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An Improved Method to Compute the Inverse Mellin Transform of Holonomic Sequences

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We describe a new method to compute the inverse Mellin transform of holonomic sequences, that is based on a method to compute the Mellin transform of holonomic functions. The method is implemented in the computer algebra package HarmonicSums.

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1. Introduction

There have been several methods proposed to compute the inverse Mellin transform of special sequences, for instance in [1] an algorithm (using rewrite rules) to compute the inverse Mellin transform of harmonic sums was stated. This algorithm was extended in [2] to generalized harmonic sums such as S-sums and cyclotomic sums. A different approach to compute inverse Mellin transforms of binomial sums was described in [3]. In [4] a method to compute the inverse Mellin transform of general holonomic sequences was described. That method uses holonomic closure properties and was implemented in the computer algebra package `HarmonicSums` [2, 5, 6, 7, 8]. In the frame of this article we want show how this method can be modified in order to find a more efficient and improved method to compute the inverse Mellin transform of holonomic sequences. The resulting method has been heavily used in the frame of the work on [9].

In the following we repeat important definitions and properties (compare [3, 4, 10]). Let $\mathbb{K}$ be a field of characteristic 0. A function $f = f(x)$ is called *holonomic* (or *D-finite*) if there exist polynomials $p_d(x), p_{d-1}(x), \ldots, p_0(x) \in \mathbb{K}[x]$ (not all p_i being 0) such that the following holonomic differential equation holds:

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0. \tag{1.1}$$

We emphasize that the class of holonomic functions is rather large due to its closure properties. Namely, if we are given two such differential equations that contain holonomic functions $f(x)$ and $g(x)$ as solutions, one can compute holonomic differential equations that contain $f(x) + g(x)$, $f(x)g(x)$ or $\int_0^x f(y)dy$ as solutions. In other words any composition of these operations over known holonomic functions $f(x)$ and $g(x)$ is again a holonomic function $h(x)$. In particular, if for the inner building blocks $f(x)$ and $g(x)$ the holonomic differential equations are given, also the holonomic differential equation of $h(x)$ can be computed.
Of special importance is the connection to recurrence relations. A sequence $(f_n)_{n\geq 0}$ with $f_n \in \mathbb{K}$ is called holonomic (or *P-finite*) if there exist polynomials $p_d(n), p_{d-1}(n), \ldots, p_0(n) \in \mathbb{K}[n]$ (not all p_i being 0) such that the holonomic recurrence

$$p_d(n) f_{n+d} + \cdots + p_1(n) f_{n+1} + p_0(n) f_n = 0 \tag{1.2}$$

holds for all $n \in \mathbb{N}$ (from a certain point on). In the following we utilize the fact that holonomic functions are precisely the generating functions of holonomic sequences: if $f(x)$ is holonomic, then the coefficients f_n of the formal power series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

form a holonomic sequence. Conversely, for a given holonomic sequence $(f_n)_{n\geq 0}$, the function defined by the above sum (i.e., its generating function) is holonomic (this is true in the sense of formal power series, even if the sum has a zero radius of convergence). Note that given a holonomic differential equation for a holonomic function $f(x)$ it is straightforward to construct a holonomic recurrence for the coefficients of its power series expansion. For a recent overview of this holonomic machinery and further literature we refer to [10]. An additional property of

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holonomic functions was given for example in [4] and [3]: if the Mellin transform of a holonomic function

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x)dx = F(n) \tag{1.3}$$

is defined, i.e., the integral $\int_0^1 x^n f(x)dx$ exists, then it is a holonomic sequence. And using the properties of the Mellin transform we can easily check that

$$\mathbf{M}[x^m f^{(p)}(x)](n) = \frac{(-1)^p(n+m)!}{(n+m-p)!}\mathbf{M}[f(x)](n+m-p) + \sum_{i=0}^{p-1}\frac{(-1)^i(n+m)!}{(n+m-i)!}f^{(p-1-i)}(1). \tag{1.4}$$

Conversely, if the Mellin transform $\mathbf{M}[f(x)](n)$ of a function $f(x)$ is holonomic, then also the function $f(x)$ is holonomic. In this article we will report on an improved method to calculate the inverse Mellin transform in terms of iterated integrals, note that this method is implemented in the the package `HarmonicSums`.

The paper is organized as follows. In Section 2 we revisit a method to derive a differential equation for $f(x)$ under the assumption that a holonomic recurrence for $\mathbf{M}[f(x)](n)$ is given. In Section 3 we present a new improved method to compute the inverse Mellin transform of holonomic sequences and finally in Section 4 we give a detailed example to show the application of the method.

2. Deriving the differential equation

In this section we want to recall how we can compute a differential equation for $f(x)$ given a holonomic recurrence for $\mathbf{M}[f(x)](n)$ (compare [4]). First we state an important property of the Mellin transform that will be useful in the remainder of this section:

$$\frac{d^m}{dn^m}\mathbf{M}[f(x)](n) = \mathbf{M}[\log(x)^m f(x)](n). \tag{2.1}$$

Analyzing (1.4) we see that

$$\begin{aligned}\mathbf{M}[(-1)^p x^{m+p} f^{(p)}(x)](n) &= \frac{(n+m+p)!}{(n+m)!}\mathbf{M}[f(x)](n+m)\\ &+\sum_{i=0}^{p-1}\frac{(-1)^{i+p}(n+m+p)!}{(n+m+p-i)!}f^{(p-1-i)}(1).\end{aligned} \tag{2.2}$$

Hence we get

$$\begin{aligned}n^p\mathbf{M}[f(x)](n+m) &= \mathbf{M}[(-1)^p x^{m+p} f^{(p)}(x)](n) - a(n)\mathbf{M}[f(x)](n+m)\\ &-\sum_{i=0}^{p-1}\frac{(-1)^{i+p}(n+m+p)!}{(n+m+p-i)!}f^{(p-1-i)}(1),\end{aligned} \tag{2.3}$$

where $a(n) \in \mathbb{K}[n]$ with $\deg(a(n)) < p$. We can use this observation to compute the differential equation recursively: Let

$$p_d(n)f_{n+d} + \cdots + p_1(n)f_{n+1} + p_0(n)f_n = 0 \tag{2.4}$$

be the holonomic recurrence for $\mathbf{M}[f(x)](n)$. Let $k := \max_{0\le i\le d}(\deg(p_i(x)))$ and let c be the coefficient of n^k in the recurrence i.e.,

$$c = \sum_{i=0}^{d} c_i f_{n+i},$$

for some $c_i \in \mathbb{K}$. For $0 \le i \le d$ we replace $c_i n^k f_{n+i}$ by

$$c_i n^k f_{n+i} + \mathbf{M}[c_i(-1)^k x^{k+i} f^{(k)}(x)](n) - c_i(-1)^k \underbrace{\mathbf{M}[x^{k+i} f^{(k)}(x)](n)}_{*}$$

and apply (1.4) to $*$. Considering (2.3) we conclude that we reduced the degree of n. We apply this strategy until we have removed all appearances of f_{n+i}. At this point we deal with an equation of the form

$$\mathbf{M}[q_l(x) f^{(l)}(x) + \cdots + q_1(x) f'(x) + q_0(x) f(x)](n) + \sum_{j=0}^{k-1} r_j(n) f^{(j)}(1) = 0 \tag{2.5}$$

where $r_i(n) \in \mathbb{K}[n]$. If all $r_i(n) = 0$, we can immediately conclude that $f(x)$ has to satisfy the differential equation

$$f^{(l)}(x) + \cdots + q_1(x) f'(x) + q_0(x) f(x) = 0. \tag{2.6}$$

If not all $r_i(n) = 0$, let $m := \max_{0\le i\le k-1}(\deg(r_j(n)))$, we differentiate equation (2.5) $(m+1)$-times with respect to n. According to (2.1) we get

$$\mathbf{M}[\log(x)^m (q_l(x) f^{(l)}(x) + \cdots + q_1(x) f'(x) + q_0(x) f(x))](n) = 0,$$

and hence $f(x)$ has to satisfy the differential equation (2.6).

3. The Inverse Mellin Transform of Holonomic Sequences

In the following, we deal with the following problem:
Given a nested sum $F(n)$ of the form

$$F(n) := F_0(n) \sum_{i_1=1}^{n} F_1(i_1) \sum_{i_2=1}^{i_1} F_2(i_2) \cdots \sum_{i_k=1}^{i_{k-1}} F_k(i_k), \tag{3.1}$$

with $F_j(n)$ holonomic in n such that $\frac{F_j(y)}{F_j(y+1)} \in \mathbb{K}(\eta)(y)$[1]. Note that we could incorporate η into the field $\mathbb{K}$, but in our applications η represents a specific number with $0 < \eta < 1$ and we keep η explicit since it is important in the regularization later on.
Find, whenever possible, a representation in the form

$$H(n) = \sum_{j=0}^{k} v_j^n \left(d_{0,j} + \int_0^1 (x^n - a_j^n) \sum_{i=1}^{b_j} d_{i,j} f_{i,j}(x) dx \right), \tag{3.2}$$

[1] In this particular instance $F_j(y)$ is also called *hypergeometric* with respect to y.

such that $F(n) = H(n)$ for all $n \in \mathbb{N}$ with $n > n_0$ for some $n_0 \in \mathbb{N}$ where in our cases $v_j,\ a_j,\ d_{i,j} \in \mathbb{K}(\eta)$ and $f_{i,j}(x)$ are expressions of the form $p(\eta,x)\ g(x)$ with $p(\eta,x) \in \mathbb{K}(\eta)(x)$ and $g(x)$ is an iterated integral of the form

$$\mathrm{G}(g_1(\tau), g_2(\tau), \cdots, g_k(\tau), x) = \int_0^x g_1(\tau_1)\mathrm{G}(g_2(\tau), \cdots, g_k(\tau), \tau_1)\, d\tau_1,$$

with $g_j(x)$ holonomic in x such that $\frac{g_j(y)}{g_j'(y)} \in \mathbb{K}(\eta)(y)^2$ and with the special cases

$$\mathrm{G}(x) = 1,$$

and

$$\mathrm{G}\Big(\underbrace{\frac{1}{\tau}, \frac{1}{\tau}, \ldots, \frac{1}{\tau}}_{k \text{ times}}, x\Big) = \frac{1}{k!}\log(x)^k.$$

In order to find such a representation we start by defining

$$\bar{F}_j(n) := F_0(n) \sum_{i_1=1}^{n} F_1(i_1) \sum_{i_2=1}^{i_1} F_2(i_2) \cdots \sum_{i_j=1}^{i_{j-1}} F_j(i_j) \tag{3.3}$$

for $0 \leq j \leq k$. Hence for example $\bar{F}_k(n) = F(n)$, $\bar{F}_{k-1}(n)$ is the original sum with the innermost summation quantifier dropped, $\bar{F}_1(n) = F_0(n)\sum_{i_1=1}^{n} F_1(i_1)$ and $\bar{F}_0(n) = F_0(n)$. Now for each j with $0 \leq j \leq k$ we proceed as follows:

- Determine v_j, note that v_j^n reflects the asymptotic behavior of $F_j(n)$.
- Construct a recurrence

  $$p_d(n)f_{n+d} + \cdots + p_1(n)f_{n+1} + p_0(n)f_n = 0, \tag{3.4}$$

  such that $v_j^{-n}\bar{F}_j(n)$ is a solution of (3.4).
- Use the method from Section 2 to derive a differential equation

  $$q_{b_j}(x)f^{(b_j)}(x) + \cdots + q_1(x)f'(x) + q_0(x)f(x) = 0 \tag{3.5}$$

  for the inverse Mellin transform of $v_j^{-n}\bar{F}_j(n)$.
- Compute, if possible, a general solution[3]

  $$\sum_{i=1}^{b_j} d_{i,j} f_{i,j}(x) \tag{3.6}$$

  of (3.5) in terms of iterated integrals by using the algorithms form [11, 12, 13, 14, 15], see also [16].

[2] $g_j(y)$ is also called *hyperexponential* with respect to y.

[3] If only a subspace of the general solution can be computed, the result can be still obtained provided that such a representation exists.

- Read off $a_j \in (0,1]$ from (3.6), note that a_j is a zero of the denominator or 0.

At this point we use the ansatz

$$F(n) = \sum_{j=0}^{k} v_j^n \left(d_{0,j} + \sum_{i=1}^{b_j} d_{i,j} \int_0^1 (x^n - a_j^n) f_{i,j}(x)dx \right), \tag{3.7}$$

and it remains to determine the $d_{i,j}$. Therefore, for each j we compute

$$M_j(n) = \sum_{i=1}^{b_j} d_{i,j} \int_0^1 (x^n - a_j^n) f_{i,j}(x)dx, \tag{3.8}$$

i.e., we compute the Mellin transform of $\sum_{i=0}^{b_j} d_{i,j} f_{i,j}(x)$ for symbolic $d_{i,j}$. Hence we have

$$F(n) = \sum_{j=0}^{k} v_j^n \left(d_{0,j} + M_j(n) \right), \tag{3.9}$$

and by checking a sufficient amount of initial values we can determine the $d_{i,j}$.

4. A detailed Example

Let's consider the following expression ($0 < \eta < 1$):

$$F(n) := 4^{-n} \binom{2n}{n} \sum_{i=1}^{n} \frac{4^i \left(\frac{1}{1-\eta}\right)^i \sum_{j=1}^{i} \frac{(1-\eta)^j}{j}}{i^2 \binom{2i}{i}}.$$

We proceed as suggested in the previous section. Hence we define:

$$\begin{aligned}
\bar{F}_2(n) &:= 4^{-n} \binom{2n}{n} \sum_{i=1}^{n} \frac{4^i \left(\frac{1}{1-\eta}\right)^i}{i^2 \binom{2i}{i}} \sum_{j=1}^{i} \frac{(1-\eta)^j}{j},\\
\bar{F}_1(n) &:= 4^{-n} \binom{2n}{n} \sum_{i=1}^{n} \frac{4^i \left(\frac{1}{1-\eta}\right)^i}{i^2 \binom{2i}{i}},\\
\bar{F}_0(n) &:= 4^{-n} \binom{2n}{n}.
\end{aligned}$$

In order to find v_0, v_1 and v_2 we determine the asymptotic behavior of $\bar{F}_0(n), \bar{F}_1(n)$ and $\bar{F}_2(n)$, respectively. Since,

$$\underbrace{4^{-n}\binom{2n}{n}}_{\sim(1)^n\left(\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{n}}+O\left(\frac{1}{n}\right)\right)} \sum_{i=1}^{n} \underbrace{\frac{4^i \left(\frac{1}{1-\eta}\right)^i}{i^2 \binom{2i}{i}}}_{\sim\left(\frac{1}{1-\eta}\right)^i\left(\frac{1}{\sqrt{i^3}}+O\left(\frac{1}{i^2}\right)\right)} \sum_{j=1}^{i} \underbrace{\frac{(1-\eta)^j}{j}}_{\sim(1-\eta)^j\left(\frac{1}{j}+O\left(\frac{1}{j^2}\right)\right)},$$

we have

$$\bar{F}_2(n) \sim \left(1 \cdot \frac{1}{1-\eta} \cdot (1-\eta)\right)^n \left(\frac{\log(\eta)}{\eta^2 n^2} + 0\left(\frac{1}{n^3}\right)\right) = 1^n \left(\frac{\log(\eta)}{\eta^2 n^2} + 0\left(\frac{1}{n^3}\right)\right) \ (n \to \infty),$$

$$\bar{F}_1(n) \sim \left(1 \cdot \frac{1}{1-\eta}\right)^n \left(\frac{1}{\eta}\frac{1}{n^2} + 0\left(\frac{1}{n^3}\right)\right) = \left(\frac{1}{1-\eta}\right)^n \left(\frac{1}{\eta}\frac{1}{n^2} + 0\left(\frac{1}{n^3}\right)\right) \ (n \to \infty),$$
$$\bar{F}_0(n) \sim 1^n \left(\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right) \ (n \to \infty),$$

and hence $v_0 = v_2 = 1$ and $v_1 = \frac{1}{1-\eta}$.
Now for $j = 2$ we derive a recurrence for

$$v_2^{-n}\bar{F}_2(n) = 4^{-n}\binom{2n}{n}\sum_{i=1}^{n}\frac{4^i\left(\frac{1}{1-\eta}\right)^i}{i^2\binom{2i}{i}}\sum_{j=1}^{i}\frac{(1-\eta)^j}{j},$$

and find

$$\begin{aligned}0 = {} & (1+n)(2+n)(1+2n)f(n) + (2+n)\left(-17+6\eta-20n+7\eta n-6n^2+2\eta n^2\right)f(1+n)\\ & + \left(85-61\eta+104n-72\eta n+43n^2-29\eta n^2+6n^3-4\eta n^3\right)f(2+n)\\ & + 2(\eta-1)(3+n)^3 f(3+n).\end{aligned}$$

We compute a differential equation for the inverse Mellin transform of $v_2^{-n}\bar{F}_2(n)$ using the method presented in Section 1 and get

$$\begin{aligned}0 = {} & 2(\eta-1)f(x) + (11-14x-3\eta+14x\eta)f'(x)\\ & + \left(-5+17x-12x^2-7x\eta+12x^2\eta\right)f''(x) + 2x(1-x+x\eta)f^{(3)}(x).\end{aligned}$$

with the general solution $d_{1,2}f_{1,2}(x) + d_{2,2}f_{2,2}(x) + d_{3,2}f_{3,2}(x)$ with $d_{1,2}, d_{2,2}, d_{3,2} \in \mathbb{K}(\eta)$ and

$$f_{1,2}(x) = \frac{1}{\sqrt{1-x}\sqrt{x}},$$
$$f_{2,2}(x) = \frac{\sqrt{1-x}\sqrt{x} - 2\sqrt{1-x}x^{3/2} + 4\mathrm{G}\left(\sqrt{1-\tau}\sqrt{\tau}, x\right)}{\sqrt{1-x}\sqrt{x}},$$
$$\begin{aligned}f_{3,2}(x) = {} & \frac{2\mathrm{G}\left(\sqrt{1-\tau}\sqrt{\tau}, x\right)}{(1-\eta)\sqrt{1-x}\sqrt{x}} - \frac{\mathrm{G}\left(\frac{1}{1-\tau+\eta\tau}, x\right)}{1-\eta} + \frac{\eta\mathrm{G}\left(\frac{1}{1-\tau+\eta\tau}, x\right)}{1-\eta} + \frac{2x\mathrm{G}\left(\frac{1}{1-\tau+\eta\tau}, x\right)}{1-\eta}\\ & - \frac{2\eta x\mathrm{G}\left(\frac{1}{1-\tau+\eta\tau}, x\right)}{1-\eta} - \frac{\mathrm{G}\left(\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}, x\right)}{(1-\eta)\sqrt{1-x}\sqrt{x}} - \frac{\eta\mathrm{G}\left(\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau}, x\right)}{(1-\eta)\sqrt{1-x}\sqrt{x}}\\ & - \frac{4\mathrm{G}\left(\sqrt{1-\tau}\sqrt{\tau}, \frac{1}{1-\tau+\eta\tau}, x\right)}{(1-\eta)\sqrt{1-x}\sqrt{x}} + \frac{4\eta\mathrm{G}\left(\sqrt{1-\tau}\sqrt{\tau}, \frac{1}{1-\tau+\eta\tau}, x\right)}{(1-\eta)\sqrt{1-x}\sqrt{x}}.\end{aligned}$$

Since the solution has no pole in $0 < x < 1$ we can set $a_2 = 0$.
Next for $j = 1$ derive a recurrence for

$$v_1^{-n}\bar{F}_1(n) = (1-\eta)^n 4^{-n}\binom{2n}{n}\sum_{i=1}^{n}\frac{4^i\left(\frac{1}{1-\eta}\right)^i}{i^2\binom{2i}{i}},$$

and find

$$0 = -(\eta-1)(1+n)(1+2n)f(n) + \left(-8+6\eta-11n+7\eta n-4n^2+2\eta n^2\right)f(1+n)$$

$$+2(2+n)^2 f(2+n).$$

Now we compute a differential equation and get

$$0 = 2f(x) + 3(-1+\eta+2x)f'(x) + 2x(-1+\eta+x)f''(x),$$

for which we find the solution $d_{1,1}f_{1,1}(x) + d_{2,1}f_{2,1}(x)$ with $d_{1,1}, d_{2,1} \in \mathbb{K}(\eta)$ and

$$\begin{aligned} f_{1,1}(x) &= \frac{1}{\sqrt{1-\eta-x}\sqrt{x}}, \\ f_{2,1}(x) &= \frac{\left(\sqrt{1-\eta-x}\sqrt{x} - \eta\sqrt{1-\eta-x}\sqrt{x} - 2\sqrt{1-\eta-x}x^{3/2} + 4\mathrm{G}\left(\sqrt{1-\eta-\tau}\sqrt{\tau}, x\right)\right)}{(1-\eta)^2\sqrt{1-\eta-x}\sqrt{x}}. \end{aligned}$$

Again, since the solution has no pole in $0 < x < 1$ we can set $a_1 = 0$.
Finally for $j = 0$ we derive a recurrence for

$$v_0^{-n}\bar{F}_0(n) = 4^{-n}\binom{2n}{n},$$

and find

$$0 = (-1-2n)f(n) + 2(1+n)f(1+n).$$

We compute a differential equation and get

$$0 = (-1+2x)f(x) + 2(x-1)xf'(x) = 0,$$

which has the solution $d_{1,0}f_{1,0}(x)$ with $d_{1,0} \in \mathbb{K}(\eta)$ and

$$f_{1,0}(x) = \frac{1}{\sqrt{1-x}\sqrt{x}}.$$

Since the solution has no pole in $0 < x < 1$ we again set $a_0 = 0$. Since $v_0 = v_2$, $a_0 = a_2$ and $f_{1,0}(x) = f_{1,2}(x)$ we don't have to consider this solution separately.
Summarizing we have

$$\begin{aligned} F(n) &= \sum_{j=1}^{2} v_j^n \left(d_{0,j} + \sum_{i=1}^{b_j} d_{i,j} \int_0^1 (x^n - a_j^n) f_{i,j}(x) dx \right) \\ &= d_{0,2} + d_{1,2}\int_0^1 x^n f_{1,2}(x)dx + d_{2,2}\int_0^1 x^n f_{2,2}(x)dx + d_{3,2}\int_0^1 x^n f_{3,2}(x)dx \\ &\quad + \frac{1}{(1-\eta)^n}\left(d_{0,1} + d_{1,1}\int_0^1 x^n f_{1,1}(x)dx + d_{2,1}\int_0^1 x^n f_{2,1}(x)dx \right). \end{aligned}$$

And it remains to fix the $d_{i,j}$. Therefore we compute the Mellin transforms and get

$$\begin{aligned} F(n) &= d_{0,2} + d_{1,2}\left(-\frac{2\sqrt{\pi}}{4^n}\binom{2n}{n} \right) + d_{2,2}\left(-\frac{1}{4^n}\binom{2n}{n}\sum_{i=1}^{n}\frac{4^i}{\binom{2i}{i}i^2} + \cdots \right) \\ &\quad + d_{3,2}\left(-\frac{\binom{2n}{n}}{(1-\eta)4^n}\sum_{i=1}^{n}\frac{(1-\eta)^i\sum_{j=1}^{i}\frac{4^j}{\binom{2j}{j}(1-\eta)^j j^2}}{i} + \cdots \right) + d_{0,1}\frac{1}{(1-\eta)^n} \end{aligned}$$

$$+d_{1,1}\left(\frac{-\sqrt{\eta}\binom{2n}{n}}{4^n}\sum_{i=1}^{n}\frac{4^i}{\binom{2i}{i}(1-\eta)^i i}+\cdots\right)+d_{2,1}\left(\frac{-\binom{2n}{n}}{4^n}\sum_{i=1}^{n}\frac{4^i}{\binom{2i}{i}(1-\eta)^i i^2}+\cdots\right).$$

Now it is straightforward to determine the $d_{i,j}$ by checking initial values and we find

$$\begin{aligned}
F(n) = {} & \int_0^1 x^n \frac{1}{12\pi\sqrt{1-x}\sqrt{x}(1-\eta)^3}\Bigg(3\pi^2(1-\eta)^3-4\left(1+11\eta+11\eta^2+\eta^3\right)\\
& -192\sqrt{-\eta}(1+\eta)\mathrm{G}\left(\sqrt{1-\eta-\tau}\sqrt{\tau},1\right)\mathrm{G}\left(\frac{1}{1-\tau+\eta\tau},1\right)\\
& +384\mathrm{G}\left(\sqrt{1-\eta-\tau}\sqrt{\tau},1\right)^2\mathrm{G}\left(\frac{1}{1-\tau+\eta\tau},1\right)\\
& -24\pi(1-\eta)^3(1+\eta)\mathrm{G}\left(\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},1\right)\\
& +96\pi(1-\eta)^4\mathrm{G}\left(\frac{1}{1-\tau+\eta\tau},\sqrt{1-\tau}\sqrt{\tau},1\right)\\
& +192(1-\eta)^3(1+\eta)\mathrm{G}\left(\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\sqrt{1-\tau}\sqrt{\tau},1\right)\\
& -768(1-\eta)^4\mathrm{G}\left(\frac{1}{1-\tau+\eta\tau},\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},1\right)\Bigg)dx\\
& +\int_0^1 x^n\frac{2(1-\eta)\left(-\sqrt{1-x}\sqrt{x}+2\sqrt{1-x}x^{3/2}-4\mathrm{G}\left(\sqrt{1-\tau}\sqrt{\tau},x\right)\right)\mathrm{G}\left(\frac{1}{1-\tau+\eta\tau},1\right)}{\sqrt{1-x}\sqrt{x}}dx\\
& +\int_0^1 x^n\Bigg(-\frac{4\mathrm{G}\left(\sqrt{1-\tau}\sqrt{\tau},x\right)}{\sqrt{1-x}\sqrt{x}}+\frac{2(1+\eta)\mathrm{G}\left(\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},x\right)}{\sqrt{1-x}\sqrt{x}}\\
& +\frac{2\left(\sqrt{1-x}\sqrt{x}-2\sqrt{1-x}x^{3/2}-\sqrt{1-x}\sqrt{x}\eta+2\sqrt{1-x}x^{3/2}\eta\right)\mathrm{G}\left(\frac{1}{1-\tau+\eta\tau},x\right)}{\sqrt{1-x}\sqrt{x}}\\
& +\frac{2(4-4\eta)\mathrm{G}\left(\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau+\eta\tau},x\right)}{\sqrt{1-x}\sqrt{x}}\Bigg)dx\\
& +\frac{1}{(1-\eta)^n}\Bigg(\int_0^1 x^n\frac{2\left(\eta+\eta^2+4\sqrt{-\eta}\mathrm{G}\left(\sqrt{1-\eta-\tau}\sqrt{\tau},1\right)\right)\mathrm{G}\left(\frac{1}{1-\tau+\eta\tau},1\right)}{\sqrt{x}\sqrt{1-x-\eta}(-1+\eta)\sqrt{-\eta}}dx\\
& +\int_0^1 x^n\frac{2\mathrm{G}\left(\frac{1}{1-\tau+\eta\tau},1\right)}{\sqrt{x}\sqrt{1-x-\eta}(-1+\eta)}\Bigg(-\sqrt{x}\sqrt{1-x-\eta}+2x^{3/2}\sqrt{1-x-\eta}\\
& +\sqrt{x}\sqrt{1-x-\eta}\eta-4\mathrm{G}\left(\sqrt{1-\eta-\tau}\sqrt{\tau},x\right)\Bigg)dx\Bigg).
\end{aligned}$$

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