PROCEEDINGS OF SCIENCE

Aspects of Two-Dimensional Conformal Field Theories

Xi Yin*
Jefferson Physical Laboratory, Harvard University,
Cambridge, MA 02138 USA
xiyin@fas.harvard.edu

In this set of lecture notes, I give an introduction to the operator approach to two-dimensional conformal field theories, including some old and new results of the conformal bootstrap program in two dimensions.

Theoretical Advanced Study Institute Summer School 2017 "Physics at the Fundamental Frontier"
4 June - 1 July 2017
Boulder, Colorado

*Speaker.

1. Introduction

Quantum field theory, a pillar of modern theoretical physics, was built over rather unsatisfactory foundations. There are essentially three approaches: (1) Lagrangian and path integral, (2) Hamiltonian with UV regularization, and (3) operator algebra. The Lagrangian approach is useful for formulating effective theories and for perturbation theory, sometimes beyond perturbation theory for symmetry protected observables, and for non-perturbative formulations through lattice discretization of the Euclidean spacetime. The Hamiltonian approach is practically useful either through spatial lattice discretization or truncated conformal space approach. The operator algebra approach is the most intrinsic formulation of QFTs, and is best understood in the context of local operators in conformal field theories (CFTs). Large classes of Poincaré invariant local quantum field theories can be described as renormalization group flows starting from a CFT in the UV.[1] Thus, the classification and "solution" of CFTs are important in the study of quantum field theories.[2]

Conformal field theories in two dimensions enjoy an infinite dimensional conformal symmetry group. They are also subject to the important constraint of modular invariance, which allows for a consistency definition of a 2D CFT on any surface. While plenty of examples are known, we are far from classifying or solving most 2D CFTs. For instance, every Calabi-Yau manifold gives rise to a superconformal field theory based on a nonlinear sigma model, whose spectra of operators are generally not known beyond those protected by supersymmetry, at a generic point in their moduli space of exactly marginal deformations; furthermore, the Calabi-Yau models merely occupy a tiny corner of the landscape of 2D CFTs.

In this set of lectures, I will outline the operator definition of 2D CFTs, discuss a number of standard examples, and sketch the bootstrap method one may use to classify and solve CFTs.

2. Defining properties of a 2D CFT

A 2D unitary CFT admits a Hilbert space $\mathscr{H}$ of local operators, which by the state/operator mapping is canonically isomorphic to the Hilbert space of the states of the CFT on the cylinder $\mathbb{R}\times S^1$. For now, we will assume the CFT is *compact*, in the sense that the operator spectrum is discrete and in particular contains the identity operator 1, which is mapped to a normalizable ground state $|0\rangle$ on the cylinder. Later we will consider examples of noncompact CFTs in which this property is suitably relaxed.

Furthermore, a 2D CFT has a conserved traceless stress-energy tensor $T_{\mu\nu}$. The traceless condition $T^\mu{}_\mu = 0$ is equivalent to conformal invariance, which holds in flat 2D spacetime as an operator equation. On a curved spacetime, this condition is replaced by $T^\mu{}_\mu = -\frac{c}{12}R$ due to conformal anomaly, where c is the central charge of the CFT and R is the Ricci scalar of the 2D spacetime.[3]

[1] A piece of folklore states that all QFTs come from CFTs in the UV; this is incorrect: standard counterexamples are gauge theories in 2 and 3 dimensions.

[2] By "solving" a CFT, typically, we mean determining the spectrum of local operators and the algebra of operator product expansion.

[3] Of course, in this case one may insist on speaking of the traceless part of $T_{\mu\nu}$, but such an operator would not be conserved.

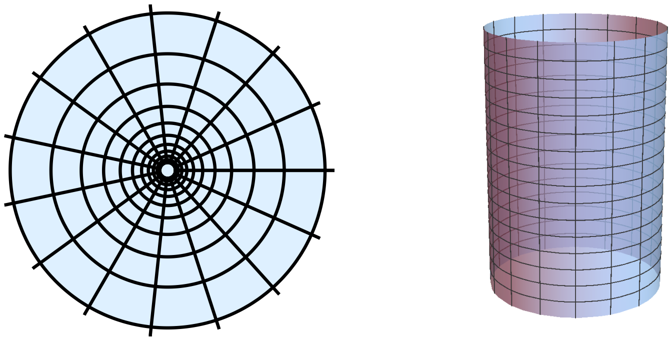


Figure 1: A local operator at the origin of the plane is mapped to a state of the CFT on the circle/cylinder under the state/operator correspondence.

Under operator product expansion, the stress-energy tensor generates the Virasoro algebra. The space of local operators $\mathscr{H}$ forms a representation of the Virasoro algebra, which can be decomposed into a direct sum of irreducible representations, each of which contains a lowest dimension "primary" operator.

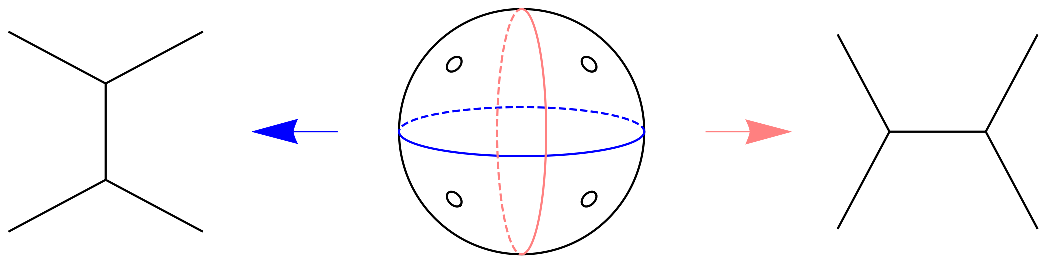


Figure 2: The associativity of the OPE is equivalent to the crossing symmetry of sphere 4-point functions.

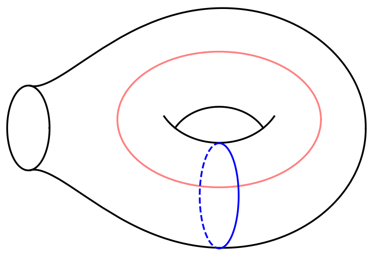


Figure 3: Modular covariance of the torus one point function is another key consistency condition on 2D CFTs.

There is a well defined correlation function of local operators at separated points on the plane (or the conformally equivalent Riemann sphere), which is entirely specified by the operator product expansion. By the completeness of local operators as states on the circle, via a plumbing construction one can construct the correlation functions of local operators on any Riemann surface. The consistency of the CFT on a general Riemann surface demands that the correlation functions computed from plumbing constructions based on different pair-of-pants decompositions must agree. This is in fact equivalent to the associativity of the OPE (crossing invariance of sphere 4-point functions) together with modular covariance of torus 1-point functions of primaries [1].

2.1 Virasoro algebra

On the Euclidean plane, which we parameterize with complex coordinates $(z,\bar{z})$ with the line element $ds^2 = dzd\bar{z}$, the stress-energy tensor has two independent components $T(z) \equiv T_{zz}(z)$ and $\widetilde{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z})$. The OPE of a pair of stress-energy tensors, say the holomorphic components, take the form

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0), \tag{2.1}$$

where we have omitted non-singular terms in the $z \to 0$ limit. The order z^{-2} term on the RHS is determined by the Ward identity associated with dilatation symmetry generated by the Noether current $zT(z)$, while the order z^{-1} term is determined by the Ward identity associated with translation symmetry generated by $T(z)$. Assuming the absence of negative dimension operators, and that the only dimension zero operator being proportional to the identity, (2.1) is the most general possibility for the TT OPE (note that a z^{-3} term would be in conflict with the commutativity of the OPE).

In the presence of a local operator $\mathscr{O}(0)$ at the origin, $T(z)$ is holomorphic on $\mathbb{C}\backslash\{0\}$. Thus, $T(z)$ admits the Laurent series representation

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}, \quad L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z), \tag{2.2}$$

that is convergent (in the sense of correlation functions) on an annulus centered at the origin where no other local operators are inserted. Likewise, we define $\bar{L}_n$ as the anti-holomorphic Laurent coefficients of $\widetilde{T}(\bar{z})$. L_n generate the Virasoro algebra with central charge c,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}, \tag{2.3}$$

and similarly $\bar{L}_n$ generate another copy of the Virasoro algebra, of possibly a different central charge $\widetilde{c}$.

A general infinitesimal conformal transformation is generated by the Noether current

$$(j_z, \widetilde{j}_{\bar{z}}) = (-\varepsilon(z)T(z), -\widetilde{\varepsilon}(\bar{z})\widetilde{T}(\bar{z})). \tag{2.4}$$

The corresponding transformation of a local operator $\mathscr{O}(z,\bar{z})$ is

$$\delta_\varepsilon \mathscr{O}(z,\bar{z}) = -\oint_{C_z} \frac{dw}{2\pi i}\varepsilon(w)T(w)\mathscr{O}(z,\bar{z}) + \oint_{C_z} \frac{d\bar{w}}{2\pi i}\widetilde{\varepsilon}(\bar{w})\widetilde{T}(\bar{w})\mathscr{O}(z,\bar{z}), \tag{2.5}$$

where C_z is a counterclockwise contour encircling z. In particular, from this one can deduce the infinitesimal conformal transformation of the stress-energy tensor $T(z)$ itself,

$$\delta_\varepsilon T(z) = -\frac{c}{12}\partial^3\varepsilon(z) - 2\partial\varepsilon(z)T(z) - \varepsilon(z)\partial T(z) \tag{2.6}$$

PoS(TASI2017)003

We may obtain a finite conformal transformation by composing together infinitesimal ones. It is useful to characterize these in terms of an associated holomorphic diffeomorphism, whose infinitesimal form is

$$z \mapsto z'(z) = z + \varepsilon(z). \tag{2.7}$$

Now passing to the finite conformal transformation represented by $z' = f(z)$, $T(z)$ is transformed into $T'(z)$ such that

$$\left(\frac{\partial z'}{\partial z}\right)^2 T'(z') = T(z) - \frac{c}{12}\{z', z\}, \tag{2.8}$$

where $\{z', z\} = \frac{2\partial_z^3 z' \partial_z z' - 3(\partial_z^2 z')^2}{2(\partial_z z')^2}$ is the Schwarzian derivative.

A particularly basic and important example of conformal transformation is the exponential map $z = e^{-iw}$ from the Euclidean cylinder parameterized by w to the complex plane parameterized by z,

$$T_{ww}(w) = (\partial_w z)^2 T_{zz}(z) + \frac{c}{24}. \tag{2.9}$$

The Laurent expansion of $T_{zz}(z)$ in the presence of an operator at the origin now turns into a Fourier expansion of $T_{ww}(w)$ on the cylinder, of the form

$$T_{ww}(w) = -\sum_{n=-\infty}^{\infty} L_n e^{inw} + \frac{c}{24}. \tag{2.10}$$

Likewise, we have

$$\widetilde{T}_{\bar w \bar w}(\bar w) = -\sum_{n=-\infty}^{\infty} \bar L_n e^{-in\bar w} + \frac{\widetilde{c}}{24}. \tag{2.11}$$

Writing $w = \sigma + i\tau$, we can analytically continue to Lorentzian signature by taking $\tau = -it$. After doing so, $w = \sigma + \tau$ and $\bar w = \sigma - t$ are now two independent real parameters. Furthermore, in the Lorentzian signature, the stress-energy tensor components T_{ww} and $\widetilde{T}_{\bar w \bar w}$ are Hermitian operators. Consequently, the Hermitian conjugation of the operator L_n is L_{-n}, and likewise $\bar L_n^\dagger = \bar L_{-n}$. The states of a unitary CFT on the (Lorentzian) cylinder by definition form unitary representations of the holomorphic and anti-holomorphic Virasoro algebra.

2.2 Representations of Virasoro algebra

We will restrict our attention to representations of the Virasoro algebra with the property that L_0 is bounded from below, which in particular is true for all unitary representations. Irreducible representations of this type are generated by a primary state $|h\rangle$ on the cylinder with the lowest L_0 eigenvalue h, that obeys

$$L_0|h\rangle = h|h\rangle, \quad L_n|h\rangle = 0,\ n > 0. \tag{2.12}$$

Under the state/operator mapping, $|h\rangle$ maps to a primary operator $\mathscr{O}_h$ at the origin. The primary operator $\mathscr{O}_h(z)$ (omitted anti-holormophic dependence) transforms under a general conformal transformation according to

$$\mathscr{O}'_h(z') = (\partial_z z')^{-h} \mathscr{O}_h(z). \tag{2.13}$$

A basis for the general Viraosro descendants is

$$L_{-N}|h\rangle \equiv L_{-n_1} \cdots L_{-n_k}|h\rangle, \tag{2.14}$$

where we denote by N the integer partition $\{n_1, n_2, \cdots, n_k\}$, say in descending order $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$. We will refer to $|N| = \sum_{j=1}^k n_j$ as the level of the state. The level n Gram matrix is defined as the matrix of the inner product of level N basis states,

$$G^{(n)}_{NM} = \langle h | L^\dagger_{-N} L_{-M} | h \rangle, \quad |N| = |M| = n, \tag{2.15}$$

with the normalization convention $\langle h|h\rangle = 1$. The matrix element $G^{(n)}_{NM}$ is a polynomial function of c and h. Of particular interest is the determinant of $G^{(n)}$, known as the Kac determinant, which is given by the formula [2, 3]

$$\det G^{(n)} = K_n \prod_{1 \leq rs \leq n} (h - h_{r,s})^{P(n-rs)}. \tag{2.16}$$

Here K_n is a positive constant, $P(m)$ is the partition number, obeying $\sum_{m=0}^\infty P_m q^m = \prod_{k=1}^\infty (1-q^k)^{-1}$, and $h_{r,s}$ are given by

$$h_{r,s} = \frac{c-1}{24} + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2, \quad \alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}. \tag{2.17}$$

It will be useful later to define $c = 1 + 6Q^2$, $Q \equiv b + b^{-1}$, and write

$$h_{r,s} = \frac{Q^2}{4} - \frac{1}{4}(rb + sb^{-1})^2. \tag{2.18}$$

Note that for real b, $c > 25$ and $\alpha_\pm$ are purely imaginary.

The intuition behind (2.16) is that a primary ν_h of weight $h = h_{r,s}$ admits a null Virasoro descendant at level rs, which may be written in the form

$$\chi_{rs} = \sum_{|N|=rs} \chi^N_{rs} L_{-N} \nu_{h_{r,s}}, \tag{2.19}$$

where the coefficients χ^N_{rs} are normalized such that $\chi^{\{1,\cdots,1\}}_{rs} = 1$. At a higher level $n > rs$, there will be $P(n-rs)$ descendants of the form $L_{-M}\chi_{rs}$ with $|M| = n - rs$ that also have zero norm, giving rise to an order $P(n-rs)$ zero of the Kac determinant.

It follows from $h_{1,1} = 0$ that a unitary representation must have $h \geq 0$ ($h = 0$ corresponds to the identity/vacuum representation). If $c > 1$, $h_{r,s}$ are either non-positive real numbers (for $c \geq 25$) or complex (for $1 < c < 25$), and $\det G^{(n)}$ being a real function of h cannot have zeros on the positive real axis, and therefore must be positive for all $h > 0$.

If $c = 1$, $\det G^{(n)}$ is non-negative for $h \geq 0$, but it does have zeros along the positive real axis since some of the $h_{r,s}$'s are positive real numbers; these give rise to null states in a representation of the $c = 1$ Virasoro algebra.

For $c < 1$, one can show that if h is not equal to one of the $h_{r,s}$'s, $\det G^{(n)}$ will be negative for some n. A careful analysis leads to the following discrete possible unitarity representations with $c < 1$ [4, 5]:

$$c = 1 - \frac{6}{m(m+1)}, \quad h = h_{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)} \text{ with } 1 \leq r \leq m-1, \ 1 \leq s \leq m, \tag{2.20}$$

where $m = 2, 3, \cdots$.

2.3 Associativity of OPE

The operator product expansion takes the general form

$$\mathscr{O}_i(z_1,\bar{z}_1)\mathscr{O}_j(z_2,\bar{z}_2) = \sum_k A_{ij}^k(z_{13},z_{23};\bar{z}_{13},\bar{z}_{23})\mathscr{O}_k(z_3,\bar{z}_3), \tag{2.21}$$

where the summation is over all operators in the CFT, labeled by the index k. Under a rescaling around z_3, of the form $z_{i3} \to \lambda z_{i3}$, $\bar{z}_{i3} \to \bar{\lambda}\bar{z}_{i3}$, the coefficient A_{ij}^k scales like $\lambda^{h_k-h_i-h_j}\bar{\lambda}^{\tilde{h}_k-\tilde{h}_i-\tilde{h}_j}$. (2.21) as an operator equation holds when inserted in a correlation function, where the expansion in z_{i3} and $\bar{z}_{i3}$ has a radius of convergence R, determined by the property that no other operator lie within the disc of radius R centered at z_3. This doesn't mean that the OPE is not applicable at all outside of the radius of convergence, but rather that we have to rearrange the basis $\mathscr{O}_k$ in a suitable way to make the expansion well defined in a correlation function. We will characterize this more precisely later through the analytic continuation of conformal blocks.

Now let us consider the product of three operators,

$$\mathscr{O}_i(z_1,\bar{z}_1)\mathscr{O}_j(z_2,\bar{z}_2)\mathscr{O}_k(z_3,\bar{z}_3). \tag{2.22}$$

We can first take the OPE of $\mathscr{O}_i\mathscr{O}_j$, expanding around the point z_4 in terms of $\mathscr{O}_m(z_4)$, and then take the OPE of $\mathscr{O}_m(z_4)$ with $\mathscr{O}_k(z_3)$, provided that z_4 is chosen appropriately so that $|z_{34}| > \max\{|z_{14}|,|z_{24}|\}$, or we can first take the OPE of $\mathscr{O}_j\mathscr{O}_k$, expanding around z_5 in terms of $\mathscr{O}_n(z_5)$, and then take the OPE of $\mathscr{O}_i(z_1)$ with $\mathscr{O}_n(z_5)$, provided that $|z_{15}| > \max\{|z_{25}|,|z_{35}|\}$. By construction, these two expansion should lead to the same result, which implies a set of consistency conditions on the coefficients A_{ij}^k.

We can exploit the full conformal symmetry by organizing the OPE according to primaries and descendants. For instance, the OPE of a pair of primaries take the form

$$\phi_i(z_1,\bar{z}_1)\phi_j(z_2,\bar{z}_2) = \sum_k C_{ij}^k \sum_{N,M} B_N(h_i,h_j;h_k|z_{13},z_{23})B_M(\tilde{h}_i,\tilde{h}_j;\tilde{h}_k|\bar{z}_{13},\bar{z}_{23})L_{-N}\bar{L}_{-M}\phi_k(z_3,\bar{z}_3), \tag{2.23}$$

where the summation is over primaries ϕ_k as well as the integer partitions N,M labeled holomorphic and anti-holomorphic Virasoro descendants. The functions $B_N(h_i,h_j;h_k|z_{13},z_{23})$ are entirely determined by conformal Ward identities and depend only on the central charge c which enters the Virasoro algebra. The coefficients C_{ij}^k are structure constants. If we choose the basis of primaries ϕ_k to have normalization 2-point function (on the plane or Riemann sphere), namely

$$\langle\phi_i(z_1,\bar{z}_1)\phi_j(z_2,\bar{z}_2)\rangle = \frac{\delta_{ij}}{z_{12}^{2h_i}\bar{z}_{12}^{2\tilde{h}_i}}, \tag{2.24}$$

then $C_{ijk} = C_{ij}^k$ is the coefficient of the three-point function $\langle\phi_i\phi_j\phi_k\rangle$ and is completely symmetric in i,j,k.

The set of primary operators ϕ_i together with the structure constants C_{ijk} determine the operator product algebra, thereby all correlation functions on the plane or sphere. They are subject to highly nontrivial constraints due to the associativity of the OPE, which we will explore later.

2.4 Modular invariance

Since every two-dimensional Riemannian manifold is conformally flat, a 2D CFT is canonically defined on any Riemann surface, up to a universal conformal anomaly which introduces a dependence on the metric within a conformal class. Furthermore, the correlation function of any set of local operators inserted on a general Riemann surface in a given CFT is determined via the plumbing construction by the structure constants C_{ijk}. The consistency of the CFT on the Riemann surface, which may be constructed by different plumbings, leads to an important set of consistent conditions known as modular invariance.[4]

A genus g surface as a smooth manifold can be constructed by gluing pair-of-pants, or three-holed spheres. A general genus $g > 1$ Riemann surface can be constructed by plumbing together $2g-2$ three-holed spheres. The genus one case can be constructed by plumbing the inner and outer boundaries of an annulus. The plumbing construction glues together a pair of circular boundaries of three-holed spheres (or two-holed discs) by an $PSL(2,\mathbb{C})$ map. A typical plumbing map takes the form

$$z' = q/z, \quad q \in \mathbb{C}^{\times}, \tag{2.25}$$

which identifies the boundary $|z| = r_1$ on one of the two-holed discs to the boundary $|z'| = r_2$ of the other two-holed disc, with $r_1 r_2 = |q|$. There is one complex modulus q associated with each of the $3g-3$ plumbing maps, giving rise to $3g-3$ complex structure moduli of the Riemann surface.

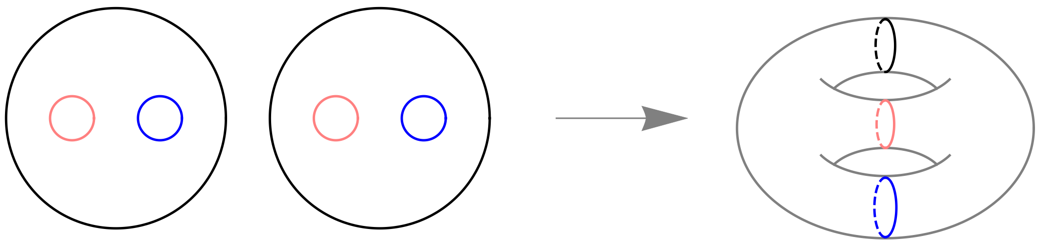


Figure 4: Plumbing construction for a genus two Riemann surface.

The plumbing construction gives a precise way to construct the partition function, and more generally, correlation functions, on an arbitrary genus g Riemann surface, based on the 3-point function of arbitrarily Virasoro descendants on the Riemann sphere, which are determined by conformal Ward identities in terms of the structure constants C_{ijk}. Modular invariance amounts to the statement that all possible plumbing constructions of the same (punctured) Riemann surface lead to the same answer for the partition function (correlation function).

The equivalence of different plumbing constructions of the same Riemann surface would be guaranteed provided that all sphere 4-point functions can be equivalently decomposed in two different channels, and that all torus 1-point functions constructed by plumbing together the inner and outer boundaries of a 1-punctured annulus are independent of choice of how the torus is cut open into an annulus. The former is equivalent to the associativity of OPE as already mentioned. The latter amounts to the modular covariance of the torus 1-point function of all primaries. Namely,

[4] A very interesting open question is the relationship between modular invariance and the notion of local Hilbert space in the sense of the "split property" [6, 7].

for every primary $\phi_k(z,\bar z)$, the torus 1-point function $\langle\phi_k(z,\bar z)\rangle_{T^2(\tau)} = f_k(\tau,\bar\tau)$ is a modular form of weight $(h_k,\tilde h_k)$. That is,

$$f_k(\tau+1,\bar\tau+1) = f_k(\tau,\bar\tau), \quad f_k(-1/\tau,-1/\bar\tau) = (-i\tau)^{h_k}(i\bar\tau)^{\tilde h_k} f_k(\tau,\bar\tau). \tag{2.26}$$

3. Virasoro conformal blocks

We will begin by considering the sphere 4-point function of primaries $\phi_i(z_i,\bar z_i)$, $i=1,2,3,4$. As already seen, writing the OPE of pairs of ϕ_i's in terms of sums of Virasoro descendants over an entire basis of primaries, the 4-point function is determined by the conformal Ward identities in terms of the structure constants C_{12k}, C_{34k}. Let us now carry out this computation concretely. Using the conformal Killing group $PSL(2,\mathbb{C})$ of the Riemann sphere, we can put the four operators at

$$z_1 = 0, \quad z_2 = z, \quad z_3 = 1, \quad z_4 = \infty. \tag{3.1}$$

To make the correlator finite and well-defined, we will rescale $\phi_4(z_4,\bar z_4)$ in the $z_4\to\infty$ limit, by defining

$$\phi_4'(\infty) = \lim_{z_4,\bar z_4\to\infty} z_4^{2h_4}\bar z_4^{2\tilde h_4}\phi_4(z_4,\bar z_4). \tag{3.2}$$

Now we are ready to compute the correlator

$$\left\langle \phi_1(0)\phi_2(z,\bar z)\phi_3(1)\phi_4'(\infty)\right\rangle. \tag{3.3}$$

In radial quantization, we may regard $\phi_1(0)$ and $\phi_4'(\infty)$ as ket and bra, and rewrite the 4-point function equivalently in the form

$$\langle\phi_4|\phi_3(1)\phi_2(z,\bar z)|\phi_1\rangle. \tag{3.4}$$

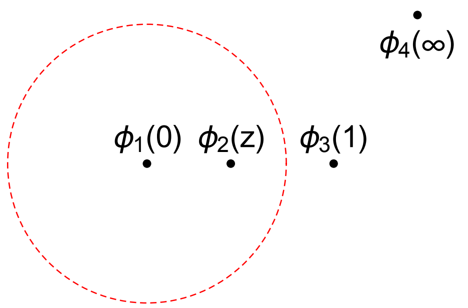


Figure 5: Inserting a complete basis of states (on the red dashed circle) in the four-point function.

Performing the OPE of $\phi_1\phi_2$ (or $\phi_3\phi_4$) is equivalent to inserting a complete basis of states in radial quantization in (3.4), which may be organized into a sum over Virasoro descendants as

$$\begin{aligned}&\langle\phi_4|\phi_3(1)\phi_2(z,\bar z)|\phi_1\rangle\\ &=\sum_k\sum_{|N|=|M|,|\tilde N|=|\tilde M|}\langle\phi_4|\phi_3(1)L_{-N}\bar L_{-\tilde N}|\phi_k\rangle G^{NM}(c,h_k)G^{\tilde N\tilde M}(c,\tilde h_k)\langle\phi_k|L^\dagger_{-M}\bar L^\dagger_{-\tilde M}\phi_2(z,\bar z)|\phi_1\rangle,\end{aligned} \tag{3.5}$$

where $G^{NM}(c,h)$ are the components of the inverse Gram matrix of Virasoro descendants of a primary of weight h. Using the dilatation symmetry on the 3-point functions, we have

$$\langle\phi_k|L^\dagger_{-M}L^\dagger_{-\tilde{M}}\phi_2(z,\bar{z})|\phi_1\rangle = z^{-h_1-h_2+h_k+|M|}\bar{z}^{-\tilde{h}_1-\tilde{h}_2+\tilde{h}_k+|\tilde{M}|}\langle\phi_k|L^\dagger_{-M}\bar{L}^\dagger_{-\tilde{M}}\phi_2(1)|\phi_1\rangle. \tag{3.6}$$

Further, we will write

$$\begin{aligned}
&\langle\phi_4|\phi_3(1)L_{-N}\bar{L}_{-\tilde{N}}|\phi_k\rangle = C_{43k}\rho(\tilde{\nu}_4,\tilde{\nu}_3,\bar{L}_{-\tilde{N}}\tilde{\nu}_k),\\
&\langle\phi_k|L^\dagger_{-M}\bar{L}^\dagger_{-\tilde{M}}\phi_2(1)|\phi_1\rangle = C_{k21}\rho(L_{-M}\nu_k,\nu_2,\nu_1)\rho(\bar{L}_{-\tilde{M}}\tilde{\nu}_k,\tilde{\nu}_2,\tilde{\nu}_1),
\end{aligned} \tag{3.7}$$

where ν_i and $\tilde{\nu}_i$ denote holomorphic and anti-holomorphic Virasoro primaries of weight h_i and $\tilde{h}_i$ respectively, and $\rho(\xi_3,\xi_2,\xi_1)$ denotes the three-point function of Virasoro descendants ξ_i with the structure constants stripped off. In particular, $\rho(\xi_3,\xi_2,\xi_1)$ is entirely fixed by conformal Ward identities in terms of the Virasoro descendatns ξ_i and the central charge c. Now we can write (3.5) as

$$\langle\phi_4|\phi_3(1)\phi_2(z,\bar{z})|\phi_1\rangle = \sum_k C_{43k}C_{k21}\mathscr{F}_c(h_1,h_2,h_3,h_4,h_k;z)\overline{\mathscr{F}_c}(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3,\tilde{h}_4,\tilde{h}_k;\bar{z}), \tag{3.8}$$

where

$$\mathscr{F}_c(h_1,h_2,h_3,h_4,h_k;z) = \sum_{|N|=|M|} z^{-h_1-h_2+h_k+|N|}\rho(\nu_4,\nu_3,L_{-N}\nu_k)G^{NM}(c,h_k)\rho(L_{-M}\nu_k,\nu_2,\nu_1) \tag{3.9}$$

is the holomorphic Virasoro conformal block for the sphere 4-point function in the $12 \to 34$ OPE channel.

So far, we have defined $\mathscr{F}_c$ as a series expansion in z. The radius of convergence of the z-expansion is 1, as $\mathscr{F}_c(z)$ has a (logarithmic) singularity at $z=1$. It is possible to analytically continue $\mathscr{F}_c(z)$ to the entire complex z-plane with a branch cut that extends from $z=1$ to $z=\infty$. The analytic structure of $\mathscr{F}_c(z)$ will be discussed in more detail later.

3.1 Recursive representation

As a simple example, let us compute $\rho(\nu_1,\nu_2,L_{-N}\nu_k)$ in the special case where the integer partition N contains only one element $N=\{n\}$, and so L_{-N} involves a singular Virasoro generator L_{-n}, $n\geq 1$. We have

$$\begin{aligned}
&\rho(\nu_1,\nu_2,L_{-n}\nu_k) = \langle\nu_1'(\infty)\nu_2(1)L_{-n}\nu_k(0)\rangle = \oint_{C_0}\frac{dz}{2\pi i}z^{-n+1}\langle\nu_1'(\infty)\nu_2(1)T(z)\nu_k(0)\rangle\\
&= -\oint_{C_\infty}\frac{dz}{2\pi i}z^{-n+1}\langle\nu_1'(\infty)\nu_2(1)T(z)\nu_k(0)\rangle - \oint_{C_1}\frac{dz}{2\pi i}z^{-n+1}\langle\nu_1'(\infty)\nu_2(1)T(z)\nu_k(0)\rangle\\
&= \langle(L_n\nu_1)'(\infty)\nu_2(1)\nu_k(0)\rangle - \langle\nu_1'(\infty)\left[h_2(-n+1)\nu_2(1)-\partial\nu_2(1)\right]\nu_k(0)\rangle\\
&= \left[h_2(n-1)-h_2-h_k+h_1\right]\rho(\nu_1,\nu_2,\nu_k) = h_2(n-1)-h_2-h_k+h_1.
\end{aligned} \tag{3.10}$$

where C_0 and C_1 are small counterclockwise contours encircling 0 and 1 respectively, whereas C_∞ is a large clockwise contour.

Let us do another example,

$$\begin{aligned}\rho(L_{-n}\nu_k,\nu_2,\nu_1) &= -\oint_{C_\infty}\frac{dz}{2\pi i}z^{n+1}\langle\nu_k'(\infty)T(z)\nu_2(1)\nu_1(0)\rangle\\ &= \oint_{C_1}\frac{dz}{2\pi i}z^{n+1}\langle\nu_k'(\infty)T(z)\nu_2(1)\nu_1(0)\rangle+\oint_{C_0}\frac{dz}{2\pi i}z^{n+1}\langle\nu_k'(\infty)T(z)\nu_2(1)\nu_1(0)\rangle\\ &= [h_2(n+1)-h_1-h_2+h_k]\,\rho(\nu_k,\nu_2,\nu_1)=h_2(n+1)-h_1-h_2+h_k.\end{aligned}\tag{3.11}$$

More generally, we can compute $\rho(\xi_3,\xi_2,\xi_1)$ for generic descendants ξ_i by repeatedly applying the following conformal Ward identities

$$\rho(L_{-n}\xi_3,\xi_2,\xi_1)=\rho(\xi_3,\xi_2,L_n\xi_1)+\sum_{m=-1}^{l(n)}\binom{n+1}{m+1}\rho(\xi_3,L_m\xi_2,\xi_1),\tag{3.12}$$

where $l(n)=n$ for $n\geq-1$, and $l(n)=\infty$ otherwise, and

$$\begin{aligned}&\rho(\xi_3,L_{-1}\xi_2,\xi_1)=[L_0(\xi_3)-L_0(\xi_2)-L_0(\xi_1)]\,\rho(\xi_3,\xi_2,\xi_1),\\ &\rho(\xi_3,L_{-n}\xi_2,\xi_1)=\sum_{m=0}^{\infty}\binom{n-2+m}{m}\left[\rho(L_{n+m}\xi_3,\xi_2,\xi_1)+(-)^n\rho(\xi_3,\xi_2,L_{m-1}\xi_1)\right],\quad n\geq 2.\end{aligned}\tag{3.13}$$

This set of identities allows us to trade a raising operator L_{-n} for $n>0$ at the beginning of the Virasoro chain in ξ_1,ξ_3 with lowering operators on ξ_1,ξ_2, or ξ_3; it also allows us to trade a raising operator at the beginning of the Virasoro chain in ξ_2 to lowering operators on ξ_1,ξ_3. Whenever we act a lowering operator on ξ_i, we can use the Virasoro algebra to reduce the length of the Virasoro chain of ξ_i. This algorithm terminates after finitely many iterations and allows us to compute any $\rho(\xi_3,\xi_2,\xi_1)$.

While such an algorithm can be implemented on a computer (using e.g. Weaver package [8]), it is highly time consuming and would be impractical beyond the first few levels. For some purposes, such as numerical bootstrap or Lorentzian continuation, it is important to know the conformal block to high accuracy, which requires going to high orders in the z expansion. A far more efficient method of computing the z-expansion of the conformal block was discovered by Zamolodchikov [9, 10], which we now describe.

The idea is to consider the analytic continuation of $\mathscr{F}_c$ in the central charge c and/or in the internal primary weight h. For generic c and h_i, $i=1,2,3,4$, the analytically continued block $\mathscr{F}_c(h_1,h_2,h_3,h_4,h;z)$ has only simple poles in h, corresponding to simple poles in the inverse Gram matrix elements $G^{NM}(c,h)$. The locations of these poles are nothing but the zeroes of the Kac determinant, namely $h=h_{r,s}(c)$, for $r,s\geq1$. Alternatively, we may hold h fixed at a generic value, and examine the poles of $\mathscr{F}_c$ in c on the complex c-plane. Again, generically we only encounter simple poles in c, whose locations

$$c_{r,s}(h)=1+6(b_{r,s}(h)+b_{r,s}(h)^{-1})^2,\quad b_{r,s}(h)=\left[\frac{rs-1+2h+\sqrt{(r-s)^2+4(rs-1)h+4h^2}}{1-r^2}\right]^{\frac{1}{2}},\tag{3.14}$$

are such that h coincides with $h_{r,s}(c_{r,s})$ for $r\geq2, s\geq1$.

For now let us focus on the analytic continuation in c. $\mathscr{F}_c$ is determined by the residues at its poles together with its behavior at large c. The large c limit of $\mathscr{F}_c$ is in fact finite and reduces to the corresponding conformal block of the global conformal group $SL(2)$,[5] which is relatively simple to compute and is in this case given by a hypergeometric function,

$$\lim_{c\to\infty}\mathscr{F}_c(h_1,h_2,h_3,h_4,h;z)=z^{-h_1-h_2+h}{}_2F_1(h+h_2-h_1,h+h_3-h_4;2h;z). \tag{3.15}$$

The residues at the poles $c=c_{r,s}(h)$ only receives contribution from Virasoro descendants of the form $L_{-N}\chi^h_{rs}$, where

$$\chi^h_{rs}=\sum_{|M|=rs}\chi^M_{rs}L_{-M}\nu_h \tag{3.16}$$

has the same coefficients χ^M_{rs} as those of the null descendant χ_{rs} of $\nu_{h_{r,s}}$ (2.19). In the $h\to h_{r,s}$ limit, the norm of χ^h_{rs} vanishes, with

$$\lim_{h\to h_{r,s}}\frac{\langle\chi^h_{rs}|\chi^h_{rs}\rangle}{h-h_{r,s}}\equiv(A^c_{rs})^{-1}. \tag{3.17}$$

There exists a closed form formula for A^c_{rs},

$$A^c_{rs}=\frac{1}{2}\prod_{m=1-r}^{r}\prod_{n=1-s}^{s}(mb+nb^{-1})^{-1},\quad (m,n)\neq(0,0),\ (r,s). \tag{3.18}$$

After rescaling by $[\langle\chi^h_{rs}|\chi^h_{rs}\rangle]^{-1}$, χ^h_{rs} behaves like a Virasoro primary in the $h\to h_{r,s}$ limit, in the sense that it is annihilated by L_n for all positive n. The contribution to the residue of the conformal block is determined by

$$\begin{aligned}&\lim_{h\to h_{r,s}}(h-h_{r,s})\mathscr{F}_c(h_1,h_2,h_3,h_4,h;z)\\&=A^c_{rs}\sum_{|N|=|M|}z^{-h_1-h_2+h_{r,s}+rs+|N|}\rho(\nu_4,\nu_3,L_{-N}\chi_{rs})G^{NM}(c,h_{r,s}+rs)\rho(L_{-M}\chi_{rs},\nu_2,\nu_1).\end{aligned} \tag{3.19}$$

Note that while χ_{rs} is a null state, its 3-point function $\rho(\nu_4,\nu_3,\chi_{rs})$ and $\rho(\chi_{rs},\nu_2,\nu_1)$ are nonzero. They are given by the “fusion polynomial"

$$\begin{aligned}\rho(\nu_1,\nu_2,\chi_{rs})&=\rho(\chi_{rs},\nu_2,\nu_1)=P^{rs}_c\begin{bmatrix}h_1\\h_2\end{bmatrix}\\&=\prod_{p=1-r\text{ step }2}^{r-1}\ \prod_{q=1-s\text{ step }2}^{s-1}\frac{\lambda_1+\lambda_2+pb+qb^{-1}}{2}\frac{\lambda_1-\lambda_2+pb+qb^{-1}}{2},\end{aligned} \tag{3.20}$$

where λ_1,λ_2 are defined by $h_i\equiv\frac{1}{4}(b+b^{-1})^2-\frac{1}{4}\lambda_i^2$. Finally, the key property that leads to a recursive formula for the residue is the factorization property

$$\rho(L_{-N}\chi_{rs},\nu_2,\nu_1)=\rho(L_{-N}\nu_{h_{r,s}+rs},\nu_2,\nu_1)\rho(\chi_{rs},\nu_2,\nu_1), \tag{3.21}$$

where $\nu_{h_{r,s}+rs}$ representations a normalized primary state of the same weight as the null state χ_{rs}. Putting these together, we can turn (3.19) into

$$\lim_{h\to h_{r,s}}(h-h_{r,s})\mathscr{F}_c(h_1,h_2,h_3,h_4,h;z)=A^c_{rs}P^{rs}_c\begin{bmatrix}h_1\\h_2\end{bmatrix}P^{rs}_c\begin{bmatrix}h_4\\h_3\end{bmatrix}\mathscr{F}_c(h_1,h_2,h_3,h_4,h_{r,s}+rs;z). \tag{3.22}$$

[5] This property has a suitable generalization beyond sphere 4-point Virasoro blocks, as we will discuss later.

Turning this into a residue in c at $c = c_{r,s}(h)$, we have

$$\mathrm{Res}_{c\to c_{r,s}(h)}\mathscr{F}_c(h_1,h_2,h_3,h_4,h;z) = -\frac{\partial c_{r,s}(h)}{\partial h} A_{rs}^{c_{r,s}} P_{c_{r,s}}^{rs}\begin{bmatrix}h_1\\h_2\end{bmatrix} P_{c_{r,s}}^{rs}\begin{bmatrix}h_4\\h_3\end{bmatrix}\mathscr{F}_{c_{r,s}}(h_1,h_2,h_3,h_4,h+rs;z). \tag{3.23}$$

Finally, we arrive at the recursion formula

$$\begin{aligned}\mathscr{F}_c(h_1,h_2,h_3,h_4,h;z) &= z^{-h_1-h_2+h}{}_2F_1(h+h_2-h_1,h+h_3-h_4;2h;z)\\ &+\sum_{r\geq 2,s\geq 1}\frac{1}{c-c_{r,s}(h)}\left[-\frac{\partial c_{r,s}(h)}{\partial h}\right] A_{rs}^{c_{r,s}} P_{c_{r,s}}^{rs}\begin{bmatrix}h_1\\h_2\end{bmatrix} P_{c_{r,s}}^{rs}\begin{bmatrix}h_4\\h_3\end{bmatrix}\mathscr{F}_{c_{r,s}}(h_1,h_2,h_3,h_4,h+rs;z).\end{aligned} \tag{3.24}$$

Suppose we have already computed the LHS up to order $z^{-h_1-h_2+h+n}$ for general c and h, we can substitute this result into the shifted blocks on the RHS, and obtain results up to order $z^{-h_1-h_2+h+n+2}$. This allows for a relatively efficient computation of the z-expansion of $\mathscr{F}_c(z)$. In the next section, we will describe an even more efficient approach.

3.2 The pillow geometry and the analytic continuation of the conformal block

To fully exhibit the analytic property of the sphere 4-point Virasoro conformal block, it is useful to map the Riemann sphere with four marked points $0,z,1,\infty$ to the pillow geometry $T^2/\mathbb{Z}_2$, parameterized by the complex coordinate $w \sim w+2\pi \sim w+2\pi\tau$, with the $\mathbb{Z}_2$ identification $w\sim -w$. It is also conventional to define the "elliptic nome" $q = e^{\pi i\tau}$. τ is related to z by

$$\tau = i\frac{K(1-z)}{K(z)}, \quad K(z) = {}_2F_1(\frac{1}{2},\frac{1}{2};1;z). \tag{3.25}$$

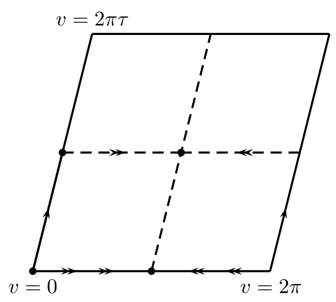


Figure 6: The pillow geometry is the quotient $T^2/\mathbb{Z}_2$. The four points on the plane $0,z,1,\infty$ are mapped to the $\mathbb{Z}_2$ fixed points $w = 0,\pi,\pi(\tau+1),\pi\tau$ respectively.

The explicit map from the coordinate u on the Riemann sphere to w on the pillow is

$$w = \frac{1}{(\theta_3(\tau))^2}\int_0^u \frac{dx}{\sqrt{x(1-x)(z-x)}}, \tag{3.26}$$

where $\theta_3(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}$ is one of the Jacobi theta functions. In the pillow frame, the four primary operators are inserted at the corners $w = 0,\pi,\pi\tau,\pi(1+\tau)$, and the conformal block can be thought of as the propagator on the cylinder [11]

$$\langle \psi_{43} | q^{L_0 - \frac{c}{24}} | \psi_{12} \rangle = \sum_{n=0}^{\infty} a_n q^{h+n-\frac{c}{24}}, \tag{3.27}$$

where $|\psi_{12}\rangle$ and $|\psi_{43}\rangle$ are the states created by a pair of primaries inserted at two corners of the pillow, projected onto the representation space of the Virasoro descendants of a primary of weight h, and the coefficients a_n are functions of the weights h_i, h and the central charge c. (3.27) differs from the sphere 4-point block by a conformal anomaly factor,

$$\mathscr{F}_c(h_1,h_2,h_3,h_4,h;z) = (\theta_3(\tau))^{\frac{c}{2}-4(h_1+h_2+h_3+h_4)} z^{\frac{c}{24}-h_1-h_2} (1-z)^{\frac{c}{24}-h_3-h_4} \langle \psi_{43} | q^{L_0-\frac{c}{24}} | \psi_{12} \rangle. \tag{3.28}$$

The pillow representation of the conformal block (3.28) exhibits several important properties. Firstly, the q-expansion of (3.27) has unit radius of convergence, and is a holomorphic function on the unit q-disc away from the origin. This covers the entire complex z-plane and more: the complex z-plane only maps to an eye-shaped region on the unit q-disc. Secondly, the matrix element (3.27) has the property that, if we take $h_1 = h_4$, $h_2 = h_3$, the coefficients a_n are norms of Virasoro descendants of $|\psi_{12}\rangle$ and are therefore non-negative (provided that the OPE is compatible with unitarity).

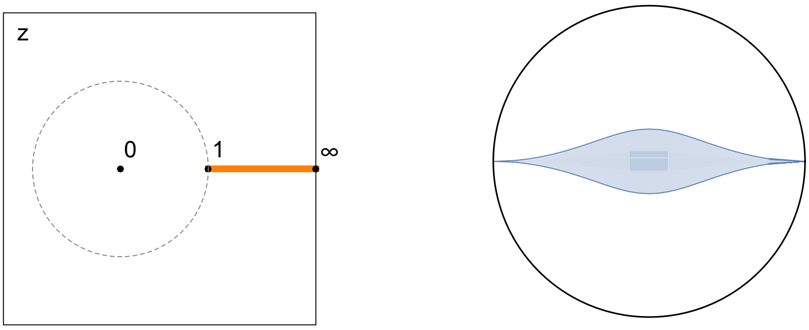


Figure 7: The complex z-plane with a branch cut along $[1,\infty)$ is mapped to the eye-shaped region on the unit q disc. The Virasoro conformal block can be analytically continued to the entire q disc.

Thirdly, (3.27) simplifies in the $h \to \infty$ limit, with c and h_i's held fixed:

$$\lim_{h\to\infty} \langle \psi_{43} | q^{L_0-\frac{c}{24}} | \psi_{12} \rangle = (16q)^{h-\frac{c}{24}} \prod_{n=1}^{\infty} (1-q^{2n})^{-\frac{1}{2}}. \tag{3.29}$$

Heuristically, the intuition here is that in the large h limit, the external operators inserted at the corners are unimportant, and the pillow block becomes the same (up to a moduli independent coefficient) as the square root of the torus character of a primary of weight h.

The knowledge of the large h limit now allows us to derive a recursion formula in h rather than in c, of the form

$$\begin{aligned} \mathscr{F}_c(h_1,h_2,h_3,h_4,h;z) &= (\theta_3(\tau))^{\frac{c}{2}-4(h_1+h_2+h_3+h_4)} z^{\frac{c}{24}-h_1-h_2} (1-z)^{\frac{c}{24}-h_3-h_4} \\ &\quad \times (16q)^{h-\frac{c}{24}} \prod_{n=1}^{\infty} (1-q^{2n})^{-\frac{1}{2}} H_c(h_1,h_2,h_3,h_4,h;q), \end{aligned} \tag{3.30}$$

where H_c obeys

$$H_c(h_1,h_2,h_3,h_4,h;q) = 1 + \sum_{r,s\geq 1} \frac{q^{rs}}{h-h_{r,s}} A^c_{rs} P^{rs}_c \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} P^{rs}_c \begin{bmatrix} h_4 \\ h_3 \end{bmatrix} H_c(h_1,h_2,h_3,h_4,h_{r,s}+rs;q). \quad (3.31)$$

In practice, we can implement this recursion formula efficiently by first using it (as a set of linear equations) to solve for $H_c(h_1,h_2,h_3,h_4,h_{r,s}+rs;q)$ for a finite set of r,s subject to $rs \leq n$, up to order q^n, and then use it to generate H_c for generic internal weight h up to order q^n.

3.3 The convergence of OPE and the crossing equation

Let us consider the sphere 4-point function in the special case $\phi_1 = \phi_4$, $\phi_2 = \phi_3$, whose conformal block decomposition in the pillow frame involves the sum

$$\sum_k C^2_{12k} \langle \psi^k_{12} | q^{L_0 - \frac{c}{24}} | \psi^k_{12} \rangle \langle \tilde{\psi}^k_{12} | \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} | \tilde{\psi}^k_{12} \rangle, \quad (3.32)$$

where we put the superscript k on $|\psi^k_{12}\rangle$ to emphasize its dependence on the internal weight h_k. The analyticity of the 4-point function for real z in the range $0 < z < 1$ implies the convergence of the conformal block sum for real q in the range $0 < q < 1$, hence over the entire q-disc via Cauchy-Schwarz inequality (since the coefficients a_n in (3.27) are non-negative). The summand in (3.32) behaves like $C^2_{12k}(16q)^{h_k+\tilde{h}_k}$ in the large internal weight limit, and thus the convergence for real q close to 1 implies that the contribution to (3.32) from operators of weights $h_k > h_*$ and $\tilde{h} > \tilde{h}_*$ would be suppressed by $|q|^{h_*+\tilde{h}_*}$ in the interior of the unit q-disc, in the large $h_*, \tilde{h}_*$ limit (one can make a similar argument if either h_k or $\tilde{h}_k$ is kept in a finite range). This result amounts to the statement that the $\phi_1\phi_2$ OPE in the pillow frame convergences exponentially fast as a sum over internal primaries.

With this understanding of the convergence of the conformal block expansion, we may decompose a general 4-point function $\langle \prod_{i=1}^4 \phi_i(z_i,\bar{z}_i) \rangle$ in three different channels:

$$s: 12 \to 34, \quad t: 14 \to 23, \quad u: 13 \to 24. \quad (3.33)$$

The equivalence of these different conformal block decompositions puts highly nontrivial constraints on the structure constants. Up to permutations of external primaries, it suffices to inspect the equivalence between s and t channel. The t-channel conformal block can be obtained from the s-channel one by sending $z \to 1-z$ while exchanging h_1 with h_3. The equivalence between the two channels can be expressed as the crossing equation

$$\begin{aligned} &\sum_k C_{12k} C_{34k} \mathscr{F}_c(h_1,h_2,h_3,h_4,h_k;z) \overline{\mathscr{F}}_c(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3,\tilde{h}_4,\tilde{h}_k;\bar{z}) \\ &= \sum_k C_{14k} C_{23k} \mathscr{F}_c(h_3,h_2,h_1,h_4,h_k;1-z) \overline{\mathscr{F}}_c(\tilde{h}_3,\tilde{h}_2,\tilde{h}_1,\tilde{h}_4,\tilde{h}_k;1-\bar{z}). \end{aligned} \quad (3.34)$$

3.4 Solution to crossing equation: minimal model

Now we will discuss one of the simplest nontrivial solutions to the crossing equation, namely the 4-point function of nontrivial primaries in the $c = \frac{1}{2}$ minimal model, also known as the 2D critical Ising model. There are only three primaries in this CFT, whose weights are

$$\begin{aligned} 1: &\ h=\tilde{h}=h_{1,1}=0, \\ \sigma: &\ h=\tilde{h}=h_{1,2}=\frac{1}{16}, \\ \varepsilon: &\ h=\tilde{h}=h_{2,1}=\frac{1}{2}. \end{aligned} \tag{3.35}$$

We will focus on the example of the 4-point function $\langle\sigma(0)\sigma(z,\bar{z})\sigma(1)\sigma'(\infty)\rangle \equiv f(z,\bar{z})$. The conformal block in this case is particularly simple, due to the fact that σ has a null descendant at level 2, of the form

$$(L_{-1}^2-\frac{3}{4}L_{-2})\sigma=(\bar{L}_{-1}^2-\frac{3}{4}\bar{L}_{-2})\sigma=0. \tag{3.36}$$

Let us insert this into the 4-point function,

$$\langle\sigma(0)(L_{-1}^2-\frac{3}{4}L_{-2})\sigma(z,\bar{z})\sigma(1)\sigma'(\infty)\rangle=0. \tag{3.37}$$

L_{-1} acting on $\sigma(z)$ can be replaced with ∂_z. We need to compute the correlator involving $L_{-2}\sigma(z,\bar{z})$, using

$$\begin{aligned} &\langle\sigma|\sigma(1)L_{-2}\sigma(z,\bar{z})|\sigma\rangle=\oint_{C_z}\frac{dw}{2\pi i}\frac{1}{w-z}\langle\sigma|\sigma(1)T(w)\sigma(z)|\sigma\rangle \\ &=h_{1,2}\left[\frac{1}{(1-z)^2}+\frac{1}{z^2}\right]\langle\sigma|\sigma(1)\sigma(z,\bar{z})|\sigma\rangle-\frac{1}{1-z}\langle\sigma|\partial\sigma(1)\sigma(z,\bar{z})|\sigma\rangle+\frac{1}{z}\langle\sigma|\sigma(1)\sigma(z,\bar{z})|\partial\sigma\rangle. \end{aligned} \tag{3.38}$$

The correlators involving $\partial\sigma$ can be computed by going back to a slightly more general form of the 4-point function, as is fixed by $PSL(2,\mathbb{C})$ invariance,

$$\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\sigma(z_3,\bar{z}_3)\sigma'(\infty)\rangle=\frac{1}{|z_{31}|^{4h_{1,2}}}f(\frac{z_{21}}{z_{31}},\frac{\bar{z}_{21}}{\bar{z}_{31}}), \tag{3.39}$$

and so

$$\begin{aligned} \langle\sigma(0)\sigma(z,\bar{z})\partial\sigma(1)\sigma'(\infty)\rangle &= \partial_{z_3}\frac{1}{|z_{31}|^{4h_{1,2}}}f(\frac{z_{21}}{z_{31}},\frac{\bar{z}_{21}}{\bar{z}_{31}})\Big|_{z_1=0,z_2=z,z_3=1}=(-z\partial_z-2h_{1,2})f(z,\bar{z}), \\ \langle\partial\sigma(0)\sigma(z,\bar{z})\sigma(1)\sigma'(\infty)\rangle &= \partial_{z_1}\frac{1}{|z_{31}|^{4h_{1,2}}}f(\frac{z_{21}}{z_{31}},\frac{\bar{z}_{21}}{\bar{z}_{31}})\Big|_{z_1=0,z_2=z,z_3=1}=((z-1)\partial_z+2h_{1,2})f(z,\bar{z}). \end{aligned} \tag{3.40}$$

Putting (3.38) and (3.40) into (3.37), we derive

$$\left[\partial_z^2+\frac{3(1-2z)}{4z(1-z)}\partial_z-\frac{3}{64z^2(1-z)^2}\right]f(z,\bar{z})=0. \tag{3.41}$$

This equation has two linear independent solutions,

$$f_\pm(z)=\frac{\sqrt{1\pm\sqrt{z}}}{z^{\frac{1}{8}}(1-z)^{\frac{1}{8}}}, \tag{3.42}$$

whose linear combinations give rise to two possible conformal blocks. The fact that there are finitely many conformal blocks (namely two) in this case is an accident due to the presence of the null Virasoro descendant of σ. Expanding around $z=0$, we have

$$\begin{aligned}\mathscr{F}_0(z)&=\frac{f_+(z)+f_-(z)}{2}=z^{-\frac{1}{8}}\left(1+\frac{1}{64}z^2+\frac{1}{64}z^3+\frac{117}{8192}z^4+\cdots\right),\\ \mathscr{F}_{\frac{1}{2}}(z)&=f_+(z)-f_-(z)=z^{\frac{3}{8}}\left(1+\frac{1}{4}z+\frac{9}{64}z^2+\frac{25}{256}z^3+\frac{613}{8192}z^4+\cdots\right).\end{aligned} \tag{3.43}$$

As the notation suggests, $\mathscr{F}_h(z)$ is the conformal block associated with a weight h internal primary for $\langle\sigma\sigma\sigma\sigma\rangle$. We see here that only the identity operator and ε can appear in the $\sigma\sigma$ OPE, as a consequence of the null state differential equation. The 4-point function must then take the form

$$f(z,\bar z)=\mathscr{F}_0(z)\overline{\mathscr{F}_0}(\bar z)+C^2_{\sigma\sigma\varepsilon}\mathscr{F}_{\frac{1}{2}}(z)\overline{\mathscr{F}_{\frac{1}{2}}}(\bar z). \tag{3.44}$$

Now we must inspect crossing symmetry, which demands that $f(z,\bar z)$ is invariant under $z\to 1-z$, $\bar z\to 1-\bar z$. Defined with a branch cut running from $z=1$ to ∞, $f_\pm(z)$ obey

$$\begin{aligned}f_+(1-z)&=\frac{f_+(z)+f_-(z)}{\sqrt{2}},\\ f_-(1-z)&=\frac{f_+(z)-f_-(z)}{\sqrt{2}}.\end{aligned} \tag{3.45}$$

Thus, we have

$$f(1-z,1-\bar z)=\left|\frac{1}{\sqrt{2}}\mathscr{F}_0+\frac{1}{2\sqrt{2}}\mathscr{F}_{\frac{1}{2}}\right|^2+C^2_{\sigma\sigma\varepsilon}\left|\sqrt{2}\mathscr{F}_0-\frac{1}{\sqrt{2}}\mathscr{F}_{\frac{1}{2}}\right|^2 \tag{3.46}$$

We see that crossing symmetry is obeyed if and only if

$$C_{\sigma\sigma\varepsilon}=\frac{1}{2}. \tag{3.47}$$

In fact, $C_{\sigma\sigma\varepsilon}$ is the only nontrivial structure constant in the $c=\frac{1}{2}$ minimal model, as $C_{\varepsilon\varepsilon\sigma}$ and $C_{\varepsilon\varepsilon\varepsilon}$ can be shown to vanish by similar analysis. The CFT is thus solved completely.

There is still an extra set of consistency conditions that the CFT must satisfy, which are the modular covariance of the torus 1-point functions of primaries. This indeed holds for the $c=\frac{1}{2}$ minimal model, but we will not verify it explicitly here.

3.5 Solution to crossing equation: Liouville theory

Now we will describe another nontrivial solution to the crossing equation. Let us consider a unitary CFT with central charge $c>1$ and suppose there are only scalar Virasoro primaries. We assume $c>1$ so that there are no possible null Virasoro descendants. The assumption that all Virasoro primaries are scalars, namely $h=\tilde h$, is an extremely strong condition - in fact, this condition (conjecturally) fixes the CFT completely!

Let us first inspect the torus partition function of such a CFT, which must be modular invariant. It takes the general form

$$Z(\tau,\bar\tau)=\sum_{h,\tilde h}d_{h,\tilde h}\chi_h(\tau)\overline{\chi}_{\tilde h}(\bar\tau), \tag{3.48}$$

where $d_{h,\tilde{h}}$ is the number of primaries of weight $(h,\tilde{h})$, and χ_h is the torus character of a Virasoro representation of a primary of weight h,

$$\chi_h(\tau) = \mathrm{Tr}_{\mathscr{V}_h} q^{h-\frac{c}{24}}, \tag{3.49}$$

where $q = e^{2\pi i \tau}$, and $\mathscr{V}_h$ is the space of Virasoro descendants of v_h. In the absence of null descendants, which is guaranteed for $c > 1$ and $h \geq 0$, we have

$$\chi_h(\tau) = q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{3.50}$$

For now, let us assume that $\tilde{c} = c$, so that the anti-holomorphic Virasoro character is just the complex conjugate of $\chi_{\tilde{h}}$. Modular invariance amounts to

$$Z(\tau,\bar{\tau}) = Z(\tau+1,\bar{\tau}+1) = Z(-1/\tau,-1/\bar{\tau}). \tag{3.51}$$

The first equality is equivalent to $h - \tilde{h} \in \mathbb{Z}$ for all primaries, namely the spins are integers. This is of course satisfied if we only have scalar primaries, $h = \tilde{h}$. For the second equality, let us write

$$Z(\tau,\bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_h d_h \exp\left[-4\pi\tau_2 \left(h - \frac{c-1}{24}\right)\right], \tag{3.52}$$

and using the modular property of eta function $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$, modular invariance of the partition function can be written as

$$\frac{1}{|\tau|} \sum_h d_h \exp\left[-4\pi\frac{\tau_2}{|\tau|^2} \left(h - \frac{c-1}{24}\right)\right] = \sum_h d_h \exp\left[-4\pi\tau_2 \left(h - \frac{c-1}{24}\right)\right]. \tag{3.53}$$

To make things clearer, let us define

$$x = \tau_2, \quad y = \frac{\tau_2}{|\tau|^2}, \tag{3.54}$$

and write the above equation as

$$\sqrt{y}\sum_h d_h \exp\left[-4\pi y \left(h - \frac{c-1}{24}\right)\right] = \sqrt{x}\sum_h d_h \exp\left[-4\pi x \left(h - \frac{c-1}{24}\right)\right]. \tag{3.55}$$

Obviously, this is possible only if both sides are independent of x and y, namely

$$\sum_h d_h \exp\left[-4\pi x \left(h - \frac{c-1}{24}\right)\right] \propto x^{-\frac{1}{2}}. \tag{3.56}$$

This is in fact not possible for a discrete spectrum of primaries. Rather, we need a continuous set of primaries of weight $h \geq \frac{c-1}{24}$, with the sum replaced by an integral with a certain spectral density $\rho(h)$,

$$\sum_h d_h \to \int_{\frac{c-1}{24}}^{\infty} dh \rho(h), \tag{3.57}$$

where

$$\rho(h) \propto \frac{1}{\sqrt{h - \frac{c-1}{24}}}. \tag{3.58}$$

If we write $c = 1 + 6Q^2$ and $h = \frac{Q^2}{4} + P^2$, then

$$\int_{\frac{c-1}{24}}^{\infty} dh\rho(h) \propto \int_0^{\infty} dP. \tag{3.59}$$

Thus, we learn that the spectral density of primaries must be a flat distribution in terms of the parameter P.

Let us suppose that there aren't any further degeneracies, so that the scalar primaries are labeled by P, which we denote by V_P. It is natural to normalize the two-point functions by

$$\langle V_P(1)V_{P'}(0)\rangle = \delta(P-P'). \tag{3.60}$$

The structure constants is now a symmetric function $C(P_1,P_2,P_3)$, and the OPE takes the form

$$V_{P_1}(z,\bar{z})V_{P_2}(0) \sim \int_0^{\infty} dP_3 C(P_1,P_2,P_3)|z|^{-\frac{Q^2}{2}-2P_1^2-2P_2^2+2P_3^2} V_{P_3}(0) + \text{Virasoro descendants}. \tag{3.61}$$

The key consistency condition, as far as the sphere correlators are concerned, is the crossing equation

$$\begin{aligned}&\int_0^{\infty} dPC(P_1,P_2,P)C(P_3,P_4,P)\left|\mathscr{F}_c(h_1,h_2,h_3,h_4,h=\frac{Q^2}{4}+P^2;z)\right|^2\\ &= \int_0^{\infty} dPC(P_3,P_2,P)C(P_1,P_4,P)\left|\mathscr{F}_c(h_3,h_2,h_1,h_4,h=\frac{Q^2}{4}+P^2;1-z)\right|^2.\end{aligned} \tag{3.62}$$

Are there any real valued functions $C(P_1,P_2,P_3)$ such that this equation is obeyed for all P_i's? Remarkably, the answer is yes, and appears to be unique (up to rescaling by an overall constant). The solution is known as the Dorn-Otto-Zamolodchikov-Zamolodchikov (DOZZ) structure constants, given by [12–15]

$$C(P_1,P_2,P_3) = \frac{\Upsilon_b'(0)}{\Upsilon_b(\frac{Q}{2}+i(P_1+P_2+P_3))}\prod_{j=1}^{3}\frac{\sqrt{\Upsilon_b(2iP_j)\Upsilon_b(-2iP_j)}}{\Upsilon_b(\frac{Q}{2}+i(P_1+P_2+P_3-2P_j))}, \tag{3.63}$$

where $\Upsilon_b(x)$ is the Barnes double Gamma function, defined as the analytic continuation of

$$\Upsilon_b(x) = \exp\left\{\int_0^{\infty}\frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2\left[\left(\frac{Q}{2}-x\right)\frac{t}{2}\right]}{\sinh\frac{tb}{2}\sinh\frac{t}{2b}}\right]\right\} \tag{3.64}$$

from the domain $0 < \mathrm{Re}(x) < Q$. For generic b, $\Upsilon_b(x)$ is an entire analytic function on the complex x-plane, with simple zeroes at $x = mb + n/b$, for integers $m,n \leq 0$ or $m,n \geq 1$. Some additional useful properties are

$$\begin{aligned}&\Upsilon_b(Q-x) = \Upsilon_b(x),\\ &\Upsilon_b(x+b) = \gamma(bx)b^{1-2bx}\Upsilon_b(x),\\ &\Upsilon_b(x+b^{-1}) = \gamma(b^{-1}x)b^{\frac{2x}{b}-1}\Upsilon_b(x),\end{aligned} \tag{3.65}$$

where $\gamma(x) \equiv \Gamma(x)/\Gamma(1-x)$.

It turns out that with this set of structure constants, the modular covariance of torus 1-point functions are also obeyed [16]. The scalar-only spectrum of primaries together with DOZZ structure constants define what is known as the Liouville CFT [15, 17].

The Liouville CFT may be defined in more conventional terms by say canonical quantization, starting from the classical action

$$S_L = \frac{1}{4\pi}\int d^2z\sqrt{g}\left(g^{mn}\partial_m\phi\partial_n\phi + QR\phi + 4\pi\mu e^{2b\phi}\right), \tag{3.66}$$

where μ is an arbitrary positive constant. On the cylinder, we may decompose $\phi(\sigma,\tau)$ into its Fourier modes,

$$\phi(\sigma,\tau) = \sum_{n=-\infty}^{\infty}\phi_n(\tau)e^{in\sigma}. \tag{3.67}$$

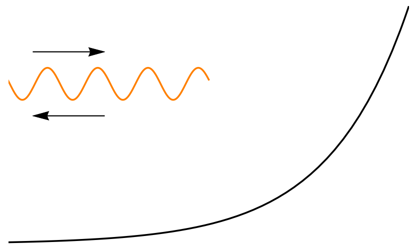


Figure 8: The scalar primary in Liouville CFT describes a scattering state off the Liouville wall in field space.

The zero mode $\phi_0(\tau)$ in particular is subject to the potential $V(\phi_0) = 2\pi\mu e^{2b\phi_0}$. If we ignore the oscillators ϕ_n with $n \neq 0$, which is an approximation that could be justified for lowest excitations in the $\phi_0 \to -\infty$ region, the energy eigenstates are scattering states described by asymptotic wave functions of the form

$$\Psi_P(\phi_0) \sim S(P)^{-\frac{1}{2}}e^{2iP\phi_0} + S(P)^{\frac{1}{2}}e^{-2iP\phi_0}, \tag{3.68}$$

whose energy is given by

$$h + \tilde{h} - \frac{c}{12} = 2P^2. \tag{3.69}$$

These are the states on the cylinder that map to the primaries V_P.

4. A brief survey of known 2D CFTs

Now we briefly recap some known constructions of unitary CFTs in two dimensions.

4.1 Narain lattice

A basic class of CFTs are constructed from the holomorphic and anti-holomorphic components of free bosons, which may be defined by the OPEs

$$X_L^i(z)X_L^j(0) \sim -\delta^{ij}\ln z, \quad X_R^i(\bar{z})X_R^j(0) \sim -\delta^{ij}\ln\bar{z}. \tag{4.1}$$

Of course, these are not single valued, and the idea is to build operators from these that obey single-valued and associative OPE, such that the operator spectrum is also modular invariant.

We begin with a set of $U(1)^{n_L}$ holomorphic currents and $U(1)^{n_R}$ anti-holomorphic currents,

$$j^i(z) = \partial X_L^i(z), \quad \tilde{j}^\ell(\bar{z}) = \bar{\partial}X_R^\ell(\bar{z}). \tag{4.2}$$

Next, consider current algebra primaries of the form

$$V_{k_L,k_R} =: e^{ik_L\cdot X_L+ik_R\cdot X_R} :, \tag{4.3}$$

where $k=(k_L,k_R)$ is viewed as a vector in $\mathbb{R}^{n,m}$. Consistency of OPE requires that k takes value in a lattice $\Gamma\subset\mathbb{R}^{n,m}$ with integral inner product

$$k\circ k' \equiv k_L\cdot k'_L - k_R\cdot k'_R \in \mathbb{Z}. \tag{4.4}$$

Modular invariance further demands that Γ is even and self-dual [18]. Let e_a be a basis of Γ, then $A_{ab}\equiv e_a\circ e_b$ is an even unimodular matrix. Such a lattice Γ must obey $n-m\equiv 0 \bmod 8$. If $n,m\geq 1$, i.e. Γ is of indefinite signature, say with $n-m=8k$, $k,m\geq 1$, then Γ is equivalent to $\Gamma_8^{\oplus k}\oplus\Gamma_{1,1}^{\oplus m}$ up to lattice isomorphic, where Γ_8 is the root lattice of E_8 and $\Gamma_{1,1}$ is the signature $(1,1)$ lattice with pairing matrix given by the Pauli matrix σ_1. For the definite signature even self-dual lattice Γ of rank $n=8k$, in the $k=1$ case Γ_8 is the only such lattice, and in the $k=2$ case there are two inequivalent ones, $\Gamma_8\oplus\Gamma_8$ and Γ_{16}.

4.2 Orbifolds

If a CFT admits a discrete symmetry group G, it is often possible to gauge G (but not always, due to 't Hooft anomaly). This construction is known as orbifold [19]. The Hilbert space of local operators (or states on the cylinder) of the orbifold CFT takes the form

$$\mathscr{H} = \bigoplus_{\text{conjugacy class } [g]} \mathscr{H}_{[g]}, \tag{4.5}$$

where the sum is over conjugacy classes of G. In particular, $\mathscr{H}_{[1]}$ is the space of G-invariant states of the original CFT, called the untwisted sectors, whereas the other $\mathscr{H}_{[g]}$ are called twisted sectors.

The twisted sectors may be formally constructed as follows. For the CFT on a Riemann surface Σ, and a given element $g\in G$, we can define a line operator L_g associated with an oriented loop around a handle of Σ, such that the states propagating through the handle on the two sides of L_g are identified up to transformation by the symmetry action of g. Since G is a symmetry, the line operators L_g are topological, in the sense that they can be continuously deformed (allowing splitting and joining) without changing the partition function or correlation function of the CFT on Σ. Several line operators L_{g_i}, $i=1,\cdots,n$, may meet at a point in cyclic order provided that $g_1\cdots g_n=1$. In particular, L_g and L_h may intersect when g and h commute.

For a pair of commuting elements g and h of G, let us denote Z_g^h the torus partition function of the original CFT with a line operator L_g around the spatial circle and another line operator L'_h around the Euclidean time circle. The partition function of the twisted sector $\mathscr{H}_{[g]}$ of the orbifold CFT is related by

$$\mathrm{Tr}_{\mathscr{H}_{[g]}} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\tilde{c}}{24}} = \frac{1}{|C_g|}\sum_{h\in C_g} Z_g^h, \tag{4.6}$$

where C_g is the commutant (centralizer) of g in G.

Generally, the twisted sector spectrum can be determined by $Z_g^1(\tau,\bar\tau)=Z_1^g(-1/\tau,-1/\bar\tau)$. In a similar way, one can determine the 3-point function between an untwisted operator and a pair of g and g^{-1} twisted operator. By the crossing symmetry between s and t channels of the four-point

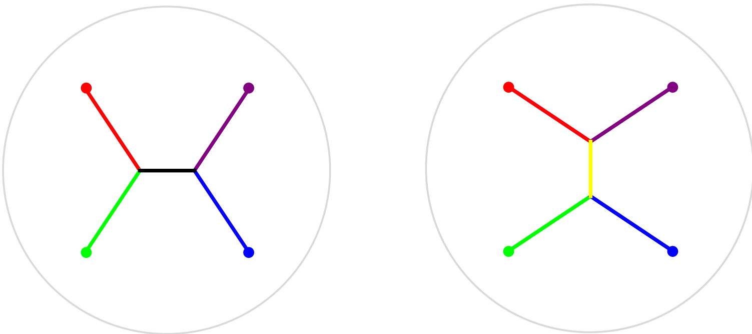


Figure 9: Sphere 4-point function of twisted sector operators $\langle \phi_{g_1}\phi_{g_2}\phi_{g_3}\phi_{g_4}\rangle$, connected by topological line operators with $g_1g_2g_3g_4 = 1$. The s channel conformal block decomposition involves structure constants $C_{g_1,g_2,(g_1g_2)^{-1}}C_{g_3,g_4,g_1g_2}$, whereas the t channel involves $C_{g_2,g_3,(g_2g_3)^{-1}}C_{g_4,g_1,g_2g_3}$ (the notation is schematic, as we only labeled the primaries in the structure constants by their twisted sectors).

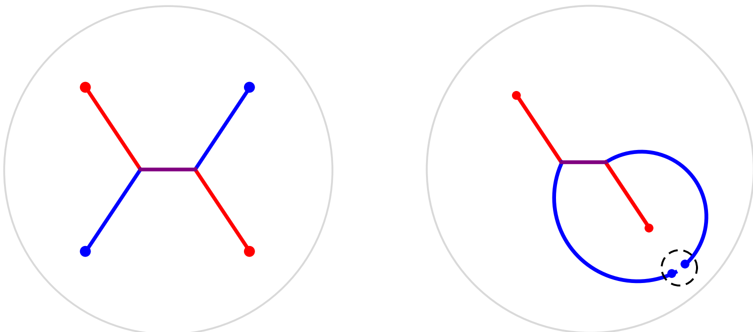


Figure 10: For a pair of commuting elements g,h, the twisted field correlator $\langle \phi_g\phi_h\phi_{g^{-1}}\phi_{h^{-1}}\rangle$ in the u channel OPE limit determines the most general twisted torus characters Z_h^g.

function of $g_1,g_2,g_2^{-1},g_1^{-1}$ twisted sector operators, we can extract the $(g_1,g_2,g_2^{-1}g_1^{-1})$ twisted operator structure constants. This is a special case of the relation shown in Figure 9. Now passing to the u-channel and considering the identity channel in the OPE of the g_2 and g_2^{-1} twisted operators, one can obtain the g_2-twisted torus character of the g_1 twisted sector, i.e. $Z_{g_1}^{g_2}$, as illustrated in Figure 10. Through this procedure, all correlators of the orbifold CFT can in principle be determined unambiguously.

As a nontrivial example, consider the Narain lattice CFT with $c = 24$ and $\tilde{c} = 0$ based on the Leech lattice L_{24}. L_{24} is an even self-dual lattice of signature $(24,0)$, with the special property that it admits no lattice vectors of length squared equal to 2. The torus partition function the L_{24} CFT is

$$Z(\tau) = j(\tau) - 720, \tag{4.7}$$

where $j(\tau)$ is the elliptic j-invariant

$$j(\tau) = \frac{E_4(\tau)^3}{(\eta(\tau))^{24}} = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots . \tag{4.8}$$

Consider the $\mathbb{Z}_2$ symmetry of L_{24} generated by g, the reflection on all lattice elements, i.e. the reflection of all 24 chiral bosons $X_L^i \to -X_L^i$. The corresponding torus character is

$$Z_1^g(\tau) = q^{-1}\prod_{n=1}^{\infty}\frac{1}{(1+q^n)^{24}} = \left[\frac{2\eta(\tau)}{\theta_2(\tau)}\right]^{12}. \tag{4.9}$$

Its modular S-transform is

$$Z_g^1(\tau) = Z_1^g(-1/\tau) = \left[\frac{2\eta(\tau)}{\theta_4(\tau)}\right]^{12}. \tag{4.10}$$

The torus partition function of the orbifold CFT $L_{24}/\mathbb{Z}_2$ is thus

$$\begin{aligned} Z_{L_{24}/\mathbb{Z}_2}(\tau) &= \frac{1}{2}\left[Z_1^1(\tau)+Z_1^g(\tau)+Z_g^1(\tau)+Z_g^g(\tau)\right] \\ &= \frac{1}{2}j(\tau) - 360 + 2^{11}(\eta(\tau))^{12}\left[\frac{1}{(\theta_2(\tau))^{12}}+\frac{1}{(\theta_4(\tau))^{12}}-\frac{1}{(\theta_3(\tau))^{12}}\right] \\ &= j(\tau)-744. \end{aligned} \tag{4.11}$$

In particular, the order q^0 term is absent, indicating that there are no conserved spin-1 currents in this CFT. The orbifold $L_{24}/\mathbb{Z}_2$ in fact admits a very large discrete symmetry group, that is isomorphic to the largest sporadic simple group, also known as the monster group. This CFT, discovered by Frenkel, Lepowsky, and Meurman [20], is hence also referred to as the monster CFT.

Note that given an element g of order n, we expect that $Z_g^1(\tau) = Z_1^g(-1/\tau)$ to be invariant under $\tau \to \tau + n$. If this fails, the orbifolding by the $\mathbb{Z}_n$ generated by g cannot be consistently constructed. This is an example of discrete 't Hooft anomaly. For instance, the monster group has 194 conjugacy classes, out of which 56 cannot be used to orbifold due to such anomalies.

4.3 WZW and coset models

A unitary 2D CFT that admits a continuous symmetry group G is expected to contain a set of conserved Noether currents j_μ^a as local primary operators, $a = 1, \cdots, \dim G$. In compact CFTs, such currents are necessarily given by holomorphic operators $j^a(z)$ of weight $(1,0)$ and anti-holomorphic operators $\tilde{j}^a(\bar{z})$ of weight $(0,1)$. Furthermore, their OPE must take the form of current algebra, say in the holomorphic case,

$$j^a(z)j^b(0) \sim \frac{\kappa^{ab}}{z^2} + \frac{i f^{ab}{}_c}{z} j^c(0). \tag{4.12}$$

Here $f^{ab}{}_c$ are the structure constants of the Lie algebra of G, and κ^{ab} is proportional to an invariant bilinear form on the Lie algebra. Unitarity demands that κ^{ab} is positive definite, which implies that G is semi-simple and compact. If G is a simple compact Lie group, we can write $\kappa^{ab} = k d^{ab}$, where d^{ab} is normalized such that its inverse d_{ab} obeys $\psi^2 \equiv d_{ab}\psi^a\psi^b = 2$ for a long root ψ.

Denote by $\widehat{G}_k$ the level k current algebra for a simple Lie group G. One can define a conserved spin-2 primary operator, known the Sugawara stress-energy tensor T^G,

$$T^G(z) = \frac{1}{2(k+h)}\lim_{w\to z}\left[d_{ab}j^a(w)j^a(z) - \frac{k\dim G}{(w-z)^2}\right], \tag{4.13}$$

where h is the dual Coxeter number, defined by the relation $-f^{ac}{}_{d}f^{bd}{}_{c} = 2h d^{ab}$. The normalization of T^G is such that the $T^G(z)T^G(0)$ OPE takes the form of that of a stress-energy tensor, with central charge

$$c^G = \frac{k \dim G}{k+h}. \tag{4.14}$$

The central charge of the CFT stress-energy tensor c is greater than or equal to c^G. If it coincides with c^G, then T^G would also be the same as the CFT stress-energy tensor $T(z)$.

It is indeed possible to construct a fully consistent CFT whose holomorphic and anti-holomorphic stress-energy tensors are $T^G(z)$ and $\widetilde{T}^G(\bar{z})$, known as the Wess-Zumino-Witten (WZW) model [21, 22]. To do so, we need to construct the spectrum of local operators, which may now be organized not only as representations of the Virasoro algebra, but as representations of the bigger current algebra (as the Virasoro generators can be built out of the currents). Each irreducible representation of the current algebra $\widehat{G}_k$ is built from a set of lowest weight current algebra primaries ϕ_i that transform in an irreducible representation R of G, with the OPE

$$j^a(z)\phi_i(0) \sim \frac{t^a_{ij}}{z}\phi_j(0). \tag{4.15}$$

The $\widehat{G}_k$ WZW model contains precisely one set of primary fields $\phi_{i\tilde{i}}(z,\bar{z})$ associated with a pair of *identical* representations (R,R) of the left and right current algebra group G, where R runs over all irreducible representations of G whose highest weight vector λ obeys $\langle \lambda, \theta \rangle \leq k$, where θ is the highest root vector of G. For $G = SU(2)$, where the representation R is labeled by a spin j, this condition is $j \leq \frac{k}{2}$.

Now given a CFT with continuous symmetry H, governed by a current algebra $\widehat{H}$, one may try to gauge H and produce a new CFT. Gauging a subgroup $H \subset G$ of the $\widehat{G}$ WZW model produces the so-called G/H coset model [23, 24]. The construction of the full operator spectrum of the coset model is rather elaborate, and we will not discuss it in detail here. Note that a particular family of coset models, $(SU(2)_k \times SU(2)_1)/SU(2)_{k+1}$, is equivalent to the A_k minimal model with $c = 1 - \frac{6}{(k+2)(k+3)}$.

4.4 Rational CFTs

If a unitary CFT admits conserved spin-s currents beyond the Virasoro algebra, these currents are necessarily holomorphic of weight $(s,0)$ or anti-holomorphic of weight $(0,s)$. Let us focus on the holomorphic operators for now. The OPEs of the holomorphic operators close on their own, giving rise to a holomorphic (i.e. chiral) vertex algebra that extends the Virasoro algebra, of the general form

$$\mathcal{O}_a(z)\mathcal{O}_b(0) = \sum_c A_{ab}{}^{c} z^{-h_a-h_b+h_c}\mathcal{O}_c(0), \tag{4.16}$$

where h_a are non-negative integers. The OPE of a holomorphic operator $\mathcal{O}_a(z)$ with a generic operator ϕ of weight $(h,\tilde{h})$ can only contain operators of weight $(h+n,\tilde{h})$ for integer n. Thus, the set of operators with anti-holomorphic weight $\tilde{h}$ form a representation of the holomorphic vertex algebra, which may or may not be irreducible. Likewise, the set of operators with holomorphic weight h form a representation of the anti-holomorphic vertex algebra. The operator spectrum of the CFT can thus be decomposed into representations R_i of the holomorphic vertex operator algebra and $\widetilde{R}_{\tilde{j}}$ of the anti-holomorphic vertex algebra.

The torus character of R_i is defined as

$$\chi_i(\tau) = \mathrm{Tr}_{R_i} q^{L_0 - \frac{c}{24}}, \tag{4.17}$$

and likewise the character $\overline{\chi}_{\tilde{i}}(\bar{\tau})$ of $\widetilde{R}_i$. The torus partition function of the CFT can be written as

$$Z(\tau, \bar{\tau}) = \sum_{i,\tilde{i}} n_{i\tilde{i}} \chi_i(\tau) \overline{\chi}_{\tilde{i}}(\bar{\tau}), \tag{4.18}$$

where $n_{i\tilde{i}}$ are non-negative integer degeneracies. A rational conformal field theory (RCFT) is one that contains finitely many representations of the holomorphic and anti-holomorphic vertex operator algebra in its spectrum, i.e. (4.18) is a finite sum. Closure of OPE then requires all weights $(h, \tilde{h})$ to be rational numbers.

Under $\tau \to \tau + 1$, we have

$$\chi_i(\tau+1) = T_{ij} \chi_j(\tau), \quad T_{ij} = \delta_{ij} e^{2\pi i (h_i - \frac{c}{24})}. \tag{4.19}$$

and similarly $\overline{\chi}_{\tilde{i}}(\bar{\tau}+1) = \widetilde{T}^*_{\tilde{i}\tilde{j}} \overline{\chi}_{\tilde{j}}(\tau)$. The modular invariance of the torus partition function demands that $n_{i\tilde{j}}$ is nonzero only when $h_i - \tilde{h}_j \in \mathbb{Z}$, and that under $\tau \to -1/\tau$, χ_i transform into linear combinations of themselves, and so do $\overline{\chi}_{\tilde{i}}$,

$$\chi_i(-1/\tau) = S_{ij} \chi_j(\tau), \quad \overline{\chi}_{\tilde{i}}(-1/\bar{\tau}) = \widetilde{S}^*_{\tilde{i}\tilde{j}} \overline{\chi}_{\tilde{j}}(\bar{\tau}), \tag{4.20}$$

such that

$$n_{i\tilde{i}} S_{ij} \widetilde{S}^*_{\tilde{i}\tilde{j}} = n_{j\tilde{j}}. \tag{4.21}$$

T and S (and similarly for $\widetilde{T}$ and $\widetilde{S}$) must form a finite dimensional representation of $\mathrm{PSL}(2,\mathbb{Z})$, namely

$$S^2 = (ST)^3 = \mathbb{I}. \tag{4.22}$$

This in particular implies that $(\det S^2)^{-3} (\det (ST)^3)^2 = \det T^6 = 1$, and it follows that the central charge c must be a rational number as well.

There isn't a complete classification of RCFTs, even though all explicitly known examples are produced by coset models or Narain lattice CFT and their products and orbifolds, at special points in their moduli space of exactly marginal deformations. The simplest class of RCFTs are meromorphic CFTs whose entire operator spectrum consists of a single holomorphic vertex operator algebra, such as the L_{24} CFT or its $\mathbb{Z}_2$ orbifold monster CFT we have discussed earlier. Even the meromorphic CFTs are far from being fully classified.

Note that all RCFTs with $c > 1$ must involve a strictly larger holomorphic vertex operator algebra than Virasoro, because the non-degenerate torus Virasoro characters do not transform in finite dimensional representations of $\mathrm{PSL}(2,\mathbb{Z})$. In other words, all $c > 1$ RCFTs admit higher spin symmetries beyond the Virasoro algebra.

4.5 NLSM

A two-dimensional nonlinear sigma model (NLSM), defined by the Lagrangian

$$S = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}G_{ij}(X)g^{ab}\partial_a X^i\partial_b X^j, \tag{4.23}$$

is classically Weyl invariant. It would give rise to a CFT if the path integral can be regularized in a manner that respects conformal symmetry, but this is far from guaranteed. We need to study its renormalization group flow by analyzing either the 1PI quantum effective action defined the path integral based on (4.23) with a suitable regularizations scheme, or a Wilsonian effective action including all possible relevant and marginal operators. The Wilsonian picture is rather murky here, as there are infinitely many relevant operators in the perturbation theory. Other classically marginal deformations are of the form

$$\frac{1}{4\pi}\int d^2\sigma B_{ij}(X)\varepsilon^{ab}\partial_a X^i\partial_b X^j + \frac{1}{8\pi}\int d^2\sigma\sqrt{g}R^{(2)}\Phi(X). \tag{4.24}$$

The first term above, the B-field, is parity odd. It appears in the Lagrangian description of the WZW model, which amounts to a NLSM on the group manifold G with a nontrivial 3-form flux $H = dB$. The second term involves the two-dimensional scalar curvature, and would be topological if $\Phi(X)$ is a constant; this term vanishes in flat 2D spacetime and serves to modify the stress-energy tensor hence modifying conformal transformations of the fields X^i. This term appears for instance in the NLSM description of coset models, the simplest nontrivial examples being the $SU(2)/U(1)$ model, and the “cigar CFT" which is equivalent to $SL(2)/U(1)$ coset model.

At least in the absence of B-field, one can straightforwardly regularize the path integral using dimensional regularization (formally working in $d = 2-\varepsilon$ dimensions), and compute perturbatively the 1PI effective action, or more directly, the trace of the stress-energy tensor

$$T^a{}_a = -\frac{\varepsilon}{2}G_{ij}(X)\partial^a X^i\partial_a X^j + \frac{\varepsilon}{4}R^{(2)}\Phi(X) - \frac{1}{2}\nabla_X^2\Phi(X). \tag{4.25}$$

To turn this into a meaningful operator equation in the quantum theory, one needs to rewrite the RHS in terms of renormalized operators and then take the $\varepsilon \to 0$ limit. In the end one arrives at an operator equation of the form

$$T^a{}_a = \frac{1}{2}\left[\beta^G_{ij}(X)\partial^a X^i\partial_a X^j\right]_r - \frac{1}{4}R^{(2)}[\beta^\Phi(X)]_r, \tag{4.26}$$

where $[\cdots]_r$ stands for a suitable operator regularization, and β^G_{ij} and β^Φ up to the leading two nontrivial perturbative orders are given by [25]

$$\begin{aligned}\beta^G_{ij} &= R_{ij} + \frac{1}{2}R_i{}^{k\ell m}R_{jk\ell m} + 2\nabla_i\nabla_j\Phi + \cdots,\\ \beta^\Phi &= -\nabla^2\Phi + (\nabla\Phi)^2 + \frac{1}{4}R^{ijk\ell}R_{ijk\ell} + \cdots.\end{aligned} \tag{4.27}$$

Conformal invariance is equivalent to the condition $\beta^G_{ij} = \beta^\Phi = 0$. It turns out that the latter are equivalent to the target space equation of motion based on an effective action $S_{eff}[G_{ij}(X), \Phi(X)]$. While these equations can be solved locally in the target space, the existence of the solution globally especially when the target space is compact is far from obvious.

For instance, the leading order condition $R_{ij} = 0$ which amounts to the vanishing of 1-loop beta function of the NLSM demands the target space to be Ricci flat. This puts highly nontrivial topological restriction on the target space. A class of even dimensional compact manifolds that admit Ricci flat metrics are Calabi-Yau manifolds. One can ask, given a Calabi-Yau manifold M with Ricci flat metric $G_{ij}(X)$, could the metric be deformed, with a dilaton profile $\Phi(X)$ turned on, such that (4.27) is solved to all order? I do not know the answer to this question, although, the answer to the superconformal version of this question is yes, as we will discuss next.

4.6 Superconformal symmetry

CFTs with supersymmetries are interesting for a number of reasons, one of being that such theories are abundant and their existence are often easy to establish due to various supersymmetry non-renormalization properties. Poincaré supersymmetry does not commute with conformal symmetry, and together they generate a superconformal symmetry group. The superconformal algebra (SCA) are characterized (but not uniquely) by the number $\mathcal{N}$ of spin $\frac{3}{2}$ currents. There are well known CFTs that admit holomorphic and anti-holomorphic SCAs of $\mathcal{N} = 1$, 2, and 4. The $\mathcal{N} = 4$ SCA come in two qualitatively different types, the small $\mathcal{N} = 4$ SCA which contains a single $SU(2)$ "R-symmetry" current algebra, and the large $\mathcal{N} = 4$ SCA which contains $SU(2) \times SU(2) \times U(1)$ current algebra.

The superconformal algebra extends the bosonic Virasoro algebra, and so a superconformal field theory (SCFT) can be viewed as an ordinary bosonic CFT with extra higher spin symmetries, with an important subtlety: the superconformal algebra involves currents of half-integer spin, which seems to violate the momentum quantization on the cylinder, and also spoils modular invariance. There are two ways to make the theory consistent: either we somehow relax the conventional notion of modular invariance slightly, by allowing ourselves to work with half-integer spin operators in the spectrum, or we insist on the integer spin condition and project out the half-integer spin operators from the spectrum. The latter is known as the Gliozzi-Scherk-Olive (GSO) projection [26], and is achieved as follows. The half-integer spin operators are anti-commuting fields, and has odd fermion number F, which is an integer defined mod 2. The projection onto integer spin operators amounts to projection onto states invariant under $(-)^F$. To do so consistently with modular invariance, one must include twisted sector states with respect to $(-)^F$. In the GSO projected theory, the half-integer spin superconformal currents are not strictly speaking part of the spectrum, but they can nonetheless be used to organized the spectrum according to representations of the SCA.

Now we will discuss a few examples of SCA (by no means exhaustive) and CFTs that realize such symmetries. We begin with the $\mathcal{N} = 1$ SCA, which contains in addition to the stress-energy tensor $T(z)$ of central charge c, a single spin $\frac{3}{2}$ anti-commuting field $G(z)$ that is a Virasoro primary. The only new independent OPE relation is

$$G(z)G(0) \sim \frac{2c}{3z^3} + \frac{2}{z}T(0). \tag{4.28}$$

Acting on the states at the origin it is useful to work with the Laurent series

$$G(z) = \sum_{r \in \mathbb{Z}+\nu} \frac{G_r}{z^{r+\frac{3}{2}}}, \tag{4.29}$$

where $\nu = \frac{1}{2}$ in the absence of branch cuts. In this case, the operator at the origin is said to be in the Neveu-Schwarz (NS) sector. In particular, the identity operator is the NS sector. The GSO projection requires twisted sector operator on which a topological line operator associated with $(-)^F$ is attached. This topological line operator creates a branch cut, and as a result $G(z)$ has a Laurent series (4.29) with $\nu = 0$. Such twisted sector operators are known as Ramond (R) sector operators.

Some basic examples of $\mathcal{N} = (1,1)$ SCFT are the $\mathcal{N} = 1$ minimal models with $c = \frac{3}{2} - \frac{12}{(k+2)(k+4)}$, $k = 1, 2, \cdots$. Interestingly, the GSO projected $\mathcal{N} = 1$ minimal models can also be realized as bosonic coset models $(SU(2)_k \times SU(2)_2)/SU(2)_{k+2}$. Some other examples are the free $c = \frac{3}{2}$ theory consisting of a compact boson and a pair of chiral and anti-chiral free fermions $\psi, \tilde{\psi}$, and the $\mathcal{N} = 1$ Liouville CFT which at the Lagrangian level may be viewed as a linear dilaton CFT together with free fermion $\psi, \tilde{\psi}$, deformed to leading order by a marginal operator of the form $\psi\tilde{\psi}e^{\beta\phi}$.

Next, let us consider $\mathcal{N} = 2$ SCA. It has a pair of spin $\frac{3}{2}$ anti-commuting fields $G^{\pm}(z)$ and a $U(1)$ R-current $J(z)$. We may normalize $J(z)$ such that its OPE takes the form

$$J(z)J(0) \sim \frac{c}{3z^2}, \tag{4.30}$$

and then $G^{\pm}$ carry charge ± 1 with respect to $J(z)$. The G^+G^+ and G^-G^- OPEs are non-singular. The only new nontrivial OPE relation is

$$G^+(z)G^-(0) \sim \frac{2c}{3z^3} + \frac{2}{z^2}J(0) + \frac{2}{z}T(0) + \frac{1}{z}\partial J(0). \tag{4.31}$$

In terms of the Laurent modes $G^{\pm}_r$, L_n, J_n, the $\mathcal{N} = 2$ SCA admits an outer automorphism known as the spectral flow,

$$G^{\pm}_r \to G^{\pm}_{r\pm\eta}, \quad J_n \to J_n + \eta\frac{c}{3}\delta_{n,0}, \quad L_n \to L_n + \eta J_n + \eta^2\frac{c}{6}\delta_{n,0}, \tag{4.32}$$

for $\eta \in \mathbb{Z}$. One may also extend the spectral flow to half-integer η, which connects NS and R sectors. However, the spectral flow need not be a symmetry of the SCFT.

Certain “nice" $\mathcal{N} = 2$ SCFTs do admit spectral flow symmetry. An example is the $(2,2)$ NLSM on a Calabi-Yau target space. In this case, we may organize the operator spectrum according the bigger symmetry generated by the $\mathcal{N} = 2$ SCA together with the spectral flow symmetry (with integer or half-integer η). This is known as the “extended $\mathcal{N} = 2$ SCA" [27, 28].

Some other examples of $\mathcal{N} = 2$ SCFT include $\mathcal{N} = 2$ minimal models, with $c = \frac{3k}{k+2}$, $k = 1, 2, \cdots$. Note that these have the same central charge as the $SU(2)_k$ WZW model, but they are not the same CFTs (in particular, they have different operator spectra). Another nontrivial example is the $\mathcal{N} = 2$ Liouville CFT [29], which at the Lagrangian level may be viewed as a compact boson and a linear dilaton, with a pair of fermions, deformed to leading order by a marginal operator of the form $\psi_+\tilde{\psi}_+e^{-\beta\varphi} + \psi_-\tilde{\psi}_-e^{-\beta\bar{\varphi}}$.

Finally, let us discuss the small $\mathcal{N} = 4$ SCA. It has four spin $\frac{3}{2}$ currents $G^{\alpha A}(z)$, where α is a doublet index with respect the $SU(2)_k$ R-currents $J_i(z)$, with $c = 6k$, and A is an extra index that transforms under an outer automorphism $SU(2)_{out}$, which is not part of the superconformal symmetry and generally not a symmetry of the SCFT. Note that the $SU(2)$ level k is an integer, and therefore c must be an integer multiple of 6. The only new nontrivial OPE relation is

$$G^{\alpha A}(z)G^{\beta B}(0) \sim \varepsilon^{AB}\left[\frac{2c}{3z^3}\varepsilon^{\alpha\beta} + \sigma_i^{\alpha\beta}\frac{2}{z^2}J_i(0) + \varepsilon^{\alpha\beta}\frac{2}{z}T(0) + \sigma_i^{\alpha\beta}\frac{1}{z}\partial J_i(0)\right]. \tag{4.33}$$

The $\mathscr{N}=4$ SCA also admits a spectral flow map

$$G_r^{\pm A} \to G_{r\pm\eta}^{\pm A}, \quad J_n^3 \to J_n^3 + \eta k\delta_{n,0}, \quad J_n^{\pm} \to J_{n+2\eta}^{\pm}, \quad L_n \to L_n + 2\eta J_n^3 + \eta^2 k\delta_{n,0}, \tag{4.34}$$

for integer η (half-integer η would relate NS and R sectors). Rather nontrivially, here the spectral flow is part of the $\mathscr{N}=4$ SCA itself (in other words, unlike the $\mathscr{N}=2$ SCA, the small $\mathscr{N}=4$ SCA is automatically “extended"). This can be shown by studying the vacuum character of the $\mathscr{N}=4$ SCA and seeing that it is invariant under spectral flow.

The representation theory of small $\mathscr{N}=4$ SCA is relatively simple to describe: the unitary representations are labeled by primaries of weight h and $SU(2)_R$ spin ℓ in a finite range [30]. In the NS sector, the non-BPS primaries have $h>\ell$ and $\ell=0,\frac{1}{2},\cdots,\frac{k-1}{2}$, while the BPS primaries have $h=\ell$ and $\ell=0,\frac{1}{2},\cdots,\frac{k}{2}$.

A class of nontrivial examples of SCFTs that admit small $\mathscr{N}=4$ SCA are supersymmetric NLSM on hyperKähler manifolds. There is also a family of $\mathscr{N}=4$ Liouville theory [31], of central charge $c=6k$, that may be constructed by marginally deforming the CFT consisting of a linear dilaton, four free fermions, and an $SU(2)_{k-1}$ WZW model.

4.7 Calabi-Yau models

The supersymmetric NLSM on a Calabi-Yau d-fold with a suitable family of metric and possibly B-field (that solve the α' corrected Ricci flatness condition) defines a family of $\mathscr{N}=(2,2)$ SCFTs with $c=3d$ and extended $\mathscr{N}=2$ SCA. In particular, the $\eta=1$ spectral flow on the holomorphic NS sector maps the identity operator (NS sector vacuum state) to a superconformal primary operator $\mathscr{O}_\Omega$ of weight $h=\frac{d}{2}$ and $U(1)_R$ charge d. Further, $\mathscr{O}_\Omega$ is annihilated by $G^+_{-\frac{1}{2}}$ and is a chiral primary. It has a natural interpretation in the NLSM description: there is a holomorphic d-form $\Omega=\frac{1}{d!}\Omega_{i_1\cdots i_d}dz^{i_1}\wedge\cdots\wedge dz^{i_d}$, from which one can construct the operator

$$\mathscr{O}_\Omega \sim \Omega_{i_1\cdots i_d}\psi^{i_1}\cdots\psi^{i_d}. \tag{4.35}$$

In the interacting NLSM the RHS a priori should be regularized, but in fact such an operator is not renormalized due to the holomorphy of Ω (note that $\bar{z}^{\bar{i}}$ and $\psi^{\bar{i}}$ do not appear).

Our knowledge of the Calabi-Yau models as SCFTs so far is largely limited to the moduli space of exactly marginal deformations and the spectrum of BPS operators, and some crude bounds on the non-BPS operator spectrum obtained from conformal bootstrap. Exceptions are special points (called Gepner points) in the moduli space where the SCFT happens to be rational and admit a description as an orbifold of product of minimal models.

Let us briefly discuss the operator spectrum. While the general representation theory of the $\mathscr{N}=2$ SCA is quite delicate [32,33], including a variety of different shortened representations, the representation theory of the extended $\mathscr{N}=2$ SCA is relatively simple to describe [27,28]. Independently on the holomorphic and anti-holomorphic side, there are non-BPS (massive) representations and BPS (massless) representations. A non-BPS representation contains a superconformal primary that carries an integer R-charge Q in the range $Q = 0, \pm 1, \pm 2, \cdots, \pm(\lfloor \frac{d}{2} \rfloor - 1), \frac{d-1}{2}$(if d odd), and obeys the BPS bound $h > \frac{|Q|}{2}$. The entire weight and charge content of the non-BPS representation is summarized by the NS character

$$\begin{aligned} ch^{\rm NS}_{h,Q}(q,y) &\equiv {\rm Tr}_{\rm non-BPS} q^{L_0 - \frac{c}{24}} y^{J_0} \\ &= q^{h-\frac{c}{24}} \sum_{m=-\infty}^{\infty} q^{\frac{d-1}{2}m^2 + mQ} y^{(d-1)m+Q} \prod_{n=1}^{\infty} \frac{(1+yq^{n-\frac{1}{2}})(1+y^{-1}q^{n-\frac{1}{2}})}{(1-q^n)^2}. \end{aligned} \tag{4.36}$$

A BPS representation on the other hand contains a chiral or anti-chiral primary that carries an integer R-charge Q in the range $Q = 0, \pm 1, \pm 2, \cdots, \pm \lfloor \frac{d-1}{2} \rfloor, \frac{d}{2}$(if d even), and saturates the BPS bound $h = \frac{|Q|}{2}$. The corresponding NS character $\chi^{\rm NS}_Q(q,y) \equiv {\rm Tr}_{\rm BPS} q^{L_0 - \frac{c}{24}} y^{J_0}$ is given by

$$\begin{aligned} &\chi^{\rm NS}_Q(q,y) = q^{\frac{Q}{2}-\frac{c}{24}} \sum_{m\in\mathbb{Z}} \frac{q^{\frac{d-1}{2}m^2+mQ} y^{(d-1)m+Q}}{1+yq^{m+\frac{1}{2}}} \prod_{n=1}^{\infty} \frac{(1+yq^{n-\frac{1}{2}})(1+y^{-1}q^{n-\frac{1}{2}})}{(1-q^n)^2}, \quad Q>0, \\ &\chi^{\rm NS}_Q(q,y) = \chi^{\rm NS}_{-Q}(q,y^{-1}), \quad Q<0, \\ &\chi^{\rm NS}_0(q,y) = q^{-\frac{c}{24}} \sum_{m\in\mathbb{Z}} \frac{(q^{-\frac{1}{2}} - q^{\frac{1}{2}}) q^{\frac{d-1}{2}m^2+m} y^{(d-1)m+1}}{(1+yq^{m-\frac{1}{2}})(1+yq^{m+\frac{1}{2}})} \prod_{n=1}^{\infty} \frac{(1+yq^{n-\frac{1}{2}})(1+y^{-1}q^{n-\frac{1}{2}})}{(1-q^n)^2}. \end{aligned} \tag{4.37}$$

An operator that belongs to BPS representations in both the holomorphic and anti-holomorphic sectors is loosely referred to as $\frac{1}{2}$-BPS, while an operator that is BPS on one side and non-BPS on the other side is referred to as $\frac{1}{4}$-BPS.

There is a supersymmetric index that is very useful in constraining the spectrum of $\frac{1}{2}$ and $\frac{1}{4}$-BPS operators, known as the elliptic genus [34]. To define this index we must work in the Ramond sector whose ground state preserves supersymmetry. The R sector character is related to the NS sector character by $\eta = \frac{1}{2}$ spectral flow on a representation of shifted primary weight and R-charge,

$$ch^{\rm R}_{h,Q}(q,y) = ch^{\rm NS}_{h+\frac{c}{24}-\frac{Q}{2}, \frac{c}{6}-Q}(q, q^{\frac{1}{2}} y), \quad \chi^{\rm R}_Q(q,y) = \chi^{\rm NS}_{\frac{c}{6}-Q}(q, q^{\frac{1}{2}} y). \tag{4.38}$$

The elliptic genus is defined as a character in the (R,R) sector, of the form

$$\chi(\tau, z) = {\rm Tr}_{RR} q^{L_0 - \frac{c}{24}} y^{J_0} \bar{q}^{\bar{L}_0 - \frac{c}{24}} e^{\pi i \bar{J}_0}, \tag{4.39}$$

where $q = e^{2\pi i \tau}$, $y = e^{2\pi i z}$. Note that the $\bar{\tau}$ dependence drops out, due to cancelation between excited states with even and odd anti-holomorphic R-charge $\bar{J}_0$. Due to spectral flow symmetry (by integer η), χ is invariant under $z \to z+1$ and $z \to z+\tau$. Furthermore, χ has PSL$(2,\mathbb{Z})$ transformation property

$$\chi(\tau+1, z) = \chi(\tau, z), \quad \chi(-1/\tau, z/\tau) = e^{2\pi i \frac{d}{2} \frac{z^2}{\tau}} \chi(\tau, z). \tag{4.40}$$

That is to say, $\chi(\tau, z)$ is a weak Jacobi form of weight 0 and index $\frac{d}{2}$. The space of weak Jacobi forms of integer index is spanned by polynomials in four basic Jacobi forms $E_4, E_6, \phi_{0,1}, \phi_{-2,1}$, where E_4, E_6 are Eisenstein series and $\phi_{0,1}, \phi_{-2,1}$ are two weak Jacobi forms of index 1. For half-integer index, one can in addition make use of two other basic weak Jacobi forms $\phi_{0,\frac{3}{2}}$ and $\phi_{-1,\frac{1}{2}}$.

In the $d = 2$ case, where an example is the SCFT with K3 target space, the elliptic genus must be proportional to $\phi_{0,1}$. By matching the Witten index with the Euler characteristic of K3, one derives [35]

$$\chi_{\mathrm{K3}}(\tau,z) = 2\phi_{0,1}(\tau,z) = 2y+20+2y^{-1}+(20y^2-128y+216-128y^{-1}+20y^{-2})q+\cdots. \tag{4.41}$$

This can be used to fix the degeneracy of all $\frac{1}{2}$ and $\frac{1}{4}$-BPS states in the K3 CFT, assuming the absence of extra conserved higher spin currents beyond the small $\mathscr{N} = 4$ SCA.

In the $d = 3$ case, which includes Calabi-Yau 3-fold models, the elliptic genus must be proportional to $\phi_{0,\frac{3}{2}}$. Matching the Witten index with the Euler characteristic gives the result

$$\chi_{\mathrm{CY}_3}(\tau,z) = (h^{1,1}-h^{2,1})\phi_{0,\frac{3}{2}}(\tau,z). \tag{4.42}$$

One can then decompose $\phi_{0,\frac{3}{2}}(\tau,z)$ into extended $\mathscr{N} = 2$ characters in the RR sector. The result is simply $\phi_{0,\frac{3}{2}} = \chi_1^R - \chi_{-1}^R$. This result has an interesting implication on the BPS spectrum: while there are (extended $\mathscr{N} = 2$) $\frac{1}{2}$-BPS multiplets that correspond to the deformation moduli of the SCFT, the $\frac{1}{4}$-BPS multiplets must contain either extra conserved higher spin currents (expected to be absent at a generic point on the conformal manifold), or the $\frac{1}{4}$-BPS multiplets must pair up and can potentially be lifted as non-BPS multiplets under marginal deformations.

4.8 Exactly marginal deformations

When a CFT admits a marginal primary $\mathscr{O}$ with weight $h = \tilde{h} = 1$, one may deform the theory by inserting $\exp\left[-\lambda\int d^2z\mathscr{O}(z,\bar{z})\right]$ into correlation functions. In a theory with Lagrangian description, this would amount to deforming the Lagrangian density by the operator $\mathscr{O}(z,\bar{z})$. While conformal invariance is preserved to first order in the deformation parameter λ, it is generally not preserved at higher orders, due to the regularization of coinciding operators $\mathscr{O}$ that break conformal symmetry. For instance, if the $\mathscr{O}(z,\bar{z})\mathscr{O}(0)$ OPE contains another marginal primary, a nontrivial beta function is generated at order λ^2.

A useful way to understand the obstructions to conformal invariance at higher orders is to view the CFT as part of the worldsheet theory of a compactified bosonic string theory, along with free bosons X^μ and b,c ghosts, where a marginal primary $\mathscr{O}$ gives rise to the vertex operator $\mathscr{V}_i = \mathscr{O}e^{ik_i\cdot X}$ of a massless scalar field in Minkowskian spacetime (parameterized by the X^μ's). The n-th order obstruction to conformal invariance is captured by the simultaneous $k_i \to 0$ limit of the n-point string tree level scattering amplitude

$$\left\langle \left[\prod_{i=1}^{n-3}\int d^2z_i\mathscr{V}_i(z_i,\bar{z}_i)\right]\mathscr{V}_{n-2}(0)\mathscr{V}_{n-1}(1)\mathscr{V}_n'(\infty)\right\rangle, \tag{4.43}$$

where the z_i-integrals may be regularized by analytic continuation in the k_i's.

Exactly marginal deformations are marginal deformations that lead to a continuous family of CFTs, parameterized by the deformation parameter λ. This requires the vanishing of the zero-momentum limit of (4.43) for all n. There is a natural notion of metric on this deformation space, given by two-point function of the marginal operators, known as the Zamolodchikov metric [36]. Therefore, this moduli space of exactly marginal deformations is a Riemannian manifold, sometimes called the conformal manifold. Basic examples are free boson CFTs based on the Narain lattice Γ_{n_L,n_R}, which admits an $n_L n_R$-dimensional conformal manifold isormorphic to $\mathrm{Aut}(\Gamma_{n_L,n_R})\backslash O(n_L,n_R)/(O(n_L)\times O(n_R))$. More generally, whenever a CFT admits a set of $U(1)^{n_L}$ holomorphic currents j^a_z, and $U(1)^{n_R}$ anti-holomorphic currents $\tilde{j}^b_{\bar{z}}$, the deformation by $j^a_z\tilde{j}^b_{\bar{z}}$ are exactly marginal.

Let us now discuss marginal deformations of a $(2,2)$ SCFT, which preserve the full superconformal symmetry to first order. Such a deformation is based on weight $(\frac{1}{2},\frac{1}{2})$ (chiral, chiral) or (chiral, anti-chiral) operators, through their descendants of the form

$$\mathscr{O} = G^-_{-\frac{1}{2}}\bar{G}^-_{-\frac{1}{2}}\phi^{+,+} + G^+_{-\frac{1}{2}}\bar{G}^+_{-\frac{1}{2}}\phi^{-,-} \quad \text{or} \quad G^-_{-\frac{1}{2}}\bar{G}^+_{-\frac{1}{2}}\phi^{+,-} + G^+_{-\frac{1}{2}}\bar{G}^-_{-\frac{1}{2}}\phi^{-,+}, \tag{4.44}$$

where $\phi^{+,+}$ represents a (chiral, chiral) primary with holomorphic and anti-holomorphic $U(1)_R$ charge $(1,1)$, and $\phi^{+,-}$ represents a (chiral, anti-chiral) primary with R-charge $(1,-1)$. To see that this preserves superconformal symmetry, we need to show that for any holomorphic function $v(z)$, the superconformal current $v(z)G^+(z)$ can be moved past $\mathscr{O}(w,\bar{w})$ up to a total derivative. Let us inspect this in the (chiral, chiral) case:

$$\begin{aligned}
&\oint \frac{dz}{2\pi i} v(z)G^+(z)\mathscr{O}(w,\bar{w}) = \oint \frac{dz}{2\pi i} v(z)\left((z-w)^{-2}G^+_{\frac{1}{2}} + (z-w)^{-1}G^+_{-\frac{1}{2}}\right) G^-_{-\frac{1}{2}}\bar{G}^-_{-\frac{1}{2}}\phi^{+,+}(w,\bar{w})\\
&= \left[\partial v(w)(2L_0+J_0) + v(w)2L_{-1}\right]\bar{G}^-_{-\frac{1}{2}}\phi^{+,+}(w,\bar{w})\\
&= 2\partial_w\left[v(w)\bar{G}^-_{-\frac{1}{2}}\phi^{+,+}(w,\bar{w})\right],
\end{aligned}\tag{4.45}$$

where we have made use of the assumption that $\phi^{+,+}$ is annihilated by G^+_r for all $r\geq -\frac{1}{2}$.

An important property of $(2,2)$ SCFTs is that marginal deformations are necessarily exactly marginal [37, 38], thereby giving rise to abundant examples of (super-)conformal manifolds. The basic reason is that when we deform the theory by say $\lambda\int d^2z G^-_{-\frac{1}{2}}\widetilde{G}^-_{-\frac{1}{2}}\phi^{+,+} + c.c.$, regularized in a supersymmetric manner, λ can only be renormalized holomorphically. This follows from writing the deformation in superspace as $\int d^2z d^2\theta^+ \lambda\Phi^{+,+}(z,\bar{z},\theta,\bar{\theta}) + c.c.$, where $\Phi^{+,+}$ is a chiral superfield whose lowest component is $\phi^{+,+}$, then promoting λ to a chiral super-parameter (also known as the "spurion"), and consider all possible local superspace integrals that could be generated under RG. It then follows from the non-singular OPE of $\phi^{+,+}$'s among themselves that λ cannot be renormalized at all.

An interesting example that exhibits some of the rich phenomena that may occur on the conformal manifold is the D1-D5 CFT [39–41], which is a SCFT with small $\mathscr{N}=(4,4)$ SCA and central charge $c=6k$ for a positive integer k. In some limits on the conformal manifold, the D1-D4 CFT can be described by the supersymmetric NLSM whose target space is the moduli space of solutions to $U(Q_5)$ self-dual Yang-Mills equation with instanton number Q_1+Q_5 on K3 (corresponding to $k=Q_1Q_5+1$) or instanton number Q_1 on T^4 (corresponding to $k=Q_1Q_5$). In particular, the case $Q_5=1$ corresponds to the orbifold SCFT $\mathrm{Sym}^{Q_1Q_5+1}(K3)$ or $\mathrm{Sym}^{Q_1Q_5}(T^4)$. When Q_1 and Q_5 are coprime, the D1-D5 CFT is expected to be compact. Arguments based on string dualities suggest that if (Q_1,Q_5) and (Q_1',Q_5') are both coprime with $Q_1Q_5=Q_1'Q_5'$, then the D1-D5 CFTs labeled by (Q_1,Q_5) and (Q_1',Q_5') lie on the same conformal manifold. This implies that conformal NLSMs on seemingly very different target space geometries are somehow connected by exactly marginal deformations through a strong coupling regime (where the target space becomes highly curved). Furthermore, there are points in the moduli space where the D1-D5 CFT develops a continuous spectrum above a dimension gap (these continuum states are the holographic duals of long closed strings that can move to infinity at finite cost of energy in global AdS_3).

4.9 RG flows

A CFT perturbed by a relevant operator will flow to a different CFT of smaller central charge c under the renormalization group. This is an important tool for exploring the relations between CFTs as well as constructing potentially new CFTs. Starting from the UV CFT, the infrared RG fixed points are generally beyond the reach of conformal perturbation theory, and so unless the RG flow is known to be integrable or protected by symmetry (typically supersymmetry), it is quite difficult to determine the infrared CFT precisely. Here we briefly mention a few classes of RG flows in two dimensions that are under control.

The first class of RG flows are integrable flows [42–47]. The simplest examples involve an RG flow to a massive theory in the IR, with a finite number of species of elementary particles and a factorized S-matrix. In this case, the QFT resulting from an integrable relevant perturbation of the UV CFT is fully determined by the IR particle spectrum and 2-body S-matrix elements. From these data, one can in principle determine the spectrum of the QFT on a circle using the thermodynamic Bethe ansatz (TBA), and recover the UV CFT from the small radius limit. Furthermore, the correlation functions of local operators may be constructed from the form factors, the latter subject to a set of analyticity constraints and crossing symmetry properties determined by the factorized S-matrix [48, 49].

The second class of RG flows are that of Landau-Ginzburg (LG) models. In the bosonic case, it is understood that the noncompact free boson ϕ CFT deformed by an operator of the form ϕ^{2n} generates a fine tuned RG flow to the minimal model with $c = 1 - \frac{6}{(n+1)(n+2)}$. This can be checked using conformal perturbation theory for large n, and for small n through the truncated conformal space approach by directly diagonalizing the deformed Hamiltonian with Rayleigh-Ritz method [50–52].

Under better control are the $(2,2)$ supersymmetric version of LG models [53], describe by the Lagrangian of a set of chiral superfields X_i, with a superpotential $W(X_i)$. Provided that there are no mass terms (i.e. quadratic terms in the fields) to begin with, the non-renormalization property of the superpotential ensures nontriviality of the infrared fixed point. The LG model is particularly useful in determining the conformal manifold as well as the structure constants of BPS operators in the IR CFT.

The third example is a class of “short" RG flows, analogous to the critical $O(N)$ vector models in three dimensions. Let us begin with the tensor product of N copies of the $c = \frac{4}{5}$ critical three-state Potts model, also known as the (A_4, D_4) minimal model. The three-state Potts model contains nonzero spin Virasoro primaries that are built from different holomorphic and anti-holomorphic degenerate primaries of the $c = \frac{4}{5}$ Virasoro algebra. It has a $\mathbb{Z}_3$ symmetry, under which the weight $(\frac{1}{15}, \frac{1}{15})$ primary σ and its conjugate σ^* transform with a phase $e^{2\pi i/3}$ and $e^{-2\pi i/3}$ respectively, while the weight $(\frac{2}{5}, \frac{2}{5})$ “energy operator" ε is uncharged. The N-fold product CFT has only two relevant operators that are invariant under $S_N \ltimes \mathbb{Z}_3^N$,

$$\sum_{i=1}^{N} \varepsilon_i, \quad \text{and} \quad \sum_{i<j}^{N} \varepsilon_i \varepsilon_j. \tag{4.46}$$

The former would generate a massive RG flow. The latter generates an RG flow to a new fixed point, at least for sufficiently large N, where one may consider an $1/N$ expansion, and the operator $\sum_{i<j}^{N} \varepsilon_i \varepsilon_j$ may be thought of as a double trace deformation. This leads to a short RG flow, under which the central charge is expected to change by an amount of order $1/N$. Indeed, it appears that even the $N = 3$ case gives a very short RG flow [54].

Finally, let us mention that another important class of RG flows in two dimensions are gauge theories, including gauged linear sigma models (GLSM) that often flow to nontrivial CFTs (such as Calabi-Yau models) in the IR [55]. These do not fit in the standard paradigm of CFT perturbed by a relevant operator, as the gauge theories are not conformal in the UV.[6] The simplest example is the pure (free) $U(1)$ gauge theory in 2D, whose energy spectrum on the circle is isomorphic to that of a free non-relativistic particle on a circle of radius $R = 1/g$ (g being the gauge coupling). Clearly, in the $R \to \infty$ limit, this spectrum does not organize into a representation of any Virasoro algebra, and the spectral density is not scale invariant (unlike in a noncompact CFT).

5. Carving out the 2D CFT landscape

We now return to the bootstrap approach to 2D CFTs and explore general constraints based on unitarity, crossing, and modular invariance.

5.1 Bounding the gap

To begin with, let us inspect constraints from the modular invariance of the torus partition function (3.48), based on the positivity of $d_{h,\tilde{h}}$. Given any linear functional α acting on the space of functions in $\tau, \bar{\tau}$, the modular crossing equation $Z(\tau,\bar{\tau}) = Z(-1/\tau,-1/\bar{\tau})$ implies

$$\sum_{h,\tilde{h}} d_{h,\tilde{h}} \alpha \left[\chi_h(\tau)\overline{\chi}_{\tilde{h}}(\bar{\tau}) - \chi_h(-1/\tau)\overline{\chi}_{\tilde{h}}(-1/\bar{\tau})\right] = 0. \tag{5.1}$$

Defining $\tau \equiv ie^{z}$, $\bar{\tau} \equiv ie^{\bar{z}}$, we may expand α on the derivative basis

$$\alpha = \sum_{m+n \text{ odd}} \alpha_{m,n} \, \partial_z^m \partial_{\bar{z}}^n \Big|_{z=\bar{z}=0}. \tag{5.2}$$

Given any hypothetical spectrum $\mathscr{I}$, if we can find an α such that $\alpha[\chi_h \overline{\chi}_{\tilde{h}}] \geq 0$ for all $(h,\tilde{h}) \in \mathscr{I}$ and say $\alpha[\chi_0 \overline{\chi}_0] > 0$ (assuming the identity operator is part of the operator spectrum), then we encounter an immediate contradiction with (5.1), hence ruling out the spectrum $\mathscr{I}$.

For instance, if $\alpha[\chi_h \overline{\chi}_{\tilde{h}}] \geq 0$ holds for all $h + \tilde{h} > \Delta_0$ and the strictly inequality holds for $h = \tilde{h} = 0$, we would rule out the possibility of a gap Δ_0 in the dimension spectrum of Virasoro primaries.

[6] Note that the c-theorem is perfectly applicable and the UV c-function is equal to that of free matter fields. However the UV limit is not the same as the theory of free matter fields, due to the gauge singlet constraint on local operators, as well as the holonomy degree of freedom when the theory is put on a spatial circle (necessary for modular invariance).

As a simple example, consider linear functionals of the form [56]

$$\alpha = a_1(\partial_z + \partial_{\bar{z}}) + a_3(\partial_z + \partial_{\bar{z}})^3\big|_{z=\bar{z}=0}. \tag{5.3}$$

We may simplify the modular crossing equation slightly by replacing

$$\chi_h(\tau) \to \hat{\chi}_h(\tau) = \tau^{\frac{1}{4}}\eta(\tau)\chi_h(\tau). \tag{5.4}$$

For $c > 1$, $\hat{\chi}_h(\tau) = \tau^{\frac{1}{4}} q^{h-\frac{c-1}{24}}$ for $h > 0$, and $\hat{\chi}_0(\tau) = \tau^{\frac{1}{4}} q^{-\frac{c-1}{24}}(1-q)$. Let us further simplify things by assuming that there are no conserved Virasoro primary currents of any spin. It is straightforward to compute

$$\alpha[\chi_h \overline{\chi}_{\tilde{h}}] = e^{-2\pi(\Delta-\frac{c-1}{12})}\left[a_1 R_1(\Delta) + a_3 R_3(\Delta)\right], \tag{5.5}$$

for $h, \tilde{h} > 0$, where $R_n(\Delta)$ are degree n polynomials in $\Delta = h + \tilde{h}$, whose coefficients depend on c, and on the vacuum character

$$\begin{aligned} \alpha[\chi_0 \overline{\chi}_0] = e^{2\pi\frac{c-1}{12}}\big[&a_1(-0.0118212 + 0.522621c) \\ &+ a_3(-0.245875 - 0.41425c + 0.811209c^2 + 0.14328c^3)\big]. \end{aligned} \tag{5.6}$$

We would like to choose a_1, a_3 such that (5.6) is positive and

$$a_1 R_1(x) + a_3 R_3(x) \geq 0 \ \ \forall x \geq \Delta_0, \tag{5.7}$$

for some Δ_0. Such a Δ_0 would give an upper bound on the gap in the operator spectrum. Δ_0 is optimized (i.e. minimized in this case) when (5.6) is taken to zero, with $a_3 < 0$, and the value of Δ_0 is then given by the largest root of $a_1 R_1(x) + a_3 R_3(x) = 0$. The result can be expanded in the large c limit as

$$\Delta_0 \approx \frac{c}{6} + 0.471824 - 0.03374\, c^{-1} + \mathscr{O}(c^{-2}). \tag{5.8}$$

This result can be improved by including higher order derivatives in z and $\bar{z}$ in the linear function α [57, 58]. For instance, we may consider linear functionals α of the form (5.2) with the truncation $m + n \leq N$, and find the strongest bound on the dimension gap within this subspace of linear functionals; the resulting bound will be denoted $\Delta^{(N)}_{\rm mod}$. We then take the limit $N \to \infty$ to find the optimal modular bound on the dimension gap, $\Delta_{\rm mod} = \lim_{N\to\infty} \Delta^{(N)}_{\rm mod}$. Numerically, $\Delta^{(N)}_{\rm mod}$ can be computed using SDPB package with reasonable efficiency up to $N = 55$ or so; we then numerically extrapolate $\Delta^{(N)}_{\rm mod}$ to $N = \infty$ to find $\Delta_{\rm mod}$ as a function of the central charge c. The result is shown in Figure 11.

It is observed that $\Delta_{\rm mod}(c)$ is exactly equal to $\frac{c}{6} + \frac{1}{3}$ for $c \in [1, 4]$. The slope of $\Delta_{\rm mod}(c)$ jumps from $\frac{1}{6}$ to $\frac{1}{8}$ at $c = 4$, and then decreases smoothly and monotonically with c. $\Delta_{\rm mod}(c)$ is strictly greater than the upper bound on the twist gap $\frac{c-1}{12}$, and thus the asymptotic slope of $\Delta_{\rm mod}(c)$ is no less than $\frac{1}{12}$. The convergence of $\Delta^{(N)}_{\rm mod}$ becomes slower with increasing c, or equivalently, the optimal linear functional α for the dimension gap bound receives more contribution from higher order $z, \bar{z}$ derivatives as c increases, suggesting that the derivative basis is inefficient at large c. It remains an open question to determine the form of the optimal linear functional α in the large c limit, and the corresponding bound $\Delta_{\rm mod}(c)$.

It is also of interest to bound the dimension of the first nontrivial scalar Virasoro primary in a compact unitary CFT. Such a bound can be obtained by working with a different hypothesis on

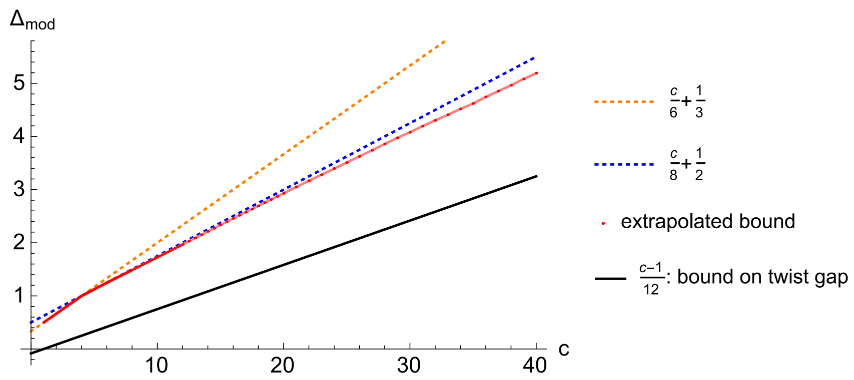


Figure 11: The modular bound on the dimension gap $\Delta_{\rm mod}$ (in red) as a function of central charge c, obtained by numerical extrapolation of $\Delta^{(N)}_{\rm mod}$ to $N=\infty$ (by fitting $31 \le N \le 55$ results with a quadratic polynomial in $1/N$) [58].

the spectrum $\mathscr{I}$, namely allowing scalar primaries of dimension $\Delta \ge \Delta_0$ only, together with the vacuum, and arbitrary weights for primaries of nonzero spin $s = |h-\tilde{h}|$ provided that they satisfy the unitarity bound $h,\tilde{h} \ge 0$. A nontrivial upper bound on the dimension gap $\Delta^{s=0}_{\rm mod}$ exists for $c<25$. The numerical results are shown in Figure 12.

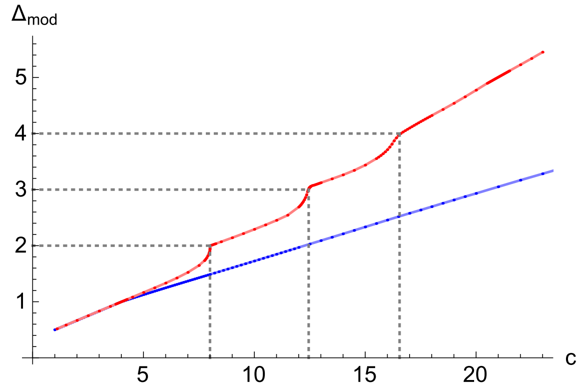


Figure 12: Modular bounds on the dimension of the lightest scalar operator $\Delta^{s=0}_{\rm mod}$ (red) as a function of the central charge c, compared with the all-spin dimension gap bound $\Delta_{\rm mod}$ (blue) [58].

By definition, $\Delta^{s=0}_{\rm mod}(c)$ coincides with $\Delta_{\rm mod}(c)$ for $c \le 4$, where the bound is less than or equal to 1. For $c>4$, $\Delta^{s=0}_{\rm mod}(c)$ is strictly greater than $\Delta_{\rm mod}(c)$, and diverges at $c=25$. The numerical results exhibits a number of interesting features. There is a kink of the bound $\Delta^{s=0}_{\rm mod}(c)$ at $c=8$, where the bound is equal to 2. This implies that all CFTs with $c<8$ must admit a nontrivial relevant scalar Virasoro primary. It is also seen that at the special values $c=1,2,\frac{14}{5},4,8$, the scalar gap bound $\Delta^{s=0}_{mod}$ is saturated by level 1 WZW models with gauge groups $SU(2)$, $SU(3)$, G_2, $SO(8)$, and E_8.

We may also bound, say, the scalar degeneracy at the gap, by considering a slightly different optimization problem. Consider linear functionals of the form (5.2), normalized by

$$\alpha[\chi_0\overline{\chi}_0] = -1. \tag{5.9}$$

We now seek to maximize $\alpha[\chi_h \overline{\chi}_{\tilde{h}}]$ for the first scalar primary at $h = \tilde{h} = \frac{1}{2}\Delta_*$, subject to

$$\alpha[\chi_h \overline{\chi}_{\tilde{h}}] \geq 0, \quad h = \tilde{h} \geq \frac{1}{2}\Delta_*, \ \text{ or } \ h \neq \tilde{h},\ h, \tilde{h} \geq 0,\ h - \tilde{h} \in \mathbb{Z}. \tag{5.10}$$

Suppose we find such a linear functional α_*, then applying it to the modular crossing equation, we have

$$\begin{aligned} 0 &= \alpha_* \left[\chi_0 \overline{\chi}_0 + d_{\text{gap}} \chi_{\frac{\Delta_0}{2}} \overline{\chi}_{\frac{\Delta_0}{2}} + \sum_{\text{the rest}} \chi_h \overline{\chi}_{\tilde{h}} \right] \\ &\geq -1 + d_{\text{gap}} \alpha_* \left[\chi_{\frac{\Delta_0}{2}} \overline{\chi}_{\frac{\Delta_0}{2}} \right], \end{aligned} \tag{5.11}$$

where d_{gap} is the scalar degeneracy at dimension Δ_0. Thus, $\left(\alpha_* \left[\chi_{\frac{\Delta_0}{2}} \overline{\chi}_{\frac{\Delta_0}{2}} \right] \right)^{-1}$ gives a rigorous upper bound of d_{gap}. Of course, Δ_0 should not exceed $\Delta_{\text{mod}}^{s=0}$. In Figure 13, we show the numerical results for the upper bound on d_{gap} when $\Delta_0 = \Delta_{\text{mod}}^{s=0}$ over a range of c.

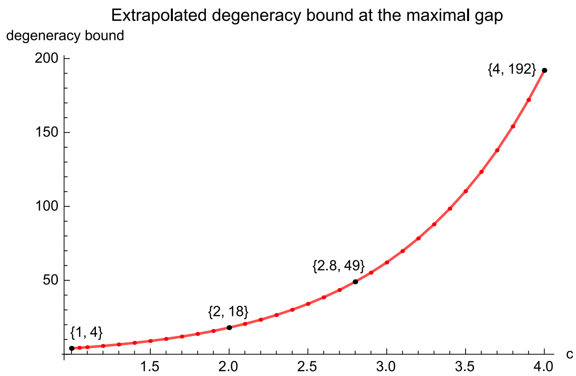


Figure 13: Upper bound on the scalar degeneracy at $\Delta_{\text{mod}}^{s=0}(c)$ for $1 \leq c \leq 4$. The black dots correspond to $SU(2)_1$, $SU(3)_1$, $(G_2)_1$ and $SO(8)_1$ WZW models, where the degeneracy bound is saturated [58].

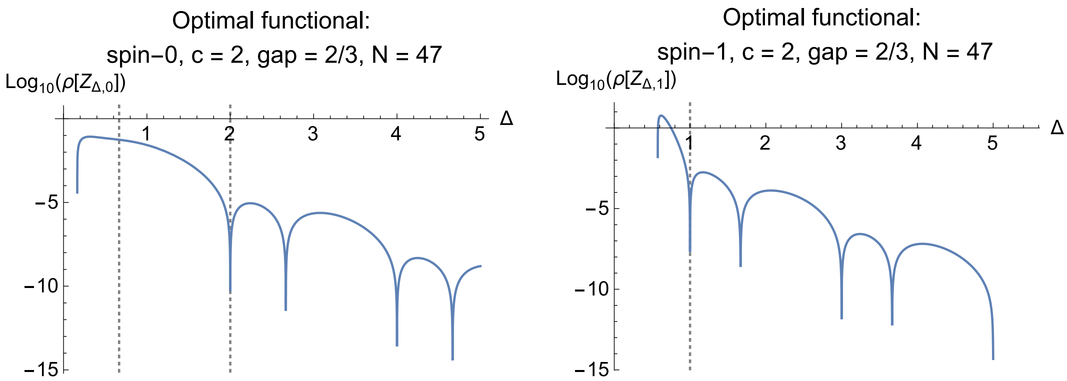


Figure 14: Logarithmic plots of the optimal functional acting on spin-0 and spin-1 characters when the degeneracy of operators at the gap is maximized and the dimension gap bound is saturated at $c = 2$. The dotted lines highlight the gap and the presence of marginal scalar primaries in the spin-0 case, as well as conserved spin-1 currents in the spin-1 case [58].

If the bound on $d_{\rm gap}$ happens to be saturated by a CFT, then we must have $\alpha_*[\chi_h\overline{\chi}_{\tilde{h}}] = 0$ for $(h,\tilde{h})$ in the rest of the CFT spectrum. In other words, the rest of the spectrum of primaries is a subset of the zeroes of $\alpha_*[\chi_h\overline{\chi}_{\tilde{h}}]$ as a function of $(h,\tilde{h})$. This is known as the "extremal functional method" [59, 60] and can be applied to all sorts of bootstrap analysis whenever a coefficient of the character or conformal block decomposition is known or conjectured to be maximized by the CFT of interest.

An example in the $c = 2$ case is shown in Figure 14, where the zeroes of the extremal functional α_* that maximizes the gap at $\Delta_0 = \frac{2}{3}$ on the spin 0 and spin 1 characters are seen to agree with the spectrum of the $SU(3)$ WZW model of the corresponding spins above the scalar gap.

5.2 Bounding the spectral function

Now we turn to the spectral function method, which allows us to extract from the crossing equation not just bounds on gap in the operator spectrum, but on the contribution from operators across the entire spectrum [61].

Consider the conformal block decomposition of the 4-point function $g(z,\bar{z}) = \langle\phi(0)\phi(z,\bar{z})\phi(1)\phi'(\infty)\rangle$ of four identical scalar primaries of dimension Δ_ϕ, of the form

$$g(z,\bar{z}) = \sum_{h,\tilde{h}} C^2_{h,\tilde{h}}\mathscr{F}_h(z)\overline{\mathscr{F}}_{\tilde{h}}(\bar{z}), \tag{5.12}$$

where $C^2_{h,\tilde{h}}$ represents the sum of OPE coefficients squared of primaries of weight $(h,\tilde{h})$ in the $\phi\phi$ OPE, and $\mathscr{F}_h(z)$ is the Virasoro conformal block with internal weight h (we have suppressed the explicit dependence on the weight of ϕ and the central charge c). The crossing equation takes the form

$$\sum_{h,\tilde{h}} C^2_{h,\tilde{h}}\left[\mathscr{F}_h(z)\overline{\mathscr{F}}_{\tilde{h}}(\bar{z}) - \mathscr{F}_h(1-z)\overline{\mathscr{F}}_{\tilde{h}}(1-\bar{z})\right] = 0, \tag{5.13}$$

or in terms of $F^{m,n}(h,\tilde{h}) \equiv \partial_z^m\partial_{\bar{z}}^n\left[\mathscr{F}_h(z)\overline{\mathscr{F}}_{\tilde{h}}(\bar{z})\right]_{z=\bar{z}=\frac{1}{2}}$,

$$\sum_{h,\tilde{h}} C^2_{h,\tilde{h}}F^{m,n}(h,\tilde{h}) = 0, \quad m,n \geq 0,\ m+n = \text{odd}. \tag{5.14}$$

Note that the conformal block itself evaluated at $z = \bar{z} = \frac{1}{2}$ is always positive, i.e. $F^{0,0}(h,\tilde{h}) > 0$, for $h,\tilde{h} \geq 0$. We define the spectral function as

$$f(\Delta_*) = \frac{\sum_{h+\tilde{h}<\Delta_*} C^2_{h,\tilde{h}}\mathscr{F}_h(\frac{1}{2})\overline{\mathscr{F}}_{\tilde{h}}(\frac{1}{2})}{g(\frac{1}{2},\frac{1}{2})}. \tag{5.15}$$

By its definition, the spectral function of a 4-point correlator of identical operators in a unitary CFT is a non-decreasing function that ranges between 0 and 1. It measures the contribution from operators up to dimension Δ_* to the correlator.

Using the crossing equation, one can obtain nontrivial upper and lower bounds on $f(\Delta_*)$, as a function of Δ_*. To do so, consider the inequality

$$\theta(\Delta_* - \Delta) - y_{0,0} - \sum_{m+n\ \text{odd}} y_{m,n}\frac{F^{m,n}(h,\tilde{h})}{F^{0,0}(h,\tilde{h})} \geq 0, \quad \forall (h,\tilde{h}) \in \mathscr{I}, \tag{5.16}$$

where $\mathscr{I}$ is a hypothetical spectrum for operators appearing in $\phi\phi$ OPE, and $y_{m,n} \in \mathbb{R}$. At any given Δ_*, if we can find a set of $y_{m,n}$'s such that (5.16) is simultaneous satisfied for all possible weights $(h,\tilde{h})$ obeying the hypothesis on $\mathscr{I}$, then by multiplying (5.16) with $C^2_{h,\tilde{h}} F^{0,0}(h,\tilde{h})$ and summing over operators in the OPE, using the crossing equation, we have

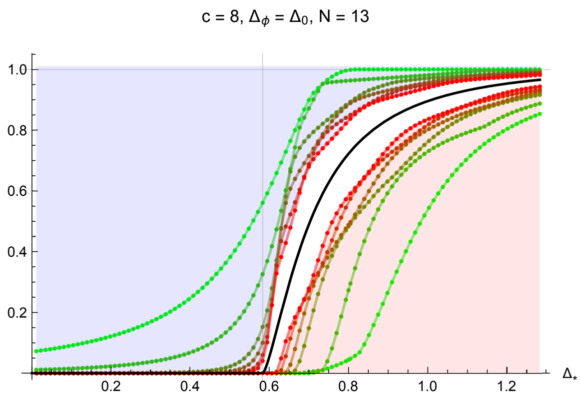
$$\sum_{h,\tilde{h}} C_{h,\tilde{h}} \left[\Theta(\Delta_* - \Delta) - y_{0,0}\right] F^{0,0}(h,\tilde{h}) = g(\frac{1}{2},\frac{1}{2})(f(\Delta_*) - y_{0,0}) \geq 0. \tag{5.17}$$

Thus, $y_{0,0}$ is a rigorous lower bound on $f(\Delta_*)$. By maximizing $y_{0,0}$ subject to (5.16), we would find an optimal lower bound on $f(\Delta_*)$. Numerically, we can do this via semidefinite programming, provided that we make the truncation $m+n \leq N$, resulting in a lower bound on the spectral function, which we denote by $f_-^{(N)}(\Delta_*)$. Similarly, by minimizing $y_{0,0}$ subject to the opposite inequality of (5.16), we would find an upper bound $f_+^{(N)}(\Delta_*)$.

As a toy example, consider the case $N = 1$, i.e. using first order derivatives only. If we could approximate $\mathscr{F}(h;z)$ by $z^{-2h_\phi+h}$, sometimes called the "scaling block", we would write $F^{1,0}/F^{0,0} \approx 2(-2h_\phi + h)$, and deduce a lower bound on the spectral function

$$y_{0,0}(x) \approx 1 - \frac{2\Delta_\phi}{x}. \tag{5.18}$$

Of course, this result is only approximate, but is qualitatively similar to the exact bound derived from (5.16). It is rather weak in the large weight limit however, where we already knew that the tail contribution from high dimension primaries must be exponentially suppressed.



(a)

Figure 15: Upper and lower bounds on the spectral function from linear functionals of increasing derivative order N (from green to red), assuming only scalar primaries for $c = 8$ with $\Delta_\phi = \frac{7}{12}$. The shaded regions are excluded and the black curve denotes the corresponding spectral function of Liouville theory [61].

Let us now inspect the $\langle\phi\phi\phi\phi\rangle$ spectral function bounds in a CFT with $c > 1$ that admits *only scalar* Virasoro primaries (i.e. no primaries of nonzero spin). We have seen that such an assumption combined with modular invariance implies that the CFT is noncompact, has a dimension gap $\frac{c-1}{12}$, and its spectral density is equal to that of Liouville CFT. Now if we know the spectral function $f(\Delta_*)$, by taking its derivative with respect to Δ_*, we would recover all the OPE coefficients. An example of numerical results for $f_{\pm}^{(N)}(\Delta_*)$ up to $N=13$ is shown in Figure 15. The bounds narrow down an allowed window, in which the Liouville spectral function lies. The upper and lower bounds could conceivably converge in the $N\to\infty$ limit, but this is hard to see numerically as the computation of the bounds becomes very time consuming at higher truncation order N.

If we make the assumption that the upper and lower bounds do agree at $N=\infty$, we may directly seek solutions to the equality

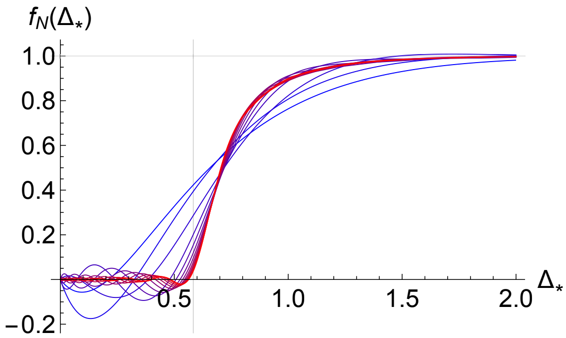
$$\theta(\Delta_*-\Delta)=y_{0,0}+\sum_{m+n\ \text{odd}} y_{m,n}\frac{F^{m,n}(\frac{\Delta}{2},\frac{\Delta}{2})}{F^{0,0}(\frac{\Delta}{2},\frac{\Delta}{2})},\quad \forall\Delta\geq 0. \tag{5.19}$$

The existence of solution for all $\Delta_*\geq 0$ amounts to the completeness of $p_{m,n}(\Delta)=\frac{F^{m,n}(\frac{\Delta}{2},\frac{\Delta}{2})}{F^{0,0}(\frac{\Delta}{2},\frac{\Delta}{2})}$ as a basis of the Hilbert space of functions of Δ over the positive real axis defined with the inner product

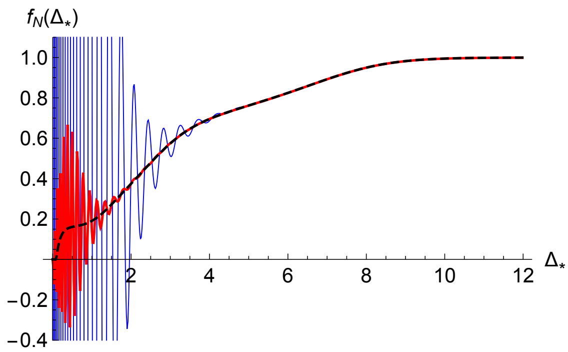
$$\langle x,y\rangle=\int_0^\infty x^*(\Delta)y(\Delta)\mu(\Delta)d\Delta \tag{5.20}$$

for some suitable measure factor $\mu(\Delta)$. Let $\rho(\Delta)$ be the spectral density, so that the conformal block decomposition of the 4-point function at $z=\bar z=\frac12$ takes the form $\int d\Delta\rho(\Delta)C_\Delta^2F^{0,0}(\frac{\Delta}{2},\frac{\Delta}{2})$. We would like to define the vector $v(\Delta)=\frac{\rho(\Delta)C_\Delta^2F^{0,0}(\frac{\Delta}{2},\frac{\Delta}{2})}{g(\frac12,\frac12)\mu(\Delta)}$ so that

$$\begin{aligned}&\langle v,p_{0,0}\rangle=1,\\&\langle v,p_{m,n}\rangle=0,\quad m,n\geq 0,\ m+n\ \text{odd}.\end{aligned} \tag{5.21}$$



(a)



(b)

Figure 16: (a) Plot of $f_N(\Delta_*)$ for $c=8$, $\Delta_\phi=\frac{7}{12}$, as N ranges from $N=1$ (blue) to $N=25$ (red) with step of 2. (b) Comparison of $f_N(\Delta_*)$ ($N=27$ in solid blue and $N=33$ in solid red) with the exact DOZZ spectral function (dashed, black) for $c=2$, $\Delta_\phi=\frac{55}{12}$. Plots taken from [61].

The measure factor $\mu(\Delta)$ should be chosen so that the inner products $\langle v,v\rangle$ and $\langle p_{n,m},p_{n,m}\rangle$ are defined by convergent integrals in Δ. Recall that the Virasoro conformal block grows like $(16q)^\Delta$ at large Δ, where q is the elliptic nome, convergence of OPE demands that $\rho(\Delta)C_\Delta^2$ decays at least as fast as $16^{-\Delta}$, and so $\mu(\Delta)v(\Delta)$ decays at least as fast as $e^{-\pi\Delta}$. $p_{m,n}(\Delta)$ grows like polynomial in Δ. Thus we can choose $\mu(\Delta)\sim e^{-\lambda\Delta}$ to ensure the convergence of both $\langle v,v\rangle$ and $\langle p_{n,m},p_{n,m}\rangle$ provided $0<\lambda<2\pi$. In practice we can choose $\lambda=\pi$. Now taking the inner product of v with (5.19), we see that the spectral function is given by

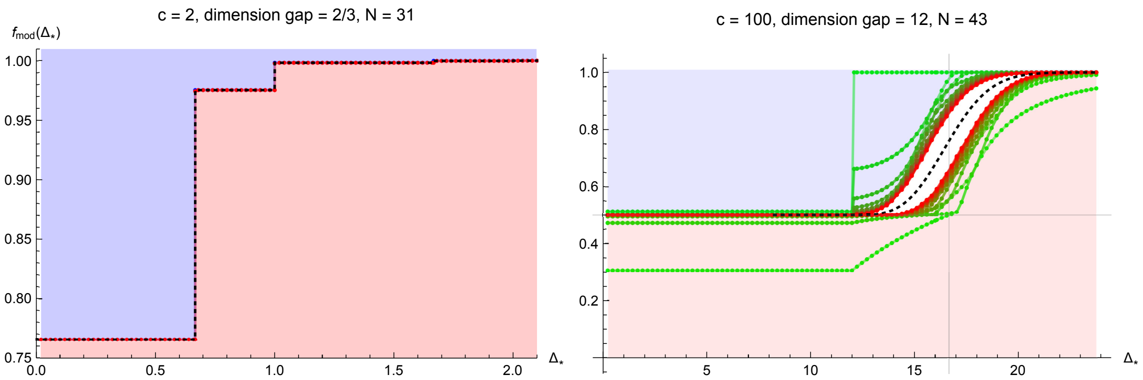
$$f(\Delta_*)=\langle v,\theta(\Delta_*-\Delta)\rangle. \tag{5.22}$$

Numerically, we may approximate v by $P_N v$, where P_N is the orthogonal projection onto the subspace spanned by $p_{0,0}$ and $p_{n,m}$ for $n+m$ odd. Now we can solve $P_N v$ simply from the linear equations

$$\begin{aligned} \langle v_N, p_{0,0} \rangle &= 1, \\ \langle v_N, p_{m,n} \rangle &= 0, \quad m,n \geq 0, \text{ odd } m+n \leq N, \end{aligned} \tag{5.23}$$

and obtain an approximate spectral function $f_N(\Delta_*) = \langle P_N v, \theta(\Delta_* - \Delta) \rangle$. Two examples are shown in Figure 16, where we see that $f_N(\Delta_*)$ stabilizes with increasing N and converges to the Liouville spectral function.

Along with similar results for mixed correlators of the form $\langle \phi_1 \phi_2 \phi_2 \phi_1 \rangle$, the uniqueness of the spectral function in $c > 1$ CFT with only scalar Virasoro primaries as suggested by the numerics would imply that the only CFT with such property is Liouville theory.



(a) (b)

Figure 17: (a) The upper (blue) and lower (red) bounds on the modular spectral function at $c = 2$ assuming a gap equal to $\Delta_{\text{mod}} = \frac{2}{3}$. (b) Upper and lower bounds on the modular spectral function assuming a gap close to Δ_{mod} at $c = 100$. The dotted black curve represents the modular spectral function of perturbative AdS_3 pure gravity computed from the thermal AdS_3 and Euclidean BTZ saddles in the gravitational path integral. Plots taken from [61].

In a similar way, we can define the modular spectral function

$$f_{\text{mod}}(\Delta_*) = \frac{\sum_{h+\tilde{h}<\Delta_*} d_{h,\tilde{h}} \chi_h(\tau = i) \overline{\chi}_{\tilde{h}}(\bar{\tau} = -i)}{Z(\tau = -\bar{\tau} = i)}, \tag{5.24}$$

and bound it from the modular crossing equation, with a suitable assumption on the spectrum $\mathscr{I}$, say a dimension gap Δ_0. If we take Δ_0 to be the modular bound on the dimension gap, $\Delta_{\text{mod}}(c)$, we expect the upper and lower bounds on $f_{\text{mod}}(\Delta_*)$ to converge and pin down the spectrum. In practice, the bounds $f^{(N)}_{\text{mod},\pm}(\Delta_*)$ obtained using up to order N derivatives on $\tau, \bar{\tau}$ converge slower with N at large c.

Two examples of the numerical bounds on the modular spectral function are shown in Figure 17. In the $c = 2$ case, assuming the maximal gap $\Delta_{\text{gap}} = \frac{2}{3}$, the bounds pin down the spectral function as that of the $SU(3)_1$ WZW model. In the $c = 100$ case, assuming a gap close to Δ_{mod}, the numerical bounds converge slowly toward the answer expected from semi-classical pure gravity in AdS_3.

5.3 Some results on superconformal theories

We will briefly describe some results on bounding the gap in the OPE of BPS operators in SCFTs in two nontrivial examples.

First, we consider the $c = 6$ $(4,4)$ SCFT defined by the supersymmetric NLSM on K3. The operator spectrum can be organized according to representations of the small $\mathcal{N} = 4$ SCA. In the *NS* sector, the representations are labeled by the weight h and $SU(2)_R$ spin ℓ of the superconformal primary. A non-BPS representation $[h,\ell]$ has $h > \ell$, and the only allow valued of $SU(2)$ spin for the $c = 6$ $\mathcal{N} = 4$ SCA is $\ell = 0$. The BPS representations on the other hand has $h = \ell$ which may take value 0 or $\frac{1}{2}$; we will simply denote the representation by $\langle \ell \rangle$. Combining the holomorphic and the anti-holomorphic sector, we have the following types of representations:

$$
\begin{aligned}
&\text{vacuum}: \ \langle 0;0\rangle, \\
&\frac{1}{2}-\text{BPS}: \ \langle \frac{1}{2};\frac{1}{2}\rangle, \\
&\frac{1}{4}-\text{BPS}: \ [s,0;\frac{1}{2}\rangle, \ \langle \frac{1}{2};s,0], \ \ s = 1,2,\cdots, \\
&\text{non}-\text{BPS}: \ [h,0;\tilde{h},0], \ \ h,\tilde{h} > 0, \ h-\tilde{h} \in \mathbb{Z}.
\end{aligned}
\tag{5.25}
$$

Here we assume to be working at a generic point on the conformal manifold, where there are no extra conserved currents beyond the superconformal algebra. Note that the $\frac{1}{4}$-BPS primaries have half-integer spin, but their half-integer level $\mathcal{N} = 4$ descendants have integer spin and would survive the GSO projection. The degeneracies of the $\frac{1}{2}$ and $\frac{1}{4}$-BPS primaries are determined by the elliptic genus. The result can be summarized by the following BPS operator content

$$
\langle 0;0\rangle \oplus 20\langle \frac{1}{2};\frac{1}{2}\rangle \oplus \bigoplus_{s=1}^{\infty} n_s \left([s,0;\frac{1}{2}\rangle \oplus \langle \frac{1}{2};s,0] \right),
\tag{5.26}
$$

with $n_1 = 90$, $n_2 = 462$, $n_3 = 1540$, etc. The 20 $\frac{1}{2}$-BPS representations give rise to 80 exactly marginal deformations preserving the $(4,4)$ SCA, corresponding to the tangent directions of the conformal manifold $\mathcal{M} \simeq \mathrm{Aut}(\Gamma_{20,4})\backslash SO(20,4)/(SO(20)\times SO(4))$.

While the BPS operator spectrum is invariant along the conformal manifold $\mathcal{M}$, except for special points where some non-BPS primaries become BPS and give rise to extra conserved higher spin currents, the non-BPS operator spectrum has highly nontrivial dependence on exactly marginal deformations. The conformal manifold $\mathcal{M}$ may be equivalently described as the moduli space of the Narain lattice $\Gamma_{20,4}$ embedded in $\mathbb{R}^{20,4}$. A lattice vector $\ell \in \Gamma_{20,4}$ maybe decomposed into its projection onto the positive and negative subspaces of $\mathbb{R}^{20,4}$, which we denote by (ℓ_L, ℓ_R). Whenever the lattice embedding is such that a lattice vector ℓ with $\ell^2 \equiv \ell_L^2 - \ell_R^2 = 2$ lies within the positive subspace, i.e. $\ell_R = 0$, the corresponding point in the moduli space $\mathcal{M}$ gives a singular K3 CFT, in that the CFT develops a continuous spectrum above a certain dimension gap. Such singularities admit an ADE classification, having to do with the enhancement of 6D gauge symmetry for type IIA string theory compactified on the K3.

An exact result that captures the moduli dependence of the K3 CFT is the following [62, 63]. Let $\phi_i^{\pm,\pm}$ be the $\frac{1}{2}$-BPS (NS,NS) primaries transforming in the doublets of holomorphic and anti-holomorphic $SU(2)$ R-symmetry. It is related by half unit spectral flow to the (R,R) primary ϕ_i^{RR}, which is a singlet under the R-symmetry and has conformal weight $(\frac{1}{4},\frac{1}{4})$. Now consider the integrated four-point function

$$\begin{aligned}&\int d^2z|z|^{-s-1}|1-z|^{-t-1}\left\langle\phi_i^{RR}(0)\phi_i^{RR}(z,\bar z)\phi_k^{RR}(1)\phi_\ell^{RR\prime}(\infty)\right\rangle\\&=2\pi\left(\frac{\delta_{ij}\delta_{k\ell}}{s}+\frac{\delta_{i\ell}\delta_{ij}}{t}+\frac{\delta_{ik}\delta_{j\ell}}{u}\right)+A_{ijk\ell}+B_{ij,k\ell}s+B_{i\ell,kj}t+B_{ik,j\ell}u+\mathscr{O}(s^2,t^2,u^2),\end{aligned}\tag{5.27}$$

where $u\equiv -s-t$, and the LHS is defined by analytic continuation from a suitable domain of s and t where the integral converges (it is also possible to expand the integral directly around $s=t=0$ by subtracting appropriate counter terms from the z-integrand). In the context of type IIB string theory compactified on K3, (5.27) has the interpretation as the string tree level scattering amplitude of four tensor multiplets in six dimensions. By supersymmetry non-renormalization results that follow from analysis of superamplitudes in 6D $(2,0)$ supergravity, which we will not review here, one can determine the coefficients $A_{ijk\ell}$ and $B_{ij,k\ell}$ as exact functions on the conformal manifold of the K3 CFT.

$A_{ijk\ell}$, for instance, is given by

$$A_{ijk\ell}=\frac{1}{16\pi^2}\left.\frac{\partial^4}{\partial y^i\partial y^j\partial y^k\partial y^\ell}\right|_{y=0}\int_{\mathscr{F}}d^2\tau\frac{\Theta_\Lambda(y|\tau,\bar\tau)}{(\eta(\tau))^{24}},\tag{5.28}$$

where $\mathscr{F}$ is the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$ acting on the upper half plane, and Λ stands for the embedded lattice $\Gamma_{20,4}\in\mathbb{R}^{20,4}$, parameterizing a point on the conformal manifold of the K3 CFT. The lattice theta function Θ_Λ is defined as

$$\Theta_\Lambda(y|\tau,\bar\tau)=e^{\frac{\pi}{2\tau_2}y^2}\sum_{\ell\in\Lambda}e^{\pi i\tau\ell_L^2-\pi i\bar\tau\ell_R^2+2\pi i\ell_L\cdot y}.\tag{5.29}$$

The auxiliary vector y lies in the positive subspace $\mathbb{R}^{20}$, whose components are in corresponding with the 20 $\frac{1}{2}$-BPS multiplets of the K3 CFT. The formula (5.28) can be understood from heterotic/type II duality as a heterotic string one-loop amplitude, hence the integration over the moduli of a torus. A similar formula for $B_{ij,k\ell}$ exists as an integral over the moduli of a genus two Riemann surface.

We will simply make use of the integrated 4-point function of a single primary, say ϕ_1^{RR}, namely A_{1111}. It turns out that A_{1111} takes non-negative real values, and diverges in two different limits: (1) a lattice vector ℓ with $\ell^2=2$ lies in the positive subspace, corresponding to the K3 developing an A_1 singularity, or (2) a null lattice vector ℓ (i.e. $\ell_L^2=\ell_R^2$) has $\ell_L^2,\ell_R^2\to 0$, corresponding to the large volume limit of the K3 surface. We would like to constrain the non-BPS operator content of the $\phi_1^{RR}\phi_1^{RR}$ OPE as a function of A_{1111}, the latter encoding dependence on the conformal manifold [64].

To proceed, we need to decompose the $\frac{1}{2}$-BPS 4-point function in terms of super-Virasoro conformal blocks of the $\mathscr{N}=(4,4)$ SCA. By a simple analysis of Ward identities one can show that the only representations that could appear in the OPE of a pair of $\frac{1}{2}$-BPS primaries are the vacuum/identity and non-BPS representations (which necessarily have vanishing $SU(2)_R$ spins). The holomorphic $c=6$ $\mathscr{N}=4$ superconformal block for the BPS correlator $\left\langle\phi^R(0)\phi^R(z)\phi^R(1)\phi^{R\prime}(\infty)\right\rangle$ (with weight $\frac{1}{4}$ external operators in the R sector) in a non-BPS channel labeled by $[h,0]$ turns to

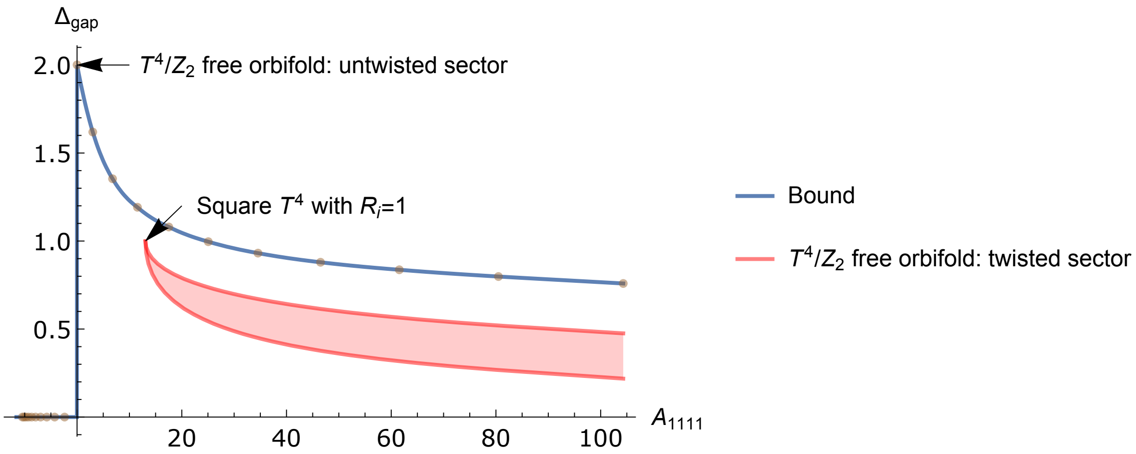


Figure 18: The solid blue curve shows the upper bound on the gap of the non-BPS primaries in $\phi_1\phi_1$ OPE, as a function of the integrated BPS four-point function A_{1111} (the orange dots are numerical data points which have already stabilized at 20 derivative order). The shaded region in red represents the gap in the OPE of $\frac{1}{2}$-BPS twist fields at a fixed point of $T^4/\mathbb{Z}_2$ with a rectangular T^4. Plot taken from [64].

be equal to the $c = 28$ bosonic Virasoro block with external weight 1 and internal weight $h+1$ up to a simple prefactor,

$$\mathscr{F}_h^{\mathscr{N}=4,R}(z) = z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}\mathscr{F}_{c=28}^{\rm Virasoro}(1,1,1,1,h+1;z). \tag{5.30}$$

The reason behind this simple relation is that (1) the $c = 6$ $\mathscr{N} = 4$ superconformal block turns out to coincide with that of the $\mathscr{N} = 2$ subalgebra (which would not be the case for $c \geq 12$) and (2) the $\mathscr{N} = 2$ blocks with external BPS operators are related to Virasoro conformal blocks in a simple way; this can be argued by studying the correlators of the $\mathscr{N} = 2$ cigar CFT. The vacuum channel block can be obtained as the $h \to 0$ limit of (5.30).

It is now straightforward to implement the constraints from the crossing equation on the correlator

$$\left\langle \phi_1^{RR}(0)\phi_1^{RR}(z,\bar z)\phi_1^{RR}(1)\phi_1^{RR\prime}(\infty)\right\rangle = \sum_{h,\tilde h} C_{h,\tilde h}^2 \mathscr{F}_h^{\mathscr{N}=4,R}(z)\overline{\mathscr{F}}_h^{\mathscr{N}=4,R}(\bar z). \tag{5.31}$$

The constraint from A_{1111} can be expressed simply as an extra linear relation on $C_{h,\tilde h}^2$, which is on the same footing as the crossing equation. The details of the analysis is too lengthy to be reproduced here. We simply summarize the result for the upper bound on the dimension gap of the first non-BPS primary in the $\phi_1\phi_1$ OPE in Figure 18.

The asymptotic value of the dimension gap bound in the $\mathscr{A}_{1111} \to \infty$ limit, while not obvious from the plot range of Figure 18, is found numerically to be $\Delta_{\rm gap} = \frac{1}{4}$. This is in fact saturated by the $\mathscr{N} = 4$ A_1 cigar CFT, which admits an exact description as the supersymmetric $SL(2,\mathbb{R})_2/U(1)$ coset, and is conjectured to capture the singular limit of the K3 CFT where the K3 develops an A_1 singularity.

Another example we will discuss very briefly is the constraint on the OPE content of $\frac{1}{2}$-BPS operators in $c = 9$ $(2,2)$ SCFTs [65], which include supersymmetric NLSM on Calabi-Yau 3-folds. As already mentioned, the latter has extended $\mathscr{N} = 2$ SCA, which puts highly nontrivial constraints on the 3-point function of BPS operators. In some cases, such as the NLSM on the quintic 3-fold, the 3-point function of say (chiral, chiral) $\frac{1}{2}$-BPS operators have been determined as an exact function on the conformal manifold. This allows us to incorporate moduli dependence into the analysis of $\frac{1}{2}$-BPS four-point functions, through the OPE coefficients of the BPS operators, thereby allowing for constraining the non-BPS operator content of the OPE using the crossing equation.

Let $\phi^{+,+}$ be a weight $(\frac{1}{2},\frac{1}{2})$ (chiral, chiral) primary, and $\phi^{-,-}$ its conjugate (anti-chiral, anti-chiral) primary. We can analyze the four-point function

$$\left\langle \phi^{+,+}(0)\phi^{-,-}(z,\bar{z})\phi^{+,+}(1)\phi^{-,-\prime}(\infty)\right\rangle \tag{5.32}$$

by its $\mathcal{N}=2$ superconformal block decomposition in the $\phi^{+,+}\phi^{+,+}$ OPE channel (CC) and in the $\phi^{+,+}\phi^{-,-}$ OPE channel (CA), and constrain the gap of the first non-BPS primary in both channels simultaneously using the $z\to 1-z$ crossing equation.

The moduli dependence can be incorporated through the OPE coefficient λ of an R-charge $(2,2)$ BPS primary in the $\phi^{+,+}\phi^{+,+}$ OPE. In Figure 19, we show an example of numerical bounds on the non-BPS spectral gap in the CC versus CA channel, as a function of λ, applied to quintic 3-fold model, where $\phi^{+,+}$ is the BPS primary associated with the Kähler deformation and the BPS OPE coefficient λ is a known function over the Kähler moduli space. Some points on the bounding surface are seen to be saturated by OPEs in free theories.

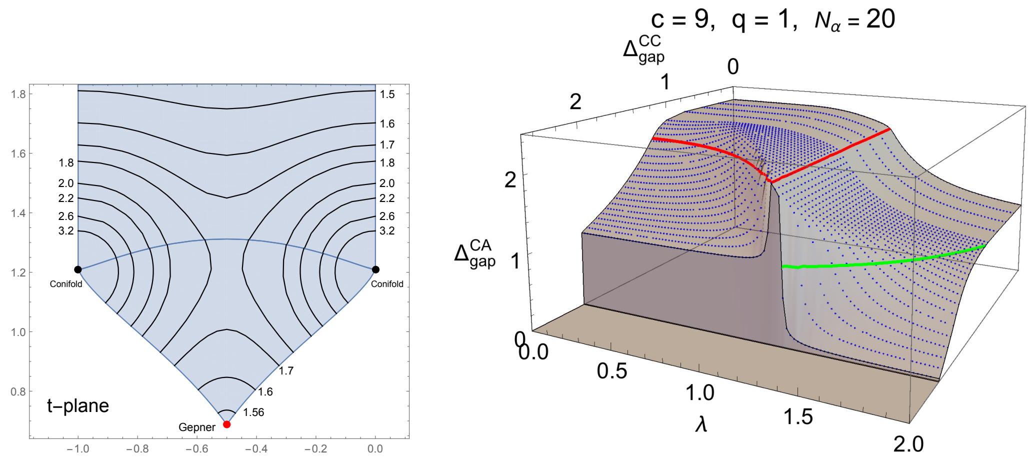


Figure 19: Left: The contour plot for the chiral ring coefficient λ over the Kähler moduli space of the quintic 3-fold [66]. The black curves are the constant λ contours. The conifold point is labeled by the black dots, while the Gepner point is shown in red. Right: Upper bounds on the gap in the CA channel Δ^{CA}_{gap} as a function of Δ^{CC}_{gap} and λ, computed using linear functionals up to 20 derivative orders. The narrow peak on the $\Delta^{CC}_{gap}=2$ cross section is expected to shrink to zero width as the derivative order is taken to infinity. Plots taken from [65].

5.4 Genus two modular bootstrap

So far, we have analyzed constraints on individual OPEs from crossing symmetry and constraints on the spectrum from the modular invariance of the partition function. One may extend the latter to the modular covariance of torus 1-point functions. Indeed, by decomposing the torus 1-point function of a primary $\phi_i(z,\bar z)$ into torus 1-point Virasoro conformal blocks,

$$\langle\phi_i(z,\bar z)\rangle_{T^2(\tau)} = \sum_j C_{ijj}\mathscr{F}_c^{T^2}(h_i,h_j,h_j;\tau)\overline{\mathscr{F}}_c^{T^2}(\tilde h_i,\tilde h_j,\tilde h_j;\bar\tau), \tag{5.33}$$

it is straightforward to write down the modular crossing equation for $\tau\to-1/\tau$ as a set of linear relations on the structure constants C_{ijj}. However, there are no a priori positivity property for C_{ijj}, and it is not easy to implement such constraints using semidefinite programming.

As for the crossing equations for the sphere 4-point functions, ultimately one would like to combine the crossing equation for all primaries in the CFT simultaneously, but this is very difficult beyond a few primaries. So while the sphere 4-point crossing equations and the modular covariance of torus 1-point functions in principle give the complete set of consistency relations for a 2D unitary CFT, we need to organize these crossing equations in a way that takes into account OPEs of a large set of primaries and satisfies certain positivity property.

This is partially achieved by consideration of the modular constraints on the partition function of the CFT on a genus two Riemann surface. Such a partition function may be decomposed into genus two Virasoro conformal blocks, whose coefficients involving either C_{ijk}^2 or $C_{ijj}C_{ikk}$. Here we will restrict our attention to the former. The moduli space of the genus two Riemann surface has complex dimension 3, and may be parameterized in a number of different ways. We will describe three parameterizations below.

The first parameterization is through the period matrix

$$\Omega = \begin{pmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{pmatrix}, \tag{5.34}$$

which obeys $\Omega=\Omega^T$ and that $\mathrm{Im}\Omega$ is positive definite. The entries Ω_{ab} are defined as follows. Pick a basis of the dimension 1 homology group $\alpha_1,\alpha_2,\beta_1,\beta_2$, with the intersection numbers $\alpha_a\cdot\alpha_b=\beta_a\cdot\beta_b=0$, $\alpha_a\cdot\beta_b=\delta_{ab}$. There are two independent holomorphic 1-forms ω_a, $a=1,2$, such that

$$\oint_{\alpha_a}\omega_b=\delta_{ab},\quad \oint_{\beta_a}\omega_b=\Omega_{ab}. \tag{5.35}$$

The basis $\{\alpha_a,\beta_a\}$ is defined up to $Sp(4,\mathbb{Z})$ transformations, of the form

$$\alpha_a\to D_{ab}\alpha_b+C_{ab}\beta_b,\quad \beta_a\to B_{ab}\alpha_b+A_{ab}\beta_b, \tag{5.36}$$

where the 2×2 matrices A,B,C,D have integer entries and obey

$$\gamma\begin{pmatrix}0 & \mathbb{I}\\ -\mathbb{I} & 0\end{pmatrix}\gamma^T=\begin{pmatrix}0 & \mathbb{I}\\ -\mathbb{I} & 0\end{pmatrix},\quad \gamma\equiv\begin{pmatrix}A & B\\ C & D\end{pmatrix}. \tag{5.37}$$

Correspondingly, the period matrix transforms by

$$\Omega\mapsto\gamma(\Omega)\equiv(A\Omega+B)(C\Omega+D)^{-1}. \tag{5.38}$$

The genus two partition function Z_2 of a CFT may be written as a function of $(\Omega,\overline{\Omega})$ over the Siegel upper half space $\mathrm{Im}\Omega\succ0$, possibly with singularities at the loci $(\gamma(\Omega))_{12}=0$ for some

$\gamma \in Sp(4,\mathbb{Z})$, up to a universal conformal anomaly factor that depends only on the central charge c. Thus, to specify Z_2, we must specify a conformal frame, or equivalently a choice of metric within its conformal equivalence class. We will discuss two particularly useful conformal frames later. Alternatively, we may avoid the conformal anomaly altogether by considering the ratio $Z_2/(Z_X)^c$, where Z_X is the genus two partition function of a given $c=1$ CFT, say a free boson, in the same conformal frame. $Z_2/(Z_X)^c$ would then be a well defined function of $(\Omega,\overline{\Omega})$, that is invariant under (5.38).

The second parameterization is the Schottky parameterization, where we realize the genus two Riemann surface as the quotient

$$(\mathbb{C}\cup\{\infty\}-\Lambda)/\langle\alpha,\beta\rangle, \tag{5.39}$$

where α,β are two loxodromic elements of PSL$(2,\mathbb{C})$, and $\langle\alpha,\beta\rangle$ is the free (non-Abelian) group generated by α and β, known as the Schottky group. Λ is the limit set of the Schottky group action. There is a useful formula that expresses the period matrix through infinite products over elements of the Schottky group, which we will not explain in detail here. Due to its connection to AdS_3 gravity, the Schottky parameterization is useful in writing the classical limit ($c\to\infty$) of genus two (and higher) Virasoro conformal blocks.

The third parameterization, which is closely related to Schottky parameterization and useful for the computation of general Virasoro conformal blocks, is based on the plumbing construction, where we form the genus two Riemann surface by gluing a pair of two-holed discs on the complex plane along their boundaries, using three PSL$(2,\mathbb{C})$ transformations. We can choose the PSL$(2,\mathbb{C})$ gluing maps to be an inversion map of the form $z\mapsto q/z$, for a complex parameter q, that identifies a pair of inner circular boundaries centered at the origin (or shifted as necessary), or a scaling map of the form $z\mapsto qz$ that identifies an inner circular boundary to an outer boundary. The three complex plumbing parameters q_i, $i=1,2,3$, are moduli of the resulting Riemann surface.

The plumbing construction also specifies a conformal frame, which we refer to as the plumbing frame. The Virasoro conformal blocks in the plumbing frame have the special property that their $c\to\infty$ limits with fixed primary weights are finite, thus permitting a recursive representation in c, generalizing Zamolodchikov's recursion formula for sphere 4-point blocks. Schematically, the general recursion formula takes the form

$$\begin{aligned}\mathcal{G}_c(\{h_a^{\rm ext}\};\{h_i\};\{q_i\}) &= \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\rm ext}\};\{h_i\};\{q_i\})\\ &\quad+\sum_j\sum_{r\geq 2,s\geq 1}\frac{Q_j^{r,s}}{c-c_{rs}(h_j)}\mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\rm ext}\};h_j\to h_j+rs;\{q_i\}),\end{aligned} \tag{5.40}$$

where $h_a^{\rm ext}$ stand for possible external weights, h_i are the internal weights, q_i are the plumbing parameters, $\mathcal{G}_\infty(0;0;\{q_j\})$ is the $c=\infty$ limit of the vacuum block in the plumbing frame, $\mathcal{G}_{SL(2)}$ is the corresponding global $SL(2)$ conformal block, the summation in the second line runs over internal weights labeled by the index j, $Q_j^{r,s}$ is a known function of h_j and its adjacent weights on the trivalent graph associated with the conformal block, and $\mathcal{G}_{c_{rs}(h_j)}$ in the summand is the conformal block evaluated at central charge $c_{rs}(h_j)$ with the internal weight h_j shifted to h_j+rs. $\mathcal{G}_{SL(2)}$ is straightforward to compute as an expansion in the plumbing parameters. The vacuum block admits a closed form expression in terms of the Schottky parameterization [67],

$$\mathscr{G}_\infty(0;0;\{q_j\}) = \prod_{\gamma\in\mathscr{P}}\prod_{n=2}^{\infty}(1-q_\gamma^n)^{-\frac{1}{2}}, \tag{5.41}$$

where $\mathscr{P}$ is the set of primitive conjugacy classes of the Schottky group. q_γ is defined as follows: pick any element of the class γ, as an element of $\mathrm{PSL}(2,\mathbb{C})$ it is conjugate to $\begin{pmatrix} q_\gamma^{1/2} & 0 \\ 0 & q_\gamma^{-1/2}\end{pmatrix}$, with $|q_\gamma|<1$.

The partition function computed in the plumbing frame is not manifestly modular invariant however, due to the conformal anomaly factor. To formulate the modular crossing equation in a computable manner, we will work in a yet different conformal frame, which we refer to as the Renyi frame, as follows [68].

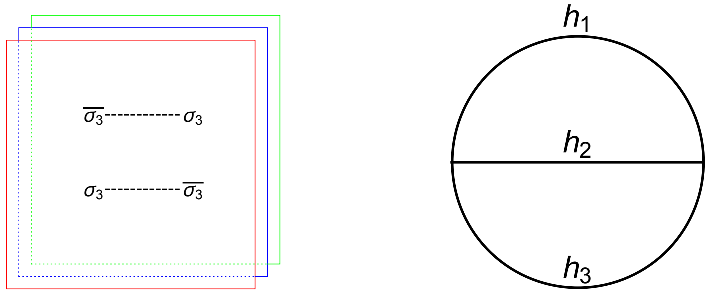


Figure 20: Left: The 3-fold cover of the Riemann sphere with four branch points is a genus-two surface. The partition function of the CFT on the covering surface can be regarded as the four-point function of $\mathbb{Z}_3$ twist fields in the 3-fold product CFT on the sphere. Right: The genus two conformal block associated with the $\sigma_3\bar{\sigma}_3$ OPE channel.

We begin by considering the “Renyi surface": a complex 1-dimensional locus in the moduli space of the genus two Riemann surface, that is a 3-fold cover of the Riemann sphere branched at four points. Explicitly, we may realize the Renyi surface as a curve in $\mathbb{P}^1\times\mathbb{P}^1$, written in affine coordinates (x,y) as

$$y^3 = \frac{(x-x_1^+)(x-x_2^+)}{(x-x_1^-)(x-x_2^-)}. \tag{5.42}$$

The genus two partition function on the Renyi surface, in a particular conformal frame specified by the covering map, is equal to the four point function of a pair of $\mathbb{Z}_3$ twist fields σ_3 and a pair of anti-twist fields $\overline{\sigma}_3$,

$$\left\langle \sigma_3(x_1^+)\sigma_3(x_2^+)\overline{\sigma}_3(x_1^-)\overline{\sigma}_3(x_2^-)\right\rangle, \tag{5.43}$$

in the 3-fold symmetric product CFT on the Riemann sphere. Its genus two conformal block decomposition takes the form

$$\left\langle \sigma_3(0)\overline{\sigma}_3(z,\bar z)\sigma_3(1)\overline{\sigma}'_3(\infty)\right\rangle = \sum_{i,j,k} C_{ijk}^2\mathscr{F}_c(h_i,h_j,h_k;z)\overline{\mathscr{F}}_c(\tilde h_i,\tilde h_j,\tilde h_k;\bar z), \tag{5.44}$$

where the conformal block $\mathscr{F}_c(h_i,h_j,h_k;z)$ can be computed as an expansion in z by directly summing over contributions from Virasoro$^{\otimes 3}$ descendants in the $\sigma_3\overline{\sigma}_3$ OPE. The Renyi frame conformal block is related to the plumbing frame block $\mathscr{G}_c$ for the Renyi surface (still parameterized by the same cross ratio z) by a conformal anomaly factor,

$$\mathscr{F}_c(h_1,h_2,h_3;z) = \exp\left[c\mathscr{F}^{cl}(z)\right]\mathscr{G}_c(h_1,h_2,h_3;z). \tag{5.45}$$

Here we record the first few terms in the z-expansion of $\mathscr{F}^{cl}$,

$$\mathscr{F}^{cl}(z) = -\frac{2}{9}\log(z) + 6\left(\frac{z}{27}\right)^2 + 162\left(\frac{z}{27}\right)^3 + 3975\left(\frac{z}{27}\right)^4 + 96552\left(\frac{z}{27}\right)^5 + 2356039\left(\frac{z}{27}\right)^6 + \cdots, \tag{5.46}$$

and of the $c\to\infty$ limit of $\mathscr{G}_c$, for generic weights h_1,h_2,h_3,

$$\begin{aligned}&\mathscr{G}_\infty(h_1,h_2,h_3;z)\\ &= \left(\frac{z}{27}\right)^{h_1+h_2+h_3}\left\{1+\left[\frac{h_1+h_2+h_3}{2}+\frac{(h_2-h_3)^2}{54h_1}+\frac{(h_3-h_1)^2}{54h_2}+\frac{(h_1-h_2)^2}{54h_3}\right]z+\cdots\right\}.\end{aligned} \tag{5.47}$$

If one of the h_i's, say h_3 vanishes, corresponding to the vacuum channel, then the conformal block exists only when $h_1 = h_2$, and is given by the $h_3 \to 0$ limit of the generic block with h_1 and h_2 set to be equal and positive. The vacuum block with $h_1 = h_2 = h_3 = 0$ is special, and is *not* given by the $h_i \to 0$ limit of the generic block, as the latter would incorrectly include contributions from null states.

The genus two conformal block for the Renyi surface may also be mapped to the 3-fold-pillow frame,

$$\mathscr{F}_c(h_1,h_2,h_3;z) = (z(1-z))^{-\frac{7c}{72}}(\theta_3(\tau))^{-\frac{5}{18}}q^{h_1+h_2+h_3-\frac{c}{8}}\sum_{n=0}^{\infty}A_n(h_1,h_2,h_3)q^n, \tag{5.48}$$

where $q = e^{\pi i\tau(z)}$ is the elliptic nome, and the coefficients $A_n(h_1,h_2,h_3)$ are positive valued provided that the unitary bound is obeyed.

We may analyze the crossing symmetry $z \to 1-z$ of (5.44), by expanding around $z = \bar{z} = \frac{1}{2}$ as in the analysis of sphere 4-point functions. As a simple but nontrivial example, just using first order derivatives in z and $\bar{z}$, the crossing equation implies a critical domain D in the space of dimension triples $(\Delta_1,\Delta_2,\Delta_3)$, such that the OPE coefficients C_{ijk} with $(\Delta_i,\Delta_j,\Delta_k)$ outside the domain D are bounded by those with weights that lie within D. An example is shown in Figure 21.

To deform away from the Renyi surface, rather than changing the geometry we can simply insert the stress-energy tensor $T(z)$ or more general Virasoro descendants of the identity, $L_{-n_1}\cdots L_{-n_k}\cdot 1$, on any of the three sheets covering the Riemann sphere. This allows us to write down the expansion of the genus two block of general moduli around the Renyi locus, thereby formulating the complete set of genus two modular crossing equations for C^2_{ijk}.

6. Summary

A good amount of knowledge of 2D CFTs has been accumulated over the last four decades. Yet, the solvable 2D CFTs are largely limited to free or rational theories, a handful of noncompact coset models, and orbifolds thereof.

Even the simplest class of rational CFTs, namely the meromorphic CFTs, are far from being classified: it is easy to constrain their spectrum from the modular invariance of the partition function, but it is quite difficult to show that there exists a set of structure constants that obey the entire

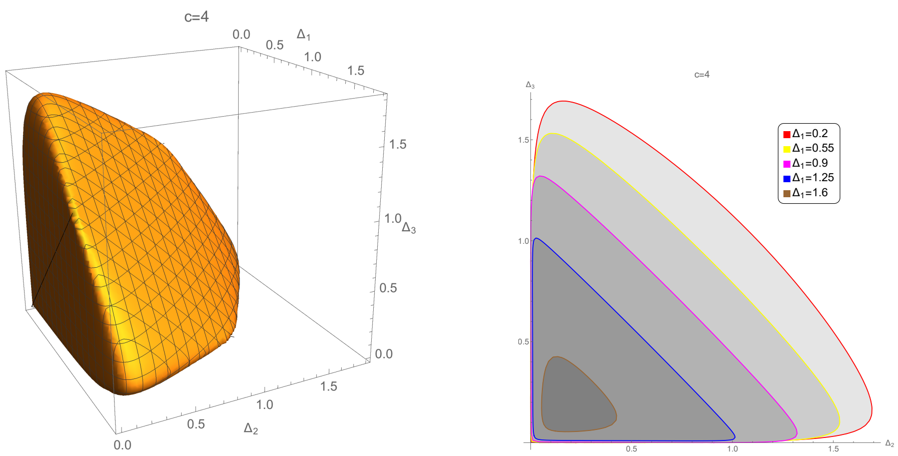


Figure 21: Left: Three-dimensional plots of the domain $D_\Delta^{(3)}$ for $c = 4$. Right: Cross-sections of this domain at constant Δ_1. Plot taken from [68].

set of crossing equations and modular covariant torus one-point functions without additional inputs (e.g. symmetry assumptions [69]).

Our knowledge of interacting, compact irrational CFTs is embarrassingly limited: the only robust constructions are based on exactly marginal deformation of $(2,2)$ superconformal theories. There isn't a single example of a unitary, compact CFT that is known to admit no extra higher spin conserved currents beyond Virasoro symmetry.[7] So either we are missing important constraints (e.g. is $\frac{c-1}{12}$ really the optimal upper bound on the twist gap [58]?), or we have no clue what the *generic* 2D CFT looks like. Clearly, we are only beginning to carve out the landscape of 2D CFTs, and much is to be learned in the decades to come.

Acknowledgements

I would like to thank the organizers of TASI 2017 for the invitation and the participating students for their enthusiasm and stimulating questions. I am grateful to Chi-Ming Chang, Minjae Cho, Scott Collier, Petr Kravchuk, Ying-Hsuan Lin, Shu-Heng Shao, David Simmons-Duffin, Yifan Wang for collaborations, and Thomas Dumitrescu, Daniel Harlow, Simeon Hellerman, Christoph Keller, Zohar Komargodski, Alex Maloney, Eric Perlmutter, Slava Rychkov, Ashoke Sen, Andy Strominger, Cumrun Vafa, for illuminating conversations. This work is supported by a Simons Investigator Award from the Simons Foundation and by DOE grant DE-FG02-91ER40654.

References

[1] G. W. Moore and N. Seiberg, *Classical and Quantum Conformal Field Theory*, *Commun. Math. Phys.* **123** (1989) 177.

[2] V. G. Kac and D. A. Kazhdan, *Structure of representations with highest weight of infinite dimensional Lie algebras*, *Adv. Math.* **34** (1979) 97–108.

[7]The critical coupled Potts model of [54] is a potential candidate, but to the best of my knowledge the possibility of it being equivalent to one of the known rational coset models is not yet ruled out.

[3] B. L. Feigin and D. B. Fuchs, *VERMA MODULES OVER THE VIRASORO ALGEBRA*, .

[4] D. Friedan, Z.-a. Qiu, and S. H. Shenker, *Conformal Invariance, Unitarity and Two-Dimensional Critical Exponents*, *Phys. Rev. Lett.* **52** (1984) 1575–1578.

[5] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*. Graduate texts in contemporary physics. Springer, New York, NY, 1997.

[6] D. Buchholz, S. Doplicher, and R. Longo, *On Noether's Theorem in Quantum Field Theory*, *Annals Phys.* **170** (1986) 1.

[7] D. Harlow, *The Ryu-Takayanagi Formula from Quantum Error Correction*, arXiv:1607.03901.

[8] D. Whalen, *An algorithm for evaluating Gram matrices in Verma modules of W-algebras*, arXiv:1412.0759.

[9] A. B. Zamolodchikov, *Conformal symmetry in two-dimensions: an explicit recurrence formula for the conformal partial wave amplitude*, *Commun. Math. Phys.* **96** (1984) 419–422.

[10] A. B. Zamolodchikov, *Conformal symmetry in two-dimensional space: Recursion representation of conformal block*, *Theoretical and Mathematical Physics* **73** (Oct., 1987) 1088–1093.

[11] J. Maldacena, D. Simmons-Duffin, and A. Zhiboedov, *Looking for a bulk point*, arXiv:1509.03612.

[12] H. Dorn and H. J. Otto, *Two and three point functions in Liouville theory*, *Nucl. Phys.* **B429** (1994) 375–388, [hep-th/9403141].

[13] A. B. Zamolodchikov and A. B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, *Nucl. Phys.* **B477** (1996) 577–605, [hep-th/9506136].

[14] B. Ponsot and J. Teschner, *Liouville bootstrap via harmonic analysis on a noncompact quantum group*, hep-th/9911110.

[15] J. Teschner, *Liouville theory revisited*, *Class. Quant. Grav.* **18** (2001) R153–R222, [hep-th/0104158].

[16] L. Hadasz, Z. Jaskolski, and P. Suchanek, *Modular bootstrap in Liouville field theory*, *Phys. Lett.* **B685** (2010) 79–85, [arXiv:0911.4296].

[17] N. Seiberg, *Notes on quantum Liouville theory and quantum gravity*, *Prog. Theor. Phys. Suppl.* **102** (1990) 319–349.

[18] K. S. Narain, *New Heterotic String Theories in Uncompactified Dimensions < 10*, *Phys. Lett.* **B169** (1986) 41–46.

[19] L. J. Dixon, D. Friedan, E. J. Martinec, and S. H. Shenker, *The Conformal Field Theory of Orbifolds*, *Nucl. Phys.* **B282** (1987) 13–73.

[20] I. Frenkel, J. Lepowsky, and A. Meurman, *VERTEX OPERATOR ALGEBRAS AND THE MONSTER*. 1988.

[21] E. Witten, *Nonabelian Bosonization in Two-Dimensions*, *Commun. Math. Phys.* **92** (1984) 455–472.

[22] D. Gepner and E. Witten, *String Theory on Group Manifolds*, *Nucl. Phys.* **B278** (1986) 493–549.

[23] P. Goddard, A. Kent, and D. I. Olive, *Unitary Representations of the Virasoro and Supervirasoro Algebras*, *Commun. Math. Phys.* **103** (1986) 105–119.

[24] K. Gawedzki and A. Kupiainen, *G/h Conformal Field Theory from Gauged WZW Model*, *Phys. Lett.* **B215** (1988) 119–123.

[25] C. G. Callan, Jr. and L. Thorlacius, *SIGMA MODELS AND STRING THEORY*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Particles, Strings and Supernovae (TASI 88) Providence, Rhode Island, June 5-July 1, 1988*, pp. 795–878, 1989.

[26] F. Gliozzi, J. Scherk, and D. I. Olive, *Supersymmetry, Supergravity Theories and the Dual Spinor Model*, *Nucl. Phys.* **B122** (1977) 253–290.

[27] S. Odake, *Extension of* $N = 2$ *Superconformal Algebra and Calabi-yau Compactification*, *Mod. Phys. Lett.* **A4** (1989) 557.

[28] S. Odake, *C = 3-d Conformal Algebra With Extended Supersymmetry*, *Mod. Phys. Lett.* **A5** (1990) 561.

[29] D. Kutasov, *Some properties of (non)critical strings*, in *Spring School on String Theory and Quantum Gravity (to be followed by Workshop) Trieste, Italy, April 15-23, 1991*, pp. 102–141, 1991. hep-th/9110041.

[30] T. Eguchi and A. Taormina, *Unitary Representations of* $N = 4$ *Superconformal Algebra*, *Phys. Lett.* **B196** (1987) 75.

[31] C. Kounnas, M. Porrati, and B. Rostand, *ON N=4 extended superLiouville theory*, *Phys. Lett.* **B258** (1991) 61–69.

[32] T. Eguchi and A. Taormina, *On the Unitary Representations of* $N = 2$ *and* $N = 4$ *Superconformal Algebras*, *Phys. Lett.* **B210** (1988) 125.

[33] W. Boucher, D. Friedan, and A. Kent, *Determinant Formulae and Unitarity for the N=2 Superconformal Algebras in Two-Dimensions or Exact Results on String Compactification*, *Phys. Lett.* **B172** (1986) 316.

[34] E. Witten, *Elliptic Genera and Quantum Field Theory*, *Commun. Math. Phys.* **109** (1987) 525.

[35] T. Eguchi, H. Ooguri, A. Taormina, and S.-K. Yang, *Superconformal Algebras and String Compactification on Manifolds with SU(N) Holonomy*, *Nucl. Phys.* **B315** (1989) 193.

[36] A. B. Zamolodchikov, *Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory*, *JETP Lett.* **43** (1986) 730–732. [Pisma Zh. Eksp. Teor. Fiz.43,565(1986)].

[37] L. J. Dixon, *SOME WORLD SHEET PROPERTIES OF SUPERSTRING COMPACTIFICATIONS, ON ORBIFOLDS AND OTHERWISE*, in *Summer Workshop in High-energy Physics and Cosmology Trieste, Italy, June 29-August 7, 1987*, 1987.

[38] D. Green, Z. Komargodski, N. Seiberg, Y. Tachikawa, and B. Wecht, *Exactly Marginal Deformations and Global Symmetries*, *JHEP* **06** (2010) 106, [arXiv:1005.3546].

[39] R. Dijkgraaf, *Instanton strings and hyperKahler geometry*, *Nucl. Phys.* **B543** (1999) 545–571, [hep-th/9810210].

[40] N. Seiberg and E. Witten, *The D1 / D5 system and singular CFT*, *JHEP* **04** (1999) 017, [hep-th/9903224].

[41] F. Larsen and E. J. Martinec, *U(1) charges and moduli in the D1 - D5 system*, *JHEP* **06** (1999) 019, [hep-th/9905064].

[42] A. B. Zamolodchikov, *Thermodynamic Bethe Ansatz in Relativistic Models. Scaling Three State Potts and Lee-yang Models*, *Nucl. Phys.* **B342** (1990) 695–720.

[43] A. B. Zamolodchikov, *TBA equations for integrable perturbed SU(2)-k x SU(2)-l / SU(2)-k+ l coset models*, *Nucl. Phys.* **B366** (1991) 122–132.

[44] A. B. Zamolodchikov, *Thermodynamic Bethe ansatz for RSOS scattering theories*, *Nucl. Phys.* **B358** (1991) 497–523.

[45] A. B. Zamolodchikov, *From tricritical Ising to critical Ising by thermodynamic Bethe ansatz*, *Nucl. Phys.* **B358** (1991) 524–546.

[46] A. B. Zamolodchikov, *On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories*, *Phys. Lett.* **B253** (1991) 391–394.

[47] A. B. Zamolodchikov, *On the thermodynamic Bethe ansatz equation in sinh-Gordon model*, *J. Phys.* **A39** (2006) 12863–12887, [hep-th/0005181].

[48] M. Karowski and P. Weisz, *Exact Form-Factors in (1+1)-Dimensional Field Theoretic Models with Soliton Behavior*, *Nucl. Phys.* **B139** (1978) 455–476.

[49] H. M. Babujian, A. Foerster, and M. Karowski, *The form factor program: A review and new results, the nested SU(N) off-shell Bethe ansatz and the 1/N expansion*, *Theor. Math. Phys.* **155** (2008) 512–522. [Teor. Mat. Fiz.155,13(2008)].

[50] V. P. Yurov and A. B. Zamolodchikov, *TRUNCATED CONFORMAL SPACE APPROACH TO SCALING LEE-YANG MODEL*, *Int. J. Mod. Phys.* **A5** (1990) 3221–3246.

[51] V. P. Yurov and A. B. Zamolodchikov, *Truncated fermionic space approach to the critical 2-D Ising model with magnetic field*, *Int. J. Mod. Phys.* **A6** (1991) 4557–4578.

[52] S. Rychkov and L. G. Vitale, *Hamiltonian truncation study of the ^{4} theory in two dimensions*, *Phys. Rev.* **D91** (2015) 085011, [arXiv:1412.3460].

[53] B. R. Greene, C. Vafa, and N. P. Warner, *Calabi-Yau Manifolds and Renormalization Group Flows*, *Nucl. Phys.* **B324** (1989) 371.

[54] V. Dotsenko, J. L. Jacobsen, M.-A. Lewis, and M. Picco, *Coupled Potts models: Self-duality and fixed point structure*, *Nucl. Phys.* **B546** (1999) 505–557.

[55] E. Witten, *Phases of N=2 theories in two-dimensions*, *Nucl. Phys.* **B403** (1993) 159–222, [hep-th/9301042].

[56] S. Hellerman, *A Universal Inequality for CFT and Quantum Gravity*, *JHEP* **08** (2011) 130, [arXiv:0902.2790].

[57] D. Friedan and C. A. Keller, *Constraints on 2d CFT partition functions*, *JHEP* **10** (2013) 180, [arXiv:1307.6562].

[58] S. Collier, Y.-H. Lin, and X. Yin, *Modular Bootstrap Revisited*, arXiv:1608.06241.

[59] S. El-Showk and M. F. Paulos, *Bootstrapping Conformal Field Theories with the Extremal Functional Method*, *Phys. Rev. Lett.* **111** (2013), no. 24 241601, [arXiv:1211.2810].

[60] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, *Solving the 3d Ising Model with the Conformal Bootstrap II. c-Minimization and Precise Critical Exponents*, *J. Stat. Phys.* **157** (2014) 869, [arXiv:1403.4545].

[61] S. Collier, P. Kravchuk, Y.-H. Lin, and X. Yin, *Bootstrapping the Spectral Function: On the Uniqueness of Liouville and the Universality of BTZ*, arXiv:1702.00423.

[62] E. Kiritsis, N. A. Obers, and B. Pioline, *Heterotic / type II triality and instantons on K(3)*, *JHEP* **01** (2000) 029, [hep-th/0001083].

[63] Y.-H. Lin, S.-H. Shao, Y. Wang, and X. Yin, *Supersymmetry Constraints and String Theory on K3*, arXiv:1508.07305.

[64] Y.-H. Lin, S.-H. Shao, D. Simmons-Duffin, Y. Wang, and X. Yin, *N=4 Superconformal Bootstrap of the K3 CFT*, arXiv:1511.04065.

[65] Y.-H. Lin, S.-H. Shao, Y. Wang, and X. Yin, *(2,2) Superconformal Bootstrap in Two Dimensions*, arXiv:1610.05371.

[66] P. Candelas, X. C. De La Ossa, P. S. Green, and L. Parkes, *A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, *Nucl. Phys.* **B359** (1991) 21–74.

[67] S. Giombi, A. Maloney, and X. Yin, *One-loop Partition Functions of 3D Gravity*, *JHEP* **08** (2008) 007, [arXiv:0804.1773].

[68] M. Cho, S. Collier, and X. Yin, *Genus Two Modular Bootstrap*, arXiv:1705.05865.

[69] D. Gaiotto, *Monster symmetry and Extremal CFTs*, arXiv:0801.0988.