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Quasi-PDFs and pseudo-PDFs

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We discuss the physical nature of quasi-PDFs, especially the reasons for the strong nonperturbative evolution pattern which they reveal in actual lattice gauge calculations. We argue that quasi-PDFs may be treated as hybrids of PDFs and the rest-frame momentum distributions of partons. The latter is also responsible for the transverse momentum dependence of TMDs. The resulting convolution structure of quasi-PDFs necessitates using large probing momenta $p_3 \gtrsim 3$ GeV to get reasonably close to the PDF limit. To deconvolute the rest-frame distribution effects, we propose to use a method based directly on the coordinate representation. We treat matrix elements $M(z_3, p_3)$ as distributions $\mathscr{M}(\nu, z_3^2)$ depending on the Ioffe-time $\nu = p_3 z_3$ and the distance parameter z_3^2. The rest-frame spatial distribution is given by $\mathscr{M}(0, z_3^2)$. Using the reduced Ioffe function $\mathfrak{M}(\nu, z_3^2) \equiv \mathscr{M}(\nu, z_3^2)/\mathscr{M}(0, z_3^2)$ we divide out the rest frame effects, including the notorious link renormalization factors. The ν-dependence remains intact and determines the shape of PDFs in the small z_3 region. The residual z_3^2 dependence of the $\mathfrak{M}(\nu, z_3^2)$ is governed by perturbative evolution. The Fourier transform of $\mathscr{M}(\nu, z_3^2)$ produces pseudo-PDFs $\mathscr{P}(x, z_3^2)$ that generalize the light-front PDFs onto spacelike intervals. On the basis of these findings we propose a new method for extraction of PDFs from lattice calculations.

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1. Introduction

The usual parton distribution functions (PDFs) $f(x)$ [1] measured in deep inelastic scattering and other inclusive processes are defined through matrix elements of certain bilocal operators on the light cone $z^2 = 0$. This fact prevents a direct extraction of these functions from Euclidean lattice gauge theory simulations. Still, recently, X. Ji [2] proposed to use separations $z = (0,0,0,z_3)$ which are purely space-like. Then one can define parton distributions in the $k_3 = yp_3$ component of the parton momentum. These quasi-PDFs $Q(y, p_3)$ approach the light-cone PDFs $f(y)$ in the limit of large hadron momenta $p_3 \to \infty$. The quasidistribution method can be also applied to distribution amplitudes (DAs). Lattice calculations of quasi-PDFs were discussed in Refs. [3, 4, 5]. The results for the pion quasi-distribution amplitudes (quasi-DAs) were reported in Ref. [6]. The lattice studies demonstrated a very strong change of quasidistributions with the probing momentum p_3, which cannot be explained by perturbative evolution.

In our recent papers [7, 8], we have demonstrated that quasi-PDFs can be obtained from the transverse momentum dependent distributions (TMDs) $\mathscr{F}(x, k_\perp^2)$. We also showed that the $k_\perp^2$-dependence of TMDs plays the major role in the nonperturbative p_3-evolution of quasi-PDFs and quasi-DAs. As. In these papers, we have based our studies on the formalism of virtuality distribution functions [9, 10]. The TMD/quasi-PDF relation allows to use simple models for TMDs for building models for the nonperturbative evolution of quasi-PDFs and quasi-DAs. The results obtained in our papers [7, 8] are in good agreement with the observed p_3-evolution patterns obtained in lattice calculations.

In the present talk, we outline the results and ideas formulated in our next paper [11]. First, it was demonstrated that the connection between TMDs and quasi-PDFs is, in fact, a mere consequence of Lorentz invariance. Thus, it may be derived in a much simpler way than in Ref. [7].

Then we show that the TMD/quasi-PDF connection formula may be rewritten in a form that allows a simple physical interpretation. Namely, it tells that when a hadron is moving, the parton k_3 momentum may be treated as coming from two sources. First, there is the motion of the hadron as a whole. It contributes the xp_3 part to the total k_3 value, and is governed by the dependence of the TMD $\mathscr{F}(x, \kappa^2)$ on its first, i.e., x, argument. The remaining part $(k_3 - xp_3)$ comes from the rest-frame momentum distribution, and is governed by the dependence of the TMD on its second argument, κ^2. Thus, the quasi-PDFs may be treated as hybrids of PDFs and momentum distributions of partons in a hadron at rest.

Since x appears in both arguments of the TMD, the quasi-PDFs have a convolution nature. This fact explains a rather complicated pattern of the change of quasi-PDFs with the probing momentum p_3, i.e., a strong nonperturbative p_3-evolution. One needs to have rather large values $p_3 \sim 3$ GeV to "stop" the nonperturbative evolution and get sufficiently close to the PDF limit.

It should be emphasized that PDFs are given by the $k_\perp$ integral of the TMDs. Since our goal is to extract PDFs, information about a particular shape of the $k_\perp$-dependence is redundant. In a sense, one would prefer a situation when this $k_\perp$-dependence is given by a delta-function $\delta(k_\perp^2)$. Then the quasi-PDF $Q(y, p_3)$ would coincide with the PDF $f(y)$ for all probing momenta p_3. However, a physical TMD is a more involved function of $k_\perp$. What is worse, this irrelevant $k_\perp$-dependence of the TMDs results in a complicated structure of quasi-PDFs, necessitating large values of p_3 just *to wipe out* information about the $k_\perp$-dependence. One may ask if there are more economical ways of eliminating the unwanted $k_\perp$ effects.

The problem is that in TMD-based momentum representation, quasi-PDFs are given by a convolution of PDF-type x-dependence and $k_\perp$-dependence. The latter is related to the momentum distribution of the hadron at rest and basically reflects the finite size of the system. So, our next idea in Ref. [11] is that the deconvolution of the finite-size effects is much simpler in the coordinate representation. To this end, we introduce the functions $\mathscr{P}(x,-z^2)$ that we call *pseudo-PDFs*. They generalize the light-cone PDFs $f(x)$ onto spacelike intervals. In particular, one can take $z=(0,0,0,z_3)$. The x-dependence of the pseudo-PDFs is obtained through Fourier transforms of the *Ioffe-time* [12] *distributions* (ITDs) [13] $\mathscr{M}(\nu,z_3^2)$ with respect to $\nu=-(pz)$. It should be noted that the rest-frame momentum distribution is determined by $\mathscr{M}(0,z_3^2)$.

The ITDs are basically given by generic matrix elements like $M(z,p)=\langle p|\phi(0)\phi(z)|p\rangle$ which are the starting point of any lattice calculation. To have the ITD formulation, we should treat $M(z,p)$ as functions of $\nu=-(pz)$ and z^2 (or $\nu=p_3z_3$ and z_3^2 if we take $z=(0,0,0,z_3)$). The large-z_3 behavior of the pseudo-PDFs is governed by the same nonperturbative physics that determines the $k_\perp$-dependence of TMDs. To get PDFs, one should either take small z_3 directly, or extrapolate $\mathscr{P}(x,z_3^2)$ to small z_3 values. In this sense, taking small z_3 for pseudo-PDFs is analogous to taking large p_3 for quasi-PDFs.

However, a serious advantage of the pseudo-PDFs is that, unlike the quasi-PDFs, they have the "canonical" $-1\leq x\leq 1$ support for all z_3^2. To access the $z_3\to 0$ limit through extrapolation, we propose to use the reduced pseudo-PDF $\mathfrak{P}(x,z_3^2)\equiv\mathscr{P}(x,z_3^2)/\mathscr{M}(0,z_3^2)$, in which the nonperturbative effects due to the rest-frame density are divided out. Thus, we argue that one should use the *reduced ITD* $\mathfrak{M}(\nu,z_3^2)\equiv\mathscr{M}(\nu,z_3^2)/\mathscr{M}(0,z_3^2)$ as the starting object for lattice calculations of PDFs. When $z_3\to 0$, the reduced ITDs obey the perturbative evolution equation, with $1/z_3$ serving as an evolution scale parameter.

2. Parton Distributions

2.1 Ioffe-time distributions and Pseudo-PDFs

Studying hard processes, experimentalists work with hadrons. Theorists work with quarks. Thus, an important object is the amplitude $T(k,p)$ describing hadron-parton transition, with p being the hadron momentum, and k that of the quark. The transition can be described also using the coordinate space for quarks. Then we deal with the matrix element of a bilocal operator. We will write it in a generic form $\langle p|\phi(0)\phi(z)|p\rangle\equiv M(z,p)$ using scalar fields notations for quarks, since the basic concept of the parton distributions is not changed by spin complications.

By Lorentz invariance, the function $M(z,p)$ depends on z through two scalar invariants, the *Ioffe time* [12] $(pz)\equiv-\nu$ and the interval z^2 (or $-z^2$ if we want a positive value for spacelike z):

$$M(z,p)=\mathscr{M}(-(pz),-z^2)\ . \tag{2.1}$$

The function $\mathscr{M}(\nu,-z^2)$ is the *Ioffe-time distribution* (ITD) [13].

It can be shown [7, 14] that, for all contributing Feynman diagrams, the Fourier transform of $\mathscr{M}(\nu,-z^2)$ with respect to (pz) has the $-1\leq x\leq 1$ support, i.e.,

$$\mathscr{M}(\nu,-z^2)=\int_{-1}^{1}dx\,e^{-ix\nu}\,\mathscr{P}(x,-z^2)\ . \tag{2.2}$$

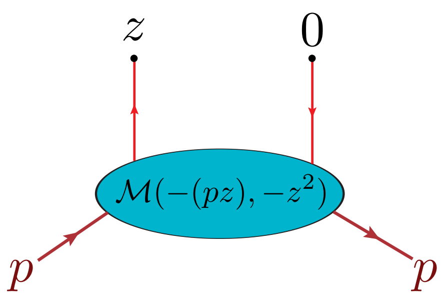

Figure 1: Ioffe-time distribution.

Note that Eq. (2.2) gives a covariant definition of x. There is no need to assume that $p^2 = 0$ or $z^2 = 0$ or take an infinite momentum frame, etc., to define x. As we will see, the function $\mathscr{P}(x,-z^2)$ generalizes the concept of the usual (or light-cone) parton distributions onto the case of non-lightlike intervals z. Following Ref. [11], we will call it pseudo-PDF. The $-1 \leq x \leq 1$ support region for pseudo-PDF is dictated by analytic properties of Feynman diagrams, and is determined by the structure of denominators of propagators. It is not affected by numerators present in non-scalar theories. The inverse transformation

$$\mathscr{P}(x,-z^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\, e^{-ix\nu}\, \mathscr{M}(\nu,-z^2) \tag{2.3}$$

may be treated as a direct definition of pseudo-PDFs as Fourier transforms of the ITDs $\mathscr{M}(\nu,-z^2)$ with respect to ν for fixed z^2. Thus, pseudo-PDFs stay just one step from the starting matrix element $M(z,p)$ (written in the form of ITD), and provide the most general object from which other parton distributions may be obtained as particular cases.

2.2 Collinear Parton Distributions, Quasi-PDFs and TMDs

Take a light-like z, say, that having just z_- component. Then $\nu = -p_+z_-$, and we can define the usual collinear (or light-cone) parton distribution $f(x) = \mathscr{P}(x,0)$

$$\mathscr{M}(-p_+z_-,0) = \int_{-1}^{1} dx\, f(x)\, e^{-ixp_+z_-} \,. \tag{2.4}$$

It has the usual interpretation of the probability that the parton carries fraction x of the hadron's p_+ momentum. Note that the $z^2 \to 0$ limit is nontrivial in QCD and other renormalizable theories, since $\mathscr{M}(\nu,z^2)$ has $\sim \ln z^2$ singularities. The latter reflect perturbative evolution of parton densities. Within the operator product expansion approach (OPE), the $\ln z^2$ singularities are subtracted, e.g., by dimensional renormalization, and then $\ln(1/z^2) \to \ln\mu^2$. Resulting PDFs depend on renormalization scale μ, $f(x) \to f(x,\mu^2)$. If one keeps z^2 spacelike, then no subtractions are needed. For pseudo-PDFs $\mathscr{P}(x,-z^2)$, the interval z^2 serves as the ultraviolet (UV) cut-off, and $-1/z^2$ is similar to the OPE scale μ^2.

Taking a spacelike $z = \{0,0,0,z_3\}$ in the frame, where the hadron momentum is $p = (E,\mathbf{0}_\perp,P)$, one can define quasi-PDFs [2] as a Fourier transform of $M(z_3,P)$ with respect to z_3

$$Q(y,P) = \frac{P}{2\pi}\int_{-\infty}^{\infty} dz_3\, e^{-iyPz_3}\, M(z_3,P). \tag{2.5}$$

It is instructive to rewrite this integral in terms of the Ioffe-time distribution

$$Q(y,P) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu \, e^{-iy\nu} \mathscr{M}(\nu, \nu^2/P^2) \ . \tag{2.6}$$

Unlike in the pseudo-PDF definition, the ν-variable appears in both arguments of the ITD. We also see that $Q(y,P)$ tends to the usual PDF $f(y)$ in the $P \to \infty$ limit, as far as $\mathscr{M}(\nu, \nu^2/P^2) \to \mathscr{M}(\nu, 0)$.

The dependence of $\mathscr{M}(\nu, -z^2)$ on ν governs the x-dependence of $f(x)$, i.e. the longitudinal momentum structure of the hadron, while its z^2-dependence is directly connected with the transverse momentum distributions (TMDs). To show this, let us introduce TMDs. Take again the frame where $p = (E, \mathbf{0}_\perp, P)$, and choose z that has $z_+ = 0$, nonzero z_- and, in addition, nonzero $z_\perp = \{z_1, z_2\}$ components. Then $z^2 = -z_\perp^2$, and the TMD is defined by

$$\mathscr{M}(\nu, z_\perp^2) = \int_{-1}^{1} dx \, e^{ix\nu} \int d^2k_\perp \, e^{-i(k_\perp z_\perp)} \mathscr{F}(x, k_\perp^2) \ . \tag{2.7}$$

The parton again carries xp_+, but it also has transverse momentum $k_\perp$ which is Fourier-conjugate to $z_\perp$. Thus, the transverse momentum dependence of TMDs is governed by the z^2-dependence of ITDs. Note that, due to the rotational invariance in $z_\perp$ plane, this TMD depends on $k_\perp^2$ only.

While the quasi-PDF is derived from a matrix element involving purely "longitudinal" $z = z_3$, the dependence of $\mathscr{M}(\nu, z_3^2)$ on z_3^2 is given by the same function that defines the TMD by Eq. (2.7). To relate quasi-PDFs and TMDs, we take $z_\perp = \{0, \nu/P\}$ in Eq. (2.7) and substitute the resulting representation into the expression (2.6) for the quasi-PDF. This gives [7, 11]

$$Q(y,P) = P\int_{-1}^{1} dx \int_{-\infty}^{\infty} dk_1 \mathscr{F}(x, k_1^2 + (y-x)^2P^2) \ . \tag{2.8}$$

According to this relation, the quasi-PDF variable y has the $-\infty < y < \infty$ support, because the components of the transverse momentum $k_\perp$ in $\mathscr{F}(x, k_\perp^2)$ are not restricted.

3. Structure of Quasi-PDFs

3.1 Momentum Distributions

Since the variable k_1 is integrated over in Eq. (2.8), it makes sense to introduce the function

$$\mathscr{R}(x, k_3) \equiv \int_{-\infty}^{\infty} dk_1 \mathscr{F}(x, k_1^2 + k_3^2) \tag{3.1}$$

depending on the remaining momentum variable k_3 only (of course, according to Eq. (3.1), $\mathscr{R}(x,k_3)$ depends on k_3 through k_3^2). Also, instead of the quasi-PDFs $Q(y,P)$ that refer to the fraction $y \equiv k_3/P$, one may consider distributions in the momentum k_3 itself: $R(k_3,P) \equiv Q(k_3/P,P)/P$. Then we can rewrite Eq. (2.8) as

$$R(k_3,P) = \int_{-1}^{1} dx \mathscr{R}(x, k_3 - xP) \ . \tag{3.2}$$

For a hadron at rest, we have a one-dimensional function

$$R(k_3, P=0) \equiv r(k_3) = \int_{-1}^{1} dx \mathscr{R}(x, k_3) \ , \tag{3.3}$$

that describes a primordial distribution of k_3 (or any other component of $\mathbf{k}$) in a rest-frame hadron. It may be directly obtained through a parameterization of the rest-frame density

$$\mathscr{M}(0,z_3^2) = \int_{-\infty}^{\infty} dk_3\, r(k_3)\, e^{ik_3z_3} \ . \tag{3.4}$$

According to Eq. (3.3), the rest-frame momentum distribution $r(k_3)$ is obtained from $\mathscr{R}(x,k_3)$ by taking the x-integral. Similarly, integrating $\mathscr{R}(x,k_3)$ over k_3 gives the collinear PDF

$$\int_{-\infty}^{\infty} dk_3 \mathscr{R}(x,k_3) = \int d^2k_\perp\, \mathscr{F}(x,k_\perp^2) = f(x) \ . \tag{3.5}$$

Now we can give the following interpretation of the formula (3.2). According to it, in a moving hadron, the parton momentum $k_3 = xP + (k_3 - xP)$ has two parts. The xP part comes from the motion of the hadron as a whole with the probability governed by x-dependence of $\mathscr{R}(x,k_3)$. The probability to get the remaining part $(k_3 - xP)$ is governed by the dependence of $\mathscr{R}(x,k_3)$ on its second argument, k_3, associated with the primordial rest-frame momentum distribution.

3.2 Factorized models for TMDs and quasi-PDFs

Both arguments of $\mathscr{R}(x,k_3 - xP)$ in Eq. (3.2) contain the integration parameter x. As a result, the shape of the momentum distributions $R(k,P)$ (and, hence, of the quasi-PDFs) is influenced by the form both of PDFs and rest-frame distributions. To illustrate the "hybrid" nature of momentum distributions and quasi-PDFs, we will use a factorized model $\mathscr{R}(x,k_3) = f(x)r(k_3)$. For the ITD, this Ansatz corresponds to the factorization assumption

$$\mathscr{M}^{\rm fact}(\nu,-z^2) = \mathscr{M}(\nu,0)\mathscr{M}(0,-z^2) \ . \tag{3.6}$$

A popular choice is a Gaussian dependence of TMDs on $k_\perp$. It gives

$$r_G(k_3) = \frac{1}{\sqrt{\pi}\Lambda} e^{-k_3^2/\Lambda^2} \ \text{ or } \ r_G(z_3^3) = e^{-z_3^2\Lambda^2/4} \ \text{ for the rest frame density.} \tag{3.7}$$

Then the factorized Gaussian model for the momentum distribution has the form

$$R_G^{\rm fact}(k_3,P) = \frac{1}{\Lambda\sqrt{\pi}} \int_{-1}^{1} dx\, f(x)\, e^{-(k_3-xP)^2/\Lambda^2} \ . \tag{3.8}$$

For PDF we choose a simple function $f(x) = 4(1-x)^3\theta(0 \le x \le 1)$ resembling valence quark distributions. From Fig. 2, one can see that the curve for $R(k,P)$ changes from a Gaussian shape for small P to a shape resembling a stretched PDF for large P. For small P/Λ values, we may approximate

$$R(k_3,P) = \int_{-1}^{1} dx\, f(x)\, r(k_3 - xP) \approx r(k_3 - \tilde{x}P) \tag{3.9}$$

($\tilde{x} =$ average x, in our model $\tilde{x} = 0.2$), i.e., for small P, the $R(k_3,P)$ curve approximately keeps its shape, but the maximum shifts to the right when P increases. For large P, we have

$$r_G(k_3 - xP) = \frac{1}{\sqrt{\pi}\Lambda} e^{-(k_3-xP)^2/\Lambda^2} \to \frac{1}{P}\,\delta(x - k_3/P) \ , \tag{3.10}$$

i.e., the combination $PR(k_3,P)$ corresponding to quasi-PDF $Q(y = k_3/P,P)$ in the large P limit converts into a scaling function $f(k_3/P) = f(y)$ coinciding with the input PDF.

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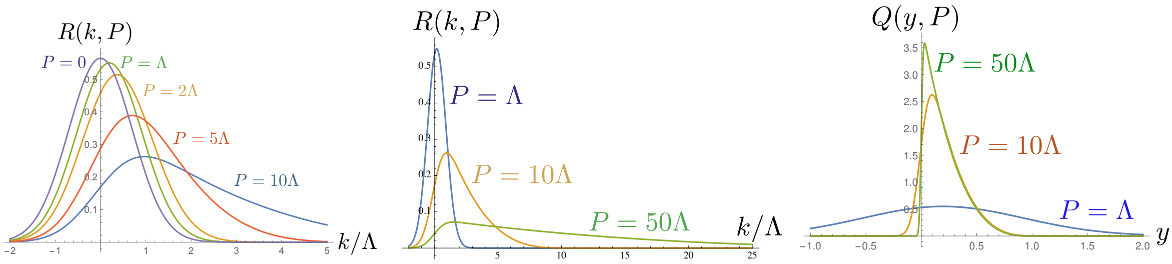

Figure 2: Momentum distribution $R(k,P)$ and quasi-PDF $Q(y,P)$ for different momentum P values.

3.3 QCD case

In QCD we deal with matrix elements of the $\mathcal{M}^\alpha(z,p) \equiv \langle p|\bar{\psi}(0)\,\gamma^\alpha\,\hat{E}(0,z;A)\,\psi(z)|p\rangle$ type, where $\hat{E}(0,z;A)$ is the standard $0 \to z$ straight-line gauge link. Due to the vector index α, the function $\mathcal{M}^\alpha(z,p)$ may be decomposed into p^α and z^α parts

$$\mathcal{M}^\alpha(z,p) = 2p^\alpha \mathcal{M}_p(-(zp),-z^2) + z^\alpha \mathcal{M}_z(-(zp),-z^2)\ . \tag{3.11}$$

In the standard definition of the TMD, we have $z_+ = 0$ and take $\alpha = +$. As a result, the z^α-part drops out, and TMD $\mathcal{F}(x,k_\perp^2)$ is related to $\mathcal{M}_p(\nu,z_\perp^2)$ by the scalar formula. To remove the z^α-contamination from quasi- PDF, we take the time component of $\mathcal{M}^\alpha(z=z_3,p)$ and define

$$\mathcal{M}^0(z_3,p) = 2p^0 \int_{-1}^{1} dy\, Q(y,P)\, e^{iyPz_3}\ . \tag{3.12}$$

Then quasi-PDF $Q(y,P)$ is related to TMD $\mathcal{F}(x,k_\perp^2)$ by the scalar formula (2.8).

4. Quasi-PDFs vs Pseudo-PDFs and Ioffe-time Distributions

According to the definition of quasi-PDF $Q(y,P)$ in Eq.(2.6), they are obtained from the ITD $\mathcal{M}(\nu,z_3^2)$ by integration over $z_3 = \nu/P$ lines in the $\{\nu,z_3\}$ plane, see Fig. 3. They tend to the horizontal $z_3 = 0$ line in the $P \to \infty$ limit, and the resulting quasi-PDFs approach PDF. It should

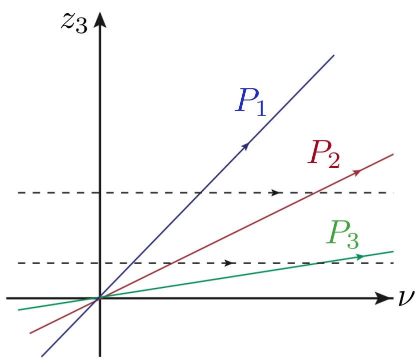

Figure 3: Lines of integration for quasi-PDF $Q(y,P)$ in the $\{\nu,z_3\}$ plane (solid lines) and for pseudo-PDFs $\mathcal{P}(x,z_3^2)$ (dashed lines).

be noted that this approach is non-trivial, since $Q(y,P)$ has perturbative evolution with respect to P for large P. In general, quasi-PDFs have the $-\infty < y < \infty$ support region. As we have seen, in the case of the soft factorized models, the support shrinks to $-1 < y < 1$ in the $P \to \infty$ limit. If one adds perturbative corrections due to hard gluon exchanges, they generate terms with the $-\infty < y < \infty$ support even in the $P \to \infty$ limit. Such terms should be removed through the use of matching conditions [2].

Pseudo-PDFs, according to their definition (2.3), are given by integration of $\mathscr{M}(\nu, z_3^2)$ over $z_3 =$const lines. They always have the $-1 \leq x \leq 1$ support. For small z_3^2, the pseudo-PDFs $\mathscr{P}(x, z_3^2)$ have perturbative evolution with respect to $1/z_3$. At the leading logarithm level, they are close to usual PDFs $f(x, C^2/z_3^2)$ with C being the matching coefficient, $C_{\overline{\rm MS}} = 2e^{-\gamma_E} \approx 1.12$.

The fact that quasi-PDFs $Q(y,P)$ are given by integration of $\mathscr{M}(\nu, z_3^2)$ over the $z_3 = \nu/P$ lines leads to their x-convolution structure, even if $\mathscr{M}(\nu, z_3^2)$ factorizes, i.e., $\mathscr{M}(\nu, z_3^2) = \mathscr{M}(\nu, 0)\mathscr{M}(0, z_3^2)$. An alternative approach [11] is to convert lattice data for $\mathscr{M}(Pz_3, z_3^2)$ into the data for $\mathscr{M}(\nu, z_3^2)$. The next step is to take the reduced function

$$\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathscr{M}(\nu, z_3^2)}{\mathscr{M}(0, z_3^2)} , \tag{4.1}$$

i.e. divide ITD $\mathscr{M}(\nu, z_3^2)$ by the rest-frame density $\mathscr{M}(0, z_3^2)$. In factorized case, the reduced ITD converts into $\mathscr{M}(\nu, 0)$, and what formally remains is to take its Fourier transform to get PDF $f(x)$. Another advantage of using the reduced ITD is that the z_3^2-dependence due to self-energy of gauge link cancels in the ratio, because the UV-induced z_3^2-dependence is multiplicative (see Refs. [15, 16, 17, 18] for recent progress in this field.)

4.1 Evolution of Ioffe-time distributions

Originally, the Ioffe-time distributions $Q(\nu, \mu^2)$ were defined [13] as functions whose Fourier transforms with respect to ν were given by usual OPE PDFs $f(x, \mu^2)$. Thus, their dependence on the renormalization parameter μ (say, $\overline{\rm MS}$ scale) is completely determined by the evolution equation for PDFs $f(x, \mu^2)$. In case of pseudo-PDFs, the parameter $1/z_3$ for small z_3 plays the role of μ. A subtlety is that $\mathscr{M}(\nu, z_3^2)$ has an extra z_3 dependence induced by the renormalization of the gauge link. However, this z_3-dependence cancels in the reduced ITD $\mathfrak{M}(\nu, z_3^2)$. As a result, for small z_3^2 we have the leading-order evolution equation

$$\frac{d}{d \ln z_3^2} \mathfrak{M}(\nu, z_3^2) = -\frac{\alpha_s}{2\pi} C_F \int_0^1 du \, B(u) \mathfrak{M}(u\nu, z_3^2)$$

with the same nonsinglet evolution kernel

$$B(u) = \left[\frac{1+u^2}{1-u}\right]_+$$

as in Ref. [13]. Examples of real and imaginary parts of ITD are shown in Fig. 4, together with functions $B \otimes \mathscr{M}$ governing their perturbative evolution. One can see that there are no perturbative evolution for $\mathscr{M}(0, z_3^2)$ [vector current is conserved]. Also, Im $\mathscr{M}(0, z_3^2) = 0$, i.e., the rest-frame distribution $\mathscr{M}(0, z_3^2) = 0$ is purely real.

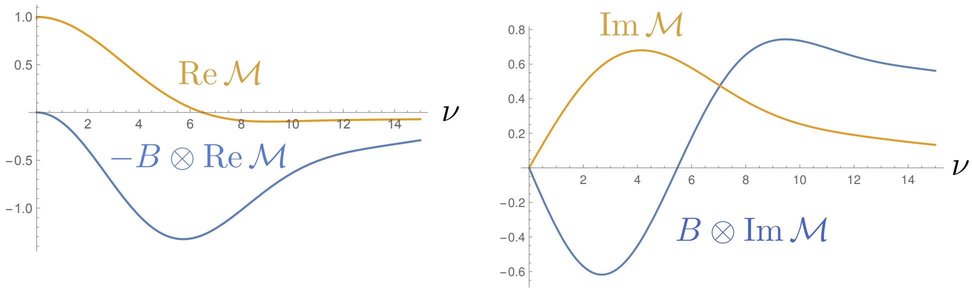

Figure 4: Real and imaginary parts of model Ioffe-time distribution $\mathscr{M}(\nu,0)$ and the function $B\otimes\mathscr{M}$ governing their evolution.

4.2 Nonfactorizable cases

The perturbative evolution produces z_3^2-dependence for the reduced ITD $\mathfrak{M}(\nu,z_3^2)$. Hence, it will unavoidably violate factorization. This z_3^2-dependence should be visible in the data as $\ln(1/z_3^2\Lambda^2)$ spikes for small z_3^2.

Take for illustration $\mathscr{P}^{\rm soft}(x,z_3^2) = f(x)e^{-z_3^2\Lambda^2/4}$ for the soft part (corresponding to the TMD $\mathscr{F}^{\rm soft}(x,k_\perp^2) = f(x)e^{-k_\perp^2/\Lambda^2}/\pi\Lambda^2)$ and choose $\alpha_s/\pi = 0.1$ for the hard part in which we use the incomplete gamma-function $\Gamma[0,z_3^2\Lambda^2/4]$ instead of a straightforward $\ln(1/z_3^2\Lambda^2)$ function. In such a model for the hard part, the evolution stops for large z_3^2.

For the real part of the ITD, the evolution effects are the largest for $\nu\sim 6$. As one can see from Fig. 5, they are clearly visible for $z_3\Lambda \simeq 1.5$. Assuming $\Lambda = 300$MeV, we obtain that $z_3\Lambda = 1.5$ for $z_3 = 1$ fm.

The reduced ITD $\mathfrak{M}(\nu,z_3^2)$ may also have residual z_3^2-dependence from the violation of factorization in the soft part. To illustrate these effects, we take the model pseudo-PDF $\mathscr{P}^{\rm soft}(x,z_3^2) = f(x)e^{-x(1-x)z_3^2\tilde\Lambda^2/4}$ corresponding to the model TMD $\mathscr{F}^{\rm soft}(x,k_\perp^2) = f(x)e^{-k_\perp^2/x(1-x)\tilde\Lambda^2}/[\pi x(1-x)\tilde\Lambda^2])$, whose dependence on $k_\perp^2$ comes through the $k_\perp^2/x(1-x)$ combination advocated by the light-front quantization proponents. To have the same $\langle k_\perp^2\rangle$ we need $\tilde\Lambda^2 = \frac{15}{2}\Lambda^2$. The $z_3 = 1$ fm distance on the right graph of Fig. 5 corresponds to $z_3^2\Lambda^2 = 2.25$.

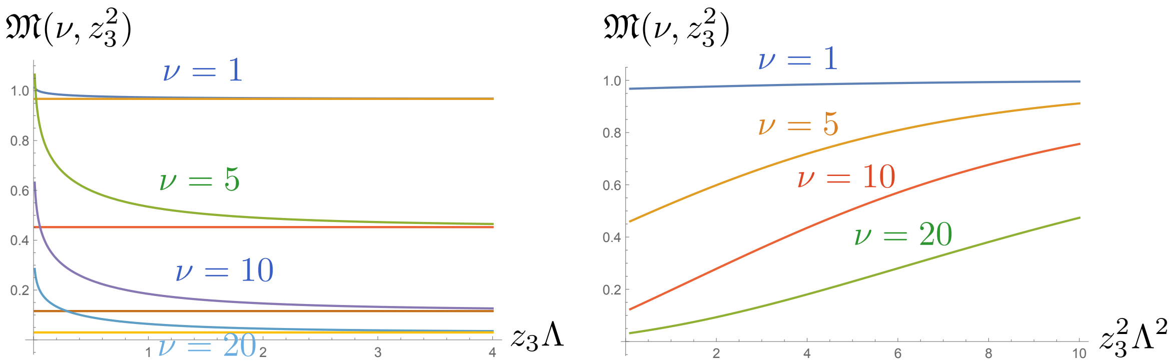

Figure 5: Violation of factorization for the reduced ITD due to perturbative evolution (left) and due to non-factorizable form of TMD (right).

In a situation when factorization is violated both by evolution and non-perturbative effects, a possible strategy is to extrapolate $\mathfrak{M}(\nu, z_3^2)$ to $z_3^2 = 0$ from not too small values of z_3^2, say, from those above 0.5 fm^2. The resulting function $\mathscr{M}^{\rm soft}(\nu, 0)$ may be treated as the Ioffe-time distribution producing the PDF $f_0(x)$ "at low normalization point". The remaining $\ln(1/z_3^2\Lambda^2)$ spikes at small z_3 will generate its evolution.

5. Summary

In this talk, we described our recent work [7, 11] on the structure of parton quasi-distributions. We found that the quasi-PDFs are hybrids of PDFs and primordial rest-frame momentum distributions. The resulting complicated convolution nature of quasi-PDFs necessitates large probing momenta $p_3 \gtrsim 3$ GeV to wipe out the primordial effects.

To avoid convolution structures, we proposed to use pseudo-PDFs $\mathscr{P}(x, z_3^2)$, the functions most closely related (by a Fourier transform) to the Ioffe-time distributions $\mathscr{M}(\nu, z_3^2)$, the primary objects both for continuum and lattice studies of parton distributions. One of the advantages of the pseudo-PDFs is that they have the same "canonical" $-1 \le x \le 1$ support as usual PDFs. Furthermore, their z_3^2-dependence for small z_3^2 is governed by a usual evolution equation.

An important ingredient of the proposed program for the pseudo-PDF-based lattice extraction of PDFs is the use of the reduced Ioffe-time distributions given by the ratio $\mathscr{M}(\nu, z_3^2)/\mathscr{M}(0, z_3^2)$ in which the z_3^2-dependence of the primordial rest-frame density $\mathscr{M}(0, z_3^2)$ is divided out from the original Ioffe-time distribution $\mathscr{M}(\nu, z_3^2)$. The use of this ratio also provides a very simple and efficient way for getting rid of the z_3^2-dependence related to the ultraviolet divergences generated by the self-energy and vertex corrections to the gauge link.

While the write-up of this talk was in preparation, the exploratory lattice studies of the pseudo-PDFs have been performed [19, 20] . Their results have completely confirmed expectations formulated in Ref. [11].

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