1^2 = p_4^2 = m^2\,, \quad p_2^2 = p_3^2 = 0\,, \\
s = (p_1+p_2)^2\,, \quad t = (p_2-p_3)^2\,, \quad u = (p_1-p_3)^2 = 2m^2 - t - s\,,
\end{gathered} \tag{2.1}
$$

where $m$ is the muon mass. Representative Feynman diagrams of the 10 relevant four-point topologies $T_i$ are shown in figure 1. All these topologies can be organised into 3 distinct integral families of the type

$$
I^{[d]}(a_1,\dots,a_9) \equiv \int \widetilde{\mathrm{d}^d k_1}\widetilde{\mathrm{d}^d k_2}\,\frac{1}{D_1^{a_1}\dots D_9^{a_9}}\,, \quad a_i \in \mathbb{Z}\,, \tag{2.2}
$$

where $k_1$ and $k_2$ are the loop momenta and $D_i$ correspond to inverse scalar propagators. The integration measure is defined as

$$
\widetilde{\mathrm{d}^d k} = \frac{\mathrm{d}^d k}{i\pi^{d/2}\Gamma_\varepsilon}\left(\frac{m^2}{\mu^2}\right)^{\varepsilon}\,, \tag{2.3}
$$

with $\mu$ being the 't Hooft scale of dimensional regularisation and $\Gamma_\varepsilon \equiv \Gamma(1+\varepsilon)$. For each individual family, the set of inverse propagators is chosen as follows:

- The first (planar) integral family, which includes the topologies $T_1$, $T_2$, $T_3$, $T_7$ and $T_8$ of figure 1, is identified by

$$
\begin{gathered}
D_1 = k_1^2 - m^2, \quad D_2 = k_2^2 - m^2, \quad D_3 = (k_1+p_1)^2, \quad D_4 = (k_2+p_1)^2, \\
D_5 = (k_1+p_1+p_2)^2, \quad D_6 = (k_2+p_1+p_2)^2, \quad D_7 = (k_1-k_2)^2, \\
D_8 = (k_1+p_4)^2, \quad D_9 = (k_2+p_4)^2 \quad .
\end{gathered} \tag{2.4}
$$

  IBPs, implemented through the computer code `Reduze` [43], reduce the integrals that belong to this family to 34 MIs, which can be chosen as shown in figure 2. In our graphical conventions, the MIs are represented by diagrams where thick lines stand for massive muon propagators, whereas thin lines stand for massless particles (electron, photon).

- The second (planar) integral family, which includes the topologies $T_4$, $T_5$, $T_9$ and $T_{10}$ depicted in figure 1, is defined by

$$
\begin{gathered}
D_1 = k_1^2 - m^2, \quad D_2 = k_2^2, \quad D_3 = (k_2+p_2)^2, \quad D_4 = (k_1+p_2)^2, \\
D_5 = (k_2+p_2-p_3)^2, \quad D_6 = (k_1+p_2-p_3)^2 - m^2, \quad D_7 = (k_1-p_1)^2, \\
D_8 = (k_2-p_1)^2 - m^2, \quad D_9 = (k_1-k_2)^2 - m^2 \quad .
\end{gathered} \tag{2.5}
$$

  The integral basis of this family contains 42 MIs. We refer the reader to [1] for the explicit definition of the basis chosen in this computation.

- For the third (non-planar) integral family, which corresponds to the topology $T_6$ of figure 1, the inverse propagators are chosen to be

$$D_1 = k_1^2 - m^2, \quad D_2 = k_2^2, \quad D_3 = (k_2 + p_2)^2, \quad D_4 = (k_1 + p_2)^2,$$
$$D_5 = (k_2 + p_2 - p_3)^2, \quad D_6 = (k_1 + p_2 - p_3)^2 - m^2, \quad D_7 = (k_1 - p_1)^2,$$
$$D_8 = (k_2 - p_1)^2 - m^2, \quad D_9 = (k_1 - k_2)^2 - m^2 \quad . \tag{2.6}$$

A basis of the 44 MIs that belong to this family is given in [2].

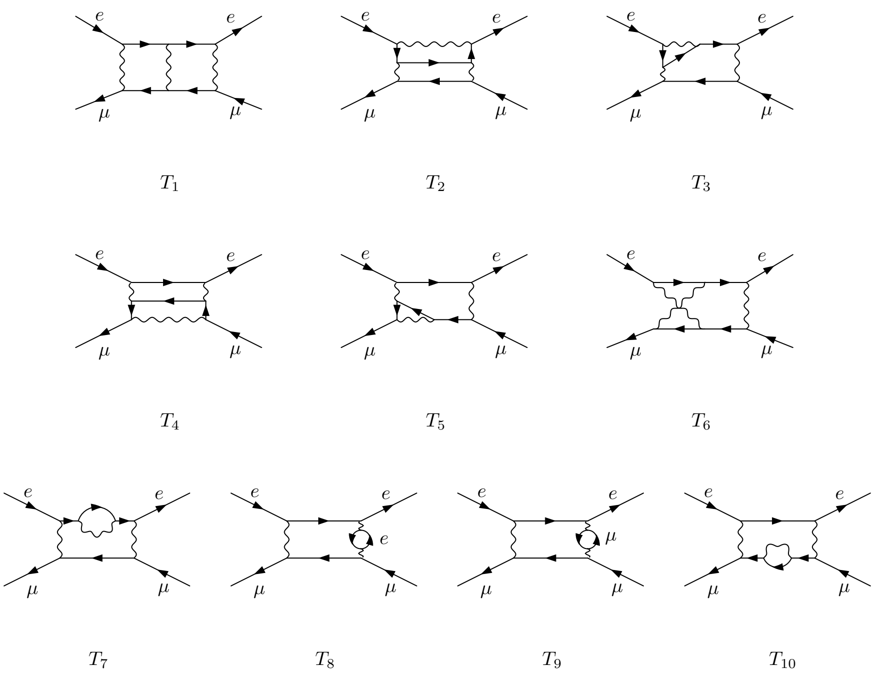


Figure 1: Two-loop four-point topologies for $\mu e$ scattering.



## 3. Differential equations

In order to determine the analytic expression of MIs identified through IBPs reduction, we solve their DEQs in the independent kinematic invariants $s$ and $t$ and $m^2$. These three dimensionful parameters can be combined into two independent dimensionless variables $x_1$ and $x_2$ that parametrise (up to a trivial scaling factor) the full dependence of the MIs on the kinematics. A suitable choice of the differentiation variables $x_i$ can greatly simplify the determination of MIs, as it can be used to remove (or, at least, simplify) non-rational terms that can appear in the DEQs. For the integrals under study, we found that the following variable definition leads to a completely rational system of DEQs:

- For the two planar integral families defined in eq. (2.4)-(2.5) we introduce $(x_1, x_2) = (x, y)$, with

$$-\frac{s}{m^2} = x, \quad -\frac{t}{m^2} = \frac{(1-y)^2}{y} \, . \tag{3.1}$$

- For the non-planar integral family defined in eq. (2.6) we choose $(x_1, x_2) = (z, y)$, with

$$\frac{u-m^2}{s-m^2} = -\frac{z^2}{w}, \qquad \frac{t}{m^2} = -\frac{(1-y)^2}{y}, \tag{3.2}$$

  where the constraint $s+t+u = 2m^2$ is understood.

### 3.1 Canonical systems of differential equations

For each integral family, we derived canonical systems of DEQs according to the algorithm described in [39, 40]: we start by empirically identifying a set of MIs that fulfils system of DEQs with a linear dependence on the regulating parameter $\varepsilon$. For the first integral family, such set corresponds to the integrals depicted in figure 2, suitably combined with $\varepsilon$-rational prefactors (we refer the reader to [1, 2] for the explicit form of such combinations as well as for the definitions of the integral basis for the other two families). Subsequently, we use the Magnus exponential matrix in order to transform these sets of MIs to integral bases that obey canonical DEQs in both $x_1$ and $x_2$. For the case of canonical DEQs, the basis change obtained through the Magnus exponential is equivalent to the Wronskian matrix (formed by the solutions of the associated homogenous equations), which has been shown to allow extensions to systems of DEQs involving elliptic solutions [44, 45, 46]. Once combined into a single total differential, the canonical DEQs in $x_1$ and $x_2$ read

$$d\mathbf{I}(\varepsilon, x_1, x_2) = \varepsilon d\mathbb{A}(x_1, x_2)\mathbf{I}(\varepsilon, x_1, x_2), \tag{3.3}$$

where $\mathbf{I}$ is a vector that collects the MIs of a given integral family and

$$d\mathbb{A}(x_1, x_2) = \sum_i \mathbb{M}_i \, d\log(\eta_i(x_1, x_2)), \tag{3.4}$$

with $\mathbb{M}_i$ being rational constant matrices. The arguments $\eta_i$ of the $d$log-form define the so-called *alphabet* of the DEQs which, for the integrals under consideration, looks as follows:

- For the planar integral families of eq. (2.4)-(2.5) the alphabet is composed of 9 letters $\eta_i(x, y)$,

$$\begin{aligned}
&\eta_1 = x, && \eta_2 = 1+x,\\
&\eta_3 = 1-x, && \eta_4 = y,\\
&\eta_5 = 1+y, && \eta_6 = 1-y,,\\
&\eta_7 = x+y, && \eta_8 = 1+xy,\\
&\eta_9 = 1-y(1-x-y),
\end{aligned} \tag{3.5}$$

  which are real and positive in the region $x > 0 \wedge 0 < y < 1$ (that corresponds to $s < 0 \wedge t < 0$).

- For the non-planar integral family of eq. (2.6) the alphabet is composed of 12 letters $\eta_i(z, y)$,

$$\begin{aligned}
&\eta_1 = y, && \eta_2 = 1+y,\\
&\eta_3 = 1-y, && \eta_4 = z,\\
&\eta_5 = 1+z, && \eta_6 = 1-z,\\
&\eta_7 = z+w, && \eta_8 = z-y,\\
&\eta_9 = z^2-y, && \eta_{10} = 1-y+y^2-z^2,\\
&\eta_{11} = 1-3y+y^2+z^2, && \eta_{12} = z^2-y^2-yz^2+y^2z^2,
\end{aligned} \tag{3.6}$$

that are real and positive in the region $0 < y < 1 \wedge \sqrt{y} < z < \sqrt{1-y+y^2}$ (or, equivalently, $s < 0 \wedge t < 0$).

All MIs are normalised in such a way that they are finite in the $\varepsilon \to 0$ limit, in such a way that $\mathbf{I}(x_1, x_2)$ admits a Taylor expansion in $\varepsilon$,

$$\mathbf{I}(\varepsilon, x_1, x_2) = \sum_{n \geq 0} \varepsilon^n \left( \sum_{i=0}^{n} \Delta^{(n-i)}(x_1, x_2; x_{1,0}, x_{2,0}) \mathbf{I}^{(i)}(x_{1,0}, x_{2,0}) \right), \tag{3.7}$$

where $\mathbf{I}^{(i)}(x_{1,0}, x_{2,0})$ is a vector of boundary constants and $\Delta^{(k)}$ the weight-$k$ operator

$$\Delta^{(k)}(x_1, x_2; x_{1,0}, x_{2,0}) = \int_\gamma \underbrace{d\mathbb{A} \ldots d\mathbb{A}}_{\text{k times}}, \qquad \Delta^{(0)}(x_1, x_2; x_{1,0}, x_{2,0}) = 1, \tag{3.8}$$

which iterates $k$ ordered integrations of the matrix-valued 1-form $d\mathbb{A}$ along a path $\gamma$ in the $x_1x_2$-plane. Since both alphabets of eq.s (3.5)-(3.6) have algebraic roots, the iterated integrals (3.8) can be directly expressed (by integrating on a piecewise smooth path which varies one variable at a time) in terms of GPLs. For all the three integral families, the solutions of the system of DEQs is derived in the unphysical region $s < 0 \wedge t < 0$, where – given the reality of the alphabets of eq.s (3.5)-(3.6) – potential imaginary parts of the solution can originate from the integration constants only. The expression of the MIs in the scattering region $s > m^2 \wedge m^2 - s < t < 0$, as well in the kinematic regions relevant for crossing-related processes, can be subsequently obtained by analytic continuation of our result.

### 3.2 Boundary constants

The iterative integration of eq. (3.3) leads to a general solution of the DEQs in terms of GPLs that depends on arbitrary integration constants. The latter are determined by imposing a suitable set of boundary conditions. On a limited number of cases, it was possible to fix the integration constants by exploiting the knowledge of the analytic expression of the MIs in special kinematic configurations, derived from either direct computation or from the solution of auxiliary, simpler, system of DEQs. For the majority of the integrals, however, it was sufficient to impose the regularity of the solution at pseudo-thresholds, i.e. singular points of the DEQs which do not correspond to physical singularities of the integrals, in order to completely determine the boundary constants. For the case under consideration, regularity conditions express the boundary constants as combinations of GPLs of argument 1 and complex weights, which arise from the kinematic limits imposed on the alphabets given in eq.s (3.5)-(3.6). We used `GiNaC` [47] to verify that, for each MI, at every order in $\varepsilon$, the corresponding combination of constant GPLs is proportional to a transcendentally uniform polynomial combination of the constants $\pi$, $\zeta_k$ and $\log 2$.

## 4. Numerical evaluation

As a validation of our result, we numerically evaluated the analytic expression of all MIs in the region $s < 0 \wedge t < 0$ by means of the `GiNac` library and checked them against independent numerical calculations. All MIs, with the only exception of the four-point non-planar integrals

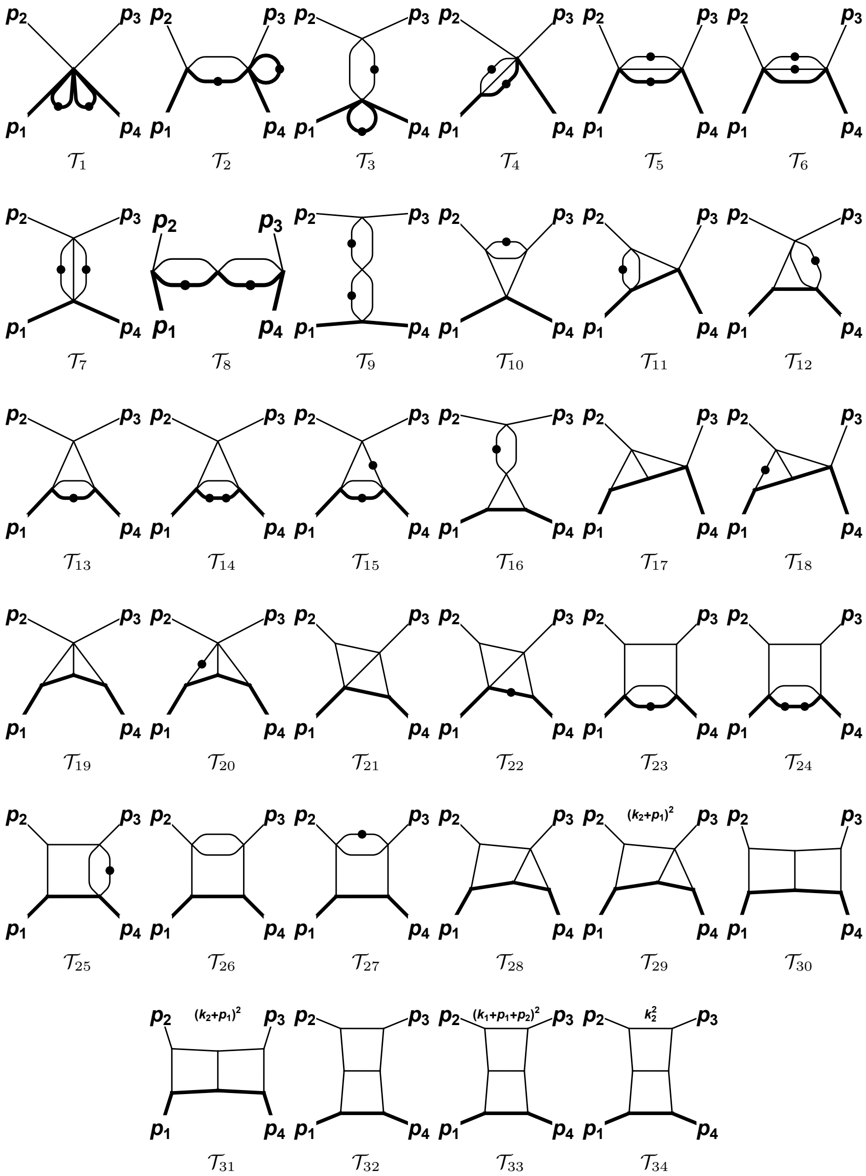


Figure 2: The master integrals of the first integral family.

that belong to the integral family defined in eq. (2.6), have been cross-checked against the results provided by the code `SecDec` [48]. For the challenging numerical evaluation of the non-planar integrals we resorted on a different strategy: after identifying an alternative set of *quasi finite* [49] MIs in $d = 6$, we evaluated their Feynman parametric representation by carrying out as many analytic integrations as possible and by numerically evaluating the leftover integrals by means of Gauss quadrature. Dimension-shifting identities and IBPs, implemented in `LiteRed` [50, 51], establish analytical relations between this set of integrals and the original MIs computed around $d = 4$.

## 5. Conclusions

In this talk, we reported on the analytic evaluation of the two-loop master integrals needed for the NNLO virtual corrections to $\mu e$ elastic scattering in QED that has been presented in [1, 2] . In the massless electron approximation, we computed the master integrals through the differential equations method, by using the Magnus exponential in order to identify a canonical set of master integrals and by deriving boundary conditions from the regularity requirements at pseudothresholds. The present results pave the way to the evaluation of the NNLO virtual amplitude, which is currently under investigation [3, 9]. The two-loop amplitude will constitute an essential part of the theoretical input required by the ambitious experimental goal of the MUonE project, which will determine the leading hadronic contribution to the muon $g-2$ by measuring the scattering of high-energy muons on atomic electrons. By crossing symmetry, the considered integrals are also relevant for muon-pair production at $e^+e^-$-colliders, as well as for the QCD corrections to heavy-quark pair production at hadron colliders.

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