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Generalised cuts and Wick Rotations

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It is proposed that the Wick rotation of the components of the loop momenta in the directions not spanned by the physical vectors can offer an useful and simple tool for the evaluation of maximally cut graphs.

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1. Introduction

In 1960 Dick Cutkosky [1] showed how to express the imaginary part of a Feynman graph by *cutting* a suitably choosen subset of its propagators, *i.e.* by replacing each of the propagators to be *cut* by a δ function of its propagator; if (k^2+m^2), is the denominator of the propagator, the replacement reads

$$\frac{1}{k^2+m^2-i\varepsilon} \to (2\pi)i\,\theta(\pm k_0)\delta(k^2+m^2)\ , \tag{1.1}$$

where the sign of k_0 (which enters quadratically in k^2) within the θ function depends on the orientation of the propagator line.

The derivation of Cutkosky was based on analytic function theory; shortly later Tini Veltman [2] gave an alternative, perhaps simpler derivation, obtained by using directly the time-space properties of the Feynman propagators (see for instance [3] for a recent and elementary description of Veltman's approach). When the Cutkosky-Veltman cuts are carried out, the graph is further cut in two separate sub-graphs, in agreement with perturbative unitarity; for that reason, the replacements (1.1) are usually referred to as *unitarity* cuts.

As, quite in general, a Feynman propagator has the form

$$\frac{1}{D-i\varepsilon} = P\left(\frac{1}{D}\right) + i\pi\,\delta(D)\ , \tag{1.2}$$

where D is quadratic in the components of momenta and mass, in close analogy with Eq.(1.1), a propagator line in which has been performed the replacement

$$\frac{1}{D-i\varepsilon} \to \delta(D) \tag{1.3}$$

is also called a *cut propagator*, even if, at variance with the *unitarity cut*, the θ function in the energy is not present.

In the *Integration by Parts* (IbP) approach [4], the $i\varepsilon$ in the denominator of the propagator does not play any role; given any IbP identity, one obtains another IbP identity with the same structure if a propagator $1/(D-i\varepsilon)$ is everywhere replaced by $1/(D+i\varepsilon)$, or, by subtracting them, if $1/(D-i\varepsilon)$ is replaced by $\delta(D)$, *i.e.* if a propagator is replaced by the corresponding *cut* propagator. Quite in general, an IbP identity can contain also a combination of amplitudes in which some of the propagators are missing, due to the ocurrence of a simplification of the kind

$$\frac{1}{D-i\varepsilon}\,D = 1\ ;$$

if that line is cut, one finds instead

$$\delta(D)\,D = 0\ ,$$

i.e. the terms without that propagator are at all missing in the IbP identity for the amplitude with the cut propagator, so that the resulting IbP identity is simpler. The simplest identity is obtained, obviously, when *all* the propagators of a graph are cut (socalled *maximally cut* graph).

In the Differential Equations approach [5, 6, 7] to the evaluation of Feynman amplitudes, the maximally cut Master Integrals satisfy an homogeneous differential equation. The direct evaluation

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of the corresponding maximally cut amplitude is much simpler than the evaluation of the original amplitude, and, when possible, can give a solution of the homogeneous differential equation. That is actually the case for the self-mass graphs with the simplest structure (the *Bubble* at 1-loop, the *Sunrise* [8] at two loop, the *Banana* [9] at 3-loop), whose solutions were indeed obtained by direct integration of the corresponding maximally cut graphs.
Maximally cut (and more in general, graphs with an arbitrary number of cut propagators) are also used as a powerful tool for the decomposition in scalar ampolitudes of multileg amplitudes in gauge theories (se for instance [10]).
In the case of the unitarity maximally cut graphs, the integration can be carried out in terms of the usual expression of the Feynman amplitude in the momentum representation; for maximal *non*-unitarity cut graphs, however, the direct, naïve integration of the usual loop momenta components can give a trivially vanishing result, as it is not possible to satisfy all the δ function requirements. A meaningful result can however be recovered by considering the maximally cut contribution of a suitable, analytically related expression, which satisfies the same IbP identities as the original Feynman amplitude; see for instance [11], where use is made of advanced mathematical methods.

In this contribution we propose an alternative elementary approach to the problem: we keep as integration variables the components of the loop momentum of the usual representation of the Feynman amplitudes, but perform a Wick rotation [12] on the loop components corresponding to the residual dimensions (*i.e.* the dimensions not spanned by the external physical vectors). It is obvious that the Wick rotated amplitude satisfies the same IbP identities as the original amplitude. In Section 2 we consider the triple cut of a Vertex amplitude and introduce the Wick rotation and in Section 3 the quadruple cut of a Box amplitude, in Section 4 we show in details as the Wick rotation works in the case of a single propagator, and finally in Section 5 how the Wick rotation reshuffles the various real and imaginary parts of the 1-loop *Bubble* amplitude in $d=2$ dimensions.

2. The triple cut of a Vertex amplitude

Let us consider as a first example the triple cut of the 1-loop Vertex amplitude, in d-continuous dimensions, for the process

$$p \to p_1 + p_2 \,,$$

with p timelike, p_1 and p_2 massless, $p_1^2 = p_2^2 = 0$, and the following kinematical configuration (in the *cms* of p)

$$\begin{aligned} s = &\ 4E^2 = -p^2 \,, \\ p &= (p_0 = 2E, p_x = 0, \vec{p}_r = 0) \,, \\ p_1 &= (E, E, 0) \,, \\ p_2 &= (E, -E, 0) \,, \end{aligned}$$

where the subscript r in p_r and in the following refers to the residual $(d-2)$ directions, not spanned by the physical vector components.
We take the three propagators of the internal lines to have all the same mass m, and the d-dimensional loop momentum k to have components $k_0, k_x, \vec{k}_r$. The triple cut of the Vertex amplitude can then be

written as

$$C_V(s) = \int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_x \; \Omega(d-2) \int_0^{\infty} k_r^{d-3} dk_r \; \delta(D_k)\delta(D_1)\delta(D_2) \; , \tag{2.1}$$

where we have introduced spherical coordinates for the residual components, k_r being their module, $\Omega(n) = 2\sqrt{\pi}/\Gamma(n/2)$ is the usual solid angle in n dimensions, and

$$\begin{aligned} D_1 &= (p_1+k)^2 + m^2 = -2Ek_0 + 2Ek_x + k^2 + m^2 \; , \\ D_2 &= (p_2-k)^2 + m^2 = +2Ek_0 + 2Ek_x + k^2 + m^2 \; , \\ D_k &= k^2 + m^2 = -k_0^2 + k_x^2 + k_r^2 + m^2 \end{aligned}$$

are the denominators of the three propagators. All the arguments of the various δ functions are forced to vanish in the integration, so that we can sum to any of the arguments any linear combination of the other arguments multiplied by a constant, or, also, by any (positive) power of the integration variables, *i.e.* the value of integral does not change under repeated replacements like, for instance

$$\delta(D_k)\; \delta(D_1) \to \delta(D_k)\; \delta(D_1 - D_k)$$

or, also,

$$\delta(D_k)\; \delta(D_1) \to \delta(D_k - k_0 D_1/2E) \; .$$

By a suitable combination of such replacements, we can rewrite Eq.(2.1) as

$$C_V(s) = \int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_x \; \Omega(d-2) \int_0^{\infty} k_r^{d-3} dk_r \; \delta(m^2 + k_r^2) \; \delta(4Ek_0) \; \delta(2Ek_x) \; . \tag{2.2}$$

The integration of the δ's is now trivial: but the final result is an uninteresting zero, as the argument of the first δ, $m^2 + k_r^2$, cannot vanish (k_r is real and positive in the whole integration range of k_d). Let us recall however that the residual components space is not spanned by the physical vectors, so that we can perform for those components a Wick rotation (or, perhaps more exactly, a Wick anti-rotation, from the Euclidean to the Minkoskian metric) carrying out for any of the residual components, say k_n, with $n = 1,..,(d-2)$, the substitutions

$$\begin{aligned} \int_{-\infty}^{\infty} dk_n &\to (-i) \int_{-\infty}^{\infty} dK_n \; , \\ k_n^2 &\to -K_n^2 \; , \end{aligned}$$

which in spherical coordinates gives

$$\begin{aligned} \int_0^{\infty} k_r^{d-3} dk_r &\to (-i)^{(d-2)} \int_0^{\infty} K_r^{d-3} dK_r \; , \\ k_r^2 &\to -K_r^2 \; . \end{aligned}$$

Carrying out the Wick rotations, $C_V(s)$ is transformed in a new quantity, say $C_V^W(s)$, which reads

$$C_V^W(s) = \int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_x \; (-i)^{(d-2)} \Omega(d-2) \int_0^{\infty} K_r^{d-3} dK_r \; \delta(m^2 - K_r^2) \; \delta(4Ek_0) \; \delta(2Ek_x) \; . \tag{2.3}$$

which is immediately integrated to give the nonvanishing, particularly simple result (the numerical constant is irrelevant for most purposes)

$$C_V^W(s) = (-i)^{(d-2)} \Omega(d-2) m^{d-4} \; \frac{1}{4s} \; . \tag{2.4}$$

3. The quadruple cut of a Box amplitude

As a second example, we take the quadruple cut of the 1-loop box amplitude, in d continuous dimensions

$$p_1 + p_2 \to p_3 + p_4 \ ,$$

with massless external lines, $p_i^2 = 0, \ i = 1,..,4$, while all the internal lines of the box have a same mass m (as in the scalar part of the QED light-light scattering). We consider the following kinematical configuration, in the *cms* system of p_1 and p_2,

$$\begin{aligned}
p_1 &= (p_{10} = E, p_{1x} = E, p_{1y} = 0, \vec{p}_{1r} = 0) \ , \\
p_2 &= (E, E, 0, 0) \ , \\
p_3 &= (E, E\cos\theta, E\sin\theta, 0) \ , \\
p_4 &= (E, E\cos\theta, -E\sin\theta, 0) \ , \\
s &= -(p_1+p_2)^2 = 4E^2 \ , \\
t &= -(p_1-p_3)^2 = -2E^2(1-\cos\theta) \ , \\
\cos\theta &= 1 + 2\frac{t}{s} \ , \\
\sin\theta &= 2\frac{\sqrt{-t(s+t)}}{s} \ .
\end{aligned}$$

The components of the external physical vectors span the time direction and the space directions x, y, while the subscript r (as in $\vec{p}_{1r}$) refers to the residual $(d-3)$ directions.
Following the previous section, one finds that the quadruple cut vanishes when using the standard loop momentum; we perform therefore the Wick rotation of the $(d-3)$ residual components of the loop momentum k, so that the quadruple cut of the box amplitude reads

$$C_B^W(s) = \int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \ (-i)^{(d-3)}\Omega(d-3) \int_0^{\infty} K_r^{d-4} dK_r \ \delta(D_k)\delta(D_1)\delta(D_2)\delta(D_3) \ , \tag{3.1}$$

where

$$\begin{aligned}
D_1 &= (p_1+k)^2 + m^2 = -2Ek_0 + 2Ek_x + k^2 + m^2 \ , \\
D_2 &= (p_2-k)^2 + m^2 = +2Ek_0 + 2Ek_x + k^2 + m^2 \ , \\
D_3 &= (p_1-p_3+k)^2 + m^2 = 2E^2(1-\cos\theta) + 2E(1-\cos\theta)k_x - 2E\sin\theta k_y + k^2 + m^2 \ , \\
D_k &= k^2 + m^2 = -k_0^2 + k_x^2 - K_r^2 + m^2 \ ,
\end{aligned}$$

are the denominators of the four propagators (in the Wick-rotated variables).
By summing to the argument of any δ functions suitable combinations of the other arguments, after a little algebra the previous integral becomes

$$\begin{aligned}
C_B^W(s) = &\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \ (-i)^{(d-3)}\Omega(d-3) \int_0^{\infty} K_r^{d-4} dK_r \\
&\delta\left(\frac{4m^2(s+t) - st}{4(s+t)} - K_r^2\right) \delta(-2Ek_0)\delta(4Ek_x)\delta\left(-t + \frac{\sqrt{-t(s+t)}}{E} k_y\right) \ ,
\end{aligned} \tag{3.2}$$

The integrations are now trivial, and give

$$C_B^W(s) = \frac{(-i)^{(d-3)}\Omega(d-3)}{\sqrt{-st[4m^2(s+t)-st]}}\left(\frac{4m^2(s+t)-st}{4(s+t)}\right)^{\frac{d-4}{2}} . \tag{3.3}$$

For comparison, let us recall the imaginary part in s of the light-light box graph in the usual variables (without the extra Wick rotation) and $d=4$ dimensions

$$\begin{aligned}\mathtt{Im}A(s,t) &= \int dk_0 dk_x dk_y dk_z\ \theta(p_{10}+k_0)\delta(D_1)\ \theta(p_{20}-k_0)\delta(D_2)\ \frac{1}{D_3D_4} \\ &= \frac{\pi}{2\sqrt{(-st)[4m^2(s+t)-st]}}\ln\frac{2m^2s-(s-4m^2)t+\sqrt{-(s-4m^2)t[4m^2(s+t)-st]}}{2m^2s-(s-4m^2)t-\sqrt{-(s-4m^2)t[4m^2(s+t)-st]}}\end{aligned} \tag{3.4}$$

Note the appearance, in Eq.s(3.3) and (3.4), of the *Mandelstam square root*

$$\sqrt{(-st)[4m^2(s+t)-st]} .$$

4. The Wick rotation

To see in more details how the Wick rotation works, consider the rather simple integral of a single Feynman propagator

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{dk_z}{-k_0^2+k_z^2+m^2-i\varepsilon} &= \frac{\pi}{\sqrt{m^2-k_0^2-i\varepsilon}} \\ &= \frac{\pi}{\sqrt{m^2-k_0^2}} ,\quad \texttt{real}\qquad\quad \texttt{if}\ m^2-k_0^2>0 , \\ &= i\frac{\pi}{\sqrt{k_0^2-m^2}} ,\quad \texttt{imaginary if}\ k_0^2-m^2>0 ,\end{aligned}$$

where the two results are obviously related to each other by analytic continuation.
But we are interested in this contribution with the integration of cut graphs, *i.e.* of δ functions, so we write the propagator as the sum of its principal value and δ function parts

$$\begin{aligned}\frac{1}{-k_0^2+k_z^2+m^2-i\varepsilon} &= P\left(\frac{1}{-k_0^2+k_z^2+m^2}\right)\quad \texttt{real} \\ &\quad + i\pi\delta(-k_0^2+k_z^2+m^2)\quad \texttt{imaginary}\end{aligned} \tag{4.1}$$

and consider separately the contributions of the two terms

$$\int_{-\infty}^{\infty} dk_z P\left(\frac{1}{-k_0^2+k_z^2+m^2}\right) = \frac{\pi}{\sqrt{m^2-k_0^2}} ,\quad \texttt{real if}\ m^2-k_0^2>0 , \tag{4.2}$$

$$= 0 ,\qquad \texttt{if}\ k_0^2-m^2>0 , \tag{4.3}$$

$$i\pi\int_{-\infty}^{\infty} dk_z\delta(-k_0^2+k_z^2+m^2) = 0 ,\qquad \texttt{if}\ m^2-k_0^2>0 , \tag{4.4}$$

$$= i\frac{\pi}{\sqrt{k_0^2-m^2}} ,\quad \texttt{imaginary if}\ k_0^2-m^2>0 . \tag{4.5}$$

In the above calculations, all the integrations are merely real (the imaginary unit i appears only as an overall constant), and the two results for each term (obviously) are not related by analytic continuation – as opposed to their sum.

We now repeat the calculations in terms of the Wick-rotated variable K_z, defined through the substitutions

$$\int_{-\infty}^{\infty} dk_z \to \int_{-\infty}^{\infty} (-i\, dK_z) \,, \qquad\qquad k_z^2 \to -K_z^2 \,.$$

We find, as in Eq.(4.1),

$$\begin{aligned}\frac{1}{-k_0^2 - K_z^2 + m^2 - i\varepsilon} &= P\left(\frac{1}{-k_0^2 - K_z^2 + m^2}\right) \quad \texttt{real} \\ &\quad + i\pi\delta(-k_0^2 - K_z^2 + m^2) \quad \texttt{imaginary}\end{aligned} \tag{4.6}$$

and, as expected,

$$\int_{-\infty}^{\infty} \frac{dk_z}{-k_0^2 + k_z^2 + m^2 - i\varepsilon} = \int_{-\infty}^{\infty} \frac{-i\, dK_z}{-k_0^2 - K_z^2 + m^2 - i\varepsilon} = \frac{\pi}{\sqrt{m^2 - k_0^2 - i\varepsilon}} \,.$$

But the contributions of the two terms of the decomposition (4.6) are

$$\int_{-\infty}^{\infty} (-i\, dK_z) P\left(\frac{1}{-k_0^2 - K_z^2 + m^2}\right) = 0 \,, \qquad \texttt{if}\ m^2 - k_0^2 > 0 \,, \tag{4.7}$$

$$= i\frac{\pi}{\sqrt{m^2 - k_0^2}} \,, \quad \texttt{imaginary if}\ k_0^2 - m^2 > 0 \,, \tag{4.8}$$

$$i\pi \int_{-\infty}^{\infty} (-i\, dK_z)\delta(-k_0^2 - K_z^2 + m^2) = \frac{\pi}{\sqrt{k_0^2 - m^2}} \,, \quad \texttt{real if}\ m^2 - k_0^2 > 0 \,, \tag{4.9}$$

$$= 0 \,, \qquad \texttt{if}\ k_0^2 - m^2 > 0 \,. \tag{4.10}$$

It can be observed that Eq.(4.2) is different from Eq.(4.7) but equal to Eq.(4.8), while Eq.(4.3) is different from Eq.(4.8) but equal to Eq.(4.7), *etc.*

5. 1-loop Bubble and Wick rotation

As a less trivial example, consider the 1-loop scalar *Bubble* amplitude with external momentum p and two different masses $M, m, M > m$, in $d = 2$ dimensions for simplicity; for p timelike, $p_0 = E > 0, p_z = 0$, defined as

$$B(E^2) = -i \int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_z \frac{1}{(D_m - i\varepsilon)(D_M - i\varepsilon)} \,, \tag{5.1}$$

where

$$\begin{aligned} D_m &= k^2 + m^2 = -k_0^2 + k_z^2 + m^2 \,, \\ D_M &= (p-k)^2 + M^2 = -(k_0 - E)^2 + k_z^2 + M^2 \,. \end{aligned}$$

By using the decomposition

$$\frac{1}{D_m - i\varepsilon} = P\left(\frac{1}{D_m}\right) + i\pi\delta(D_m), \qquad \frac{1}{D_M - i\varepsilon} = P\left(\frac{1}{D_M}\right) + i\pi\delta(D_M)\ ,$$

one can define the four quantities

$$\begin{aligned}
B(P_m P_M, E^2) &= -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_z P\left(\frac{1}{D_m}\right) P\left(\frac{1}{D_M}\right) \quad \texttt{imaginary} \\
B(P_m \delta_M, E^2) &= -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_z P\left(\frac{1}{D_m}\right)\ i\pi\delta(D_M) \quad \texttt{real} \\
B(\delta_m P_M, E^2) &= -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_z\ i\pi\delta(D_m)\ P\left(\frac{1}{D_M}\right) \quad \texttt{real} \\
B(\delta_m \delta_M, E^2) &= -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_z\ i\pi\delta(D_m)\ i\pi\delta(D_M) \quad \texttt{imaginary}
\end{aligned} \tag{5.2}$$

whose sum is, by construction, the *Bubble* amplitude

$$B(E^2) = B(P_m P_M, E^2) + B(P_m \delta_M, E^2) + B(\delta_m P_M, E^2) + B(\delta_m \delta_M, E^2)\ . \tag{5.3}$$

As the external vector p has no component along the z direction, one can perform the Wick-rotation of k_z into K_z, obtaining the corresponding quantities

$$B^W(E^2) = -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} (-i\, dK_z) \frac{1}{(D_m^W - i\varepsilon)(D_M^W - i\varepsilon)}\ , \tag{5.4}$$

with

$$\begin{aligned}
D_m^W &= k^2 + m^2 = -k_0^2 - K_z^2 + m^2\ , \\
D_M^W &= (p-k)^2 + M^2 = -(k_0 - E)^2 - K_z^2 + M^2\ ,
\end{aligned}$$

and

$$\begin{aligned}
B^W(P_m P_M, E^2) &= -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} (-i\, dK_z) P\left(\frac{1}{D_m^W}\right) P\left(\frac{1}{D_M^W}\right) \quad \texttt{real} \\
B^W(P_m \delta_M, E^2) &= -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} (-i\, dK_z) P\left(\frac{1}{D_m^W}\right)\ i\pi\delta({D_M}^W) \quad \texttt{imaginary} \\
B^W(\delta_m P_M, E^2) &= -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} (-i\, dK_z)\ i\pi\delta({D_m}^W)\ P\left(\frac{1}{D_M^W}\right) \quad \texttt{imaginary} \\
B^W(\delta_m \delta_M, E^2) &= -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} (-i\, dK_z)\ i\pi\delta(D_m^W)\ i\pi\delta(D_M^W) \quad \texttt{real.}
\end{aligned} \tag{5.5}$$

It is apparent that the four quantities defined in Eq.s(5.5) are not equal to the corresponding quantities of Eq.s(5.2), but their sum, as we will also check in a moment by an explicit calculation, is equal to $B(E^2)$

$$\begin{aligned}
B^W(E^2) &= B^W(P_m P_M, E^2) + B^W(P_m \delta_M, E^2) + B^W(\delta_m P_M, E^2) + B^W(\delta_m \delta_M, E^2) \\
&= B(E^2)\ .
\end{aligned}$$

For $E = U$, $0 < U < M - m$, defining $\sqrt{R(U^2)} = \sqrt{(M+m)^2 - U^2}\sqrt{(M-m)^2 - U^2}$, we find

$$\begin{aligned}
\sqrt{R(U^2)} \times B(P_m P_M, U^2) &= -i\pi^2 \\
\sqrt{R(U^2)} \times B^W(P_m P_M, U^2) &= +2\pi \ln \frac{\sqrt{(M+m)^2 - U^2} + \sqrt{(M-m)^2 - U^2}}{\sqrt{(M+m)^2 - U^2} - \sqrt{(M-m)^2 - U^2}} \\
\sqrt{R(U^2)} \times B(P_m \delta_M, U^2) &= -2\pi \ln \frac{\sqrt{(U+M)^2 - m^2} + \sqrt{(U-M)^2 - m^2}}{\sqrt{(U+M)^2 - m^2} - \sqrt{(U-M)^2 - m^2}} \\
\sqrt{R(U^2)} \times B^W(P_m \delta_M, U^2) &= +i\pi^2 \\
\sqrt{R(U^2)} \times B(\delta_m P_M, U^2) &= +2\pi \ln \frac{\sqrt{M^2 - (U-m)^2} + \sqrt{M^2 - (U+m)^2}}{\sqrt{M^2 - (U-m)^2} - \sqrt{M^2 - (U+m)^2}} \\
\sqrt{R(U^2)} \times B^W(\delta_m P_M, U^2) &= -i\pi^2 \\
\sqrt{R(U^2)} \times B(\delta_m \delta_M, U^2) &= +i\pi^2 \\
\sqrt{R(U^2)} \times B^W(\delta_m \delta_M, U^2) &= 0
\end{aligned} \tag{5.6}$$

whose sums are

$$B^W(U^2) = B(U^2) = \frac{2\pi}{\sqrt{R(U^2)}} \ln \frac{\sqrt{(M+m)^2 - U^2} + \sqrt{(M-m)^2 - U^2}}{\sqrt{(M+m)^2 - U^2} - \sqrt{(M-m)^2 - U^2}} .$$

Note the appearance, for ordinary as well for Wick-rotated coordinates, of terms giving imaginary contributions, which cancel out in both sums (the *Bubble* amplitude is real for energy U in the considered range $0 < U < M - m$).

As another check, for $E = Z$, $M + m < Z$, defining $\sqrt{R(Z^2)} = \sqrt{Z^2 - (M+m)^2}\sqrt{Z^2 - (M-m)^2}$ we find

$$\begin{aligned}
\sqrt{R(Z^2)} \times B(P_m P_M, Z^2) &= +i\pi^2 \\
\sqrt{R(Z^2)} \times B^W(P_m P_M, Z^2) &= -2\pi \ln \frac{\sqrt{Z^2 - (M-m)^2} + \sqrt{Z^2 - (M+m)^2}}{\sqrt{Z^2 - (M-m)^2} - \sqrt{Z^2 - (M+m)^2}} \\
\sqrt{R(Z^2)} \times B(P_m \delta_M, Z^2) &= -2\pi \ln \frac{\sqrt{(Z+M)^2 - m^2} + \sqrt{(Z-M)^2 - m^2}}{\sqrt{(Z+M)^2 - m^2} - \sqrt{(Z-M)^2 - m^2}} \\
\sqrt{R(Z^2)} \times B^W(P_m \delta_M, Z^2) &= i\pi^2 \\
\sqrt{R(Z^2)} \times B(\delta_m P_M, Z^2) &= -2\pi \ln \frac{\sqrt{(Z+m)^2 - M^2} + \sqrt{(Z-m)^2 - M^2}}{\sqrt{(Z+m)^2 - M^2} + \sqrt{(Z-m)^2 - M^2}} \\
\sqrt{R(Z^2)} \times B^W(\delta_m P_M, Z^2) &= +i\pi^2 \\
\sqrt{R(Z^2)} \times B(\delta_m \delta_M, Z^2) &= +i\pi^2 \\
\sqrt{R(Z^2)} \times B^W(\delta_m \delta_M, Z^2) &= 0
\end{aligned} \tag{5.7}$$

whose sums are

$$B^W(Z^2) = B(Z^2) = -\frac{2\pi}{\sqrt{R(Z^2)}} \ln \frac{\sqrt{Z^2 - (M-m)^2} + \sqrt{Z^2 - (M+m)^2}}{\sqrt{Z^2 - (M-m)^2} - \sqrt{Z^2 - (M+m)^2}} + \frac{2\pi^2}{\sqrt{R(Z^2)}} i .$$

Note the appearance, when the energy Z is in the range $M+m < Z < \infty$, of the expected imaginary part of the *Bubble* amplitude.
For comparison, we recall that the Cutkosky-Veltman rule gives for the *Bubble* amplitude as defined in Eq.(5.1), (with the ordinary components of the loop momentum k),

$$\mathrm{Im}B(E^2) = \frac{1}{2}\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_z\, (2\pi)\theta(k_0)\delta(D_m)\,(2\pi)\theta(p_0-k_0)\delta(D_M) = \frac{2\pi^2}{\sqrt{R(E^2)}} \,,$$

where the θ functions accompanying the δ's give a non vanishing result only for $E > (M+m)$. In contrast, the terms corresponding to the contribution of the two δ functions Eq.(5.2), and whose value is given in Eq.s(5.6) and (5.7), give, in both ranges of the energy, the same contribution $i\pi^2$ to the imaginary part, which for $0 < U < (M-m)$ is cancelled by another contribution, while for $M+m < Z < \infty$ is just one half of the actual imaginary part of the amplitude.

Conclusions
We have proposed that the Wick rotation of the residual components of the loop momenta (*i.e.* of the components not spanned by the external pysical vectors) can provide with a simple way for evaluating otherwise vanishing maximally cut Feynman amplitudes, showing how the method works at 1 loop. It is hoped that the approach might be of help even at 2 loops or more.

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