PROCEEDINGS OF SCIENCE

The QCD beta function at five loops

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In this talk we present a new result for the five-loop beta function in QCD which is valid for a general gauge group. We briefly summarise and discuss its impact on the strong coupling evolution in perturbative QCD. Furthermore we present a new result for the Higgs decay rate into gluons at N^{4}LO in perturbative QCD, which requires the five-loop beta function as input.

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1. Introduction

By governing the scale evolution of the (reduced) strong coupling constant,

$$a(\mu) = \frac{\alpha_s(\mu)}{4\pi} = \left(\frac{g_s(\mu)}{4\pi}\right)^2, \tag{1.1}$$

the beta function (in the MS scheme of dimensional regularisation [13]),

$$\beta(a) = \frac{da}{d\log\mu^2} = -\sum_{n=0}^{\infty} \beta_n a^{n+2}, \tag{1.2}$$

is of fundamental importance to QCD. Pioneering calculations of the 1-loop beta function in the 60s and early 70s [1, 2, 3, 4, 5] lead to the discovery of asymptotic freedom in QCD. Since then tremendous progress in perturbative calculations has lead to the determination of the beta function up to five loops [6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 17]. Several of these calculations were based on the use of the R^*-operation which we briefly discuss in section 2. Here we wish to report in particular on the calculation of ref [16]. This calculation made use of the background field method, where the renormalisation constant, associated to the background field Z_B, is related to that of the running coupling via [19, 20]:

$$Z_a Z_B = 1 \tag{1.3}$$

This relation allows the extraction of Z_a and therefore also β at five loops from the knowledge of the poles of the 5-loop background field self energy (see figure 1). We will briefly discuss our results in section 3.

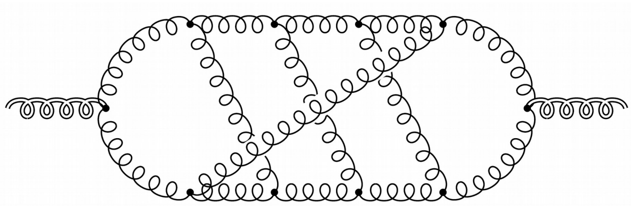


Figure 1: This figure shows a typical five-loop Feynman graph contributing to the background field self energy.

2. Calculational Method

Utilising the techniques of IR rearrangement [29], (see figure 2) and the R^* methods introduced in [32] we were able to extract these poles from five-loop Feynman graphs which can be factorised into products of trivial one-loop graphs times four-loop graphs, which in turn can be computed efficiently with the FORCER program [21, 22, 23]. In order to perform these calculations we build a computational framework based on FORM [26, 27, 28], QGRAF [24] for the generation of diagrams and the colour package of ref [25].

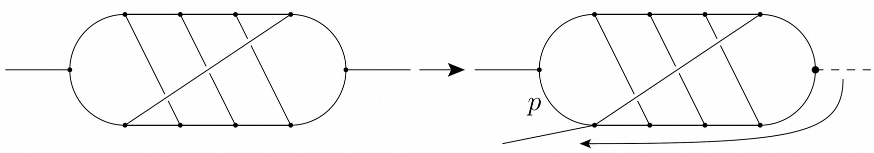


Figure 2: This figure illustrates the procedure of IR rearrangement to simplify the calculation of UV poles of Feynman integrals.

The R^*-operation

The R^*-operation acting on a Euclidean Feynman graph Γ can be written as

$$R^*(\Gamma) = \sum_{\substack{S\subseteq\Gamma,\tilde{S}\subseteq\Gamma\\ S\cap\tilde{S}=\emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma/S\setminus\tilde{S}. \tag{2.1}$$

Here the sum goes over disjoint pairs of UV and IR spinneys S and $\tilde{S}$ respectively. The UV spinney is defined, identically as in the case of the R-operation, as a possibly disjoint set of 1PI subgraphs. To define the IR spinney $\tilde{S}$ is slightly more involved than for UV spinneys and is for this reason referred to the literature [30, 31, 32]. The remaining contracted graph $\Gamma/S\setminus\tilde{S}$ is constructed by first contracting the spinney S in Γ and then deleting the lines and vertices contained in $\tilde{S}$ in Γ/S. The UV and IR counterterm operations Z and $\tilde{Z}$ are then defined recursively via:

$$Z(\Gamma) = -K\Big(\sum_{\substack{S\subsetneq\Gamma,\tilde{S}\subsetneq\Gamma\\ S\cap\tilde{S}=\emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma/S\setminus\tilde{S}\Big), \tag{2.2}$$

where one omits in the sum over UV spinneys the full graph Γ and

$$\tilde{Z}(\Gamma_0) = -K\Big(\sum_{\substack{S\subsetneq\Gamma_0,\tilde{S}\subsetneq\Gamma_0\\ S\cap\tilde{S}=\emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma_0/S\setminus\tilde{S}\Big), \tag{2.3}$$

where one omits in the sum over IR spinneys the scaleless vacuum Feynman graph Γ_0. The K-operation isolates the pole parts in the dimensional regulator ε. The identity $R^*(\Gamma_0) = 0$ can be used to find relations among IR and UV counterterms in dimensional regularisation.

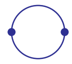
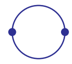
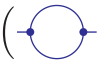
As an example let us consider the Feynman integral:

$$\Gamma = \quad = \int \frac{d^Dk_1}{i\pi^{D/2}}\frac{d^Dk_2}{i\pi^{D/2}}\frac{1}{(k_1^2)^2(k_2+P)^2(k_1+k_2)^2} \tag{2.4}$$

Here we have labeled the lines from 1-3, such that their corresponding momenta are parameterised as $q_1 = k_1, q_2 = k_2+P, q_3 = k_1+k_2$ respectively. The example features an IR divergence when the momentum is flowing through the dotted line 1 vanishes. It also features two UV divergent subgraphs, corresponding to the full graph or the subgraph which consists of lines 2 and 3. The action of the R^* operation yields:

$$R^*\Big(\quad\Big) = \quad + Z\Big(\quad\Big) + Z\Big(\quad\Big) * \quad + \tilde{Z}\Big(\quad\Big) * \quad + \tilde{Z}\Big(\quad\Big) * Z\Big(\quad\Big) * 1\,. \tag{2.5}$$

The IR counterterm can be evaluated as

$$\tilde{Z}\Big($$ $$\Big) = \tilde{Z}\Big($$  $$\Big) = -Z\Big($$  $$\Big) = K\Big($$  $$\Big)\,. \qquad (2.6)$$

3. Results

The QCD beta function at five loops

Let us now present some numeric results for the beta-function for the gauge group SU(3), fixing also the number of quark flavours n_f to a few physically relevant values:

$$\begin{aligned}
\widetilde{\beta}(\alpha_s, n_f\!=\!3) &= 1 + 0.565884\,\alpha_s + 0.453014\,\alpha_s^2 + 0.676967\,\alpha_s^3 + 0.580928\,\alpha_s^4\,,\\
\widetilde{\beta}(\alpha_s, n_f\!=\!4) &= 1 + 0.490197\,\alpha_s + 0.308790\,\alpha_s^2 + 0.485901\,\alpha_s^3 + 0.280601\,\alpha_s^4\,,\\
\widetilde{\beta}(\alpha_s, n_f\!=\!5) &= 1 + 0.401347\,\alpha_s + 0.149427\,\alpha_s^2 + 0.317223\,\alpha_s^3 + 0.080921\,\alpha_s^4\,,\\
\widetilde{\beta}(\alpha_s, n_f\!=\!6) &= 1 + 0.295573\,\alpha_s - 0.029401\,\alpha_s^2 + 0.177980\,\alpha_s^3 + 0.001555\,\alpha_s^4\,,
\end{aligned} \qquad (3.1)$$

where $\widetilde{\beta} \equiv -\beta(a_s)/(a_s^2\beta_0)$. These numbers indeed show excellent perturbative convergence of the beta-function across this physically relevant range of n_f. It appears in particular that the convergence is enhanced for increasing values of n_f at the five-loop level. It would indeed be very interesting to see whether such a pattern would continue at yet higher orders or represents a mere accident at the five loop level.

In figure 3 we show two plots illustrating the small effect of the five-loop coefficient on the scale evolution of the coupling. These effects are indeed rather mild, and show that the coupling evolution even at lower scales (which correspond to larger values of the coupling) appears now to be under excellent control.

Hadronic Higgs decays and the R-ratio at N^4LO

Let us briefly report here also on several N^4LO calculations which were completed in ref [33] using essentially the same techniques which we described for the calculation of the five-loop beta function in section 2. Namely these are the calculations of the Higgs boson decay rate into massless bottom quarks, the Higgs boson decay rate into gluons in the heavy top quark effective theory and the hadronic R-ratio mediated by an off-shell photon, all of which in massless QCD. These calculations were made possible by the use of unitarity which allows one to relate decay rates to the imaginary part of a corresponding self energy correlator. Since such self energy diagrams are real valued in Euclidean space, their imaginary pieces are suppressed by a factor of ε which originates from the analytic continuation of the phase

$$\mathrm{Im}(-p^2 - i\delta)^{-L\varepsilon} = L\pi\varepsilon\left(1 - \frac{(L\pi\varepsilon)^2}{3!} + \ldots\right)(p^2)^{-L\varepsilon}\,. \qquad (3.2)$$

This suppression factor $(L\pi\varepsilon)$ allows one to extract the decay rate from the UV divergences of the self energy to which it is related by unitarity. Consequentially the R^*-method can be applied to

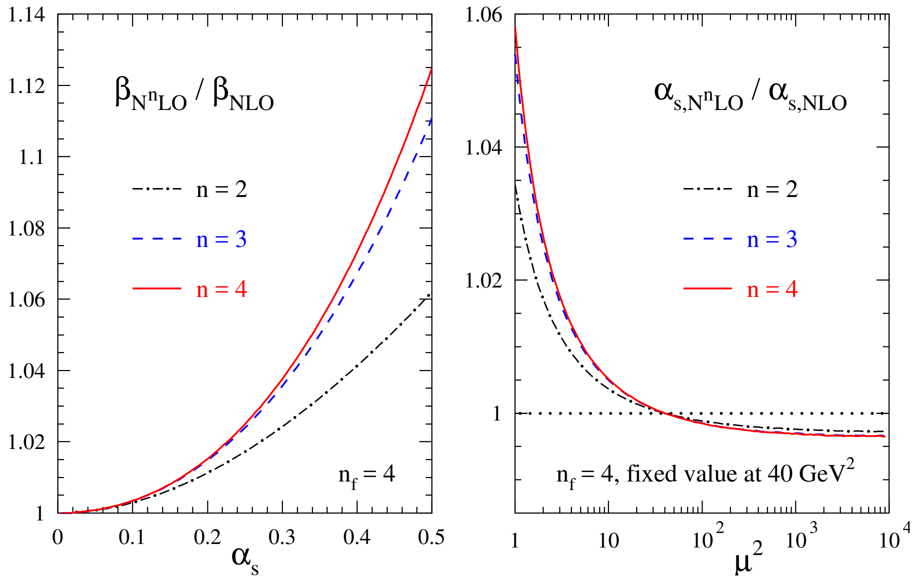


Figure 3: Left panel: The total 3-,4- and 5–loop results results for the beta function of QCD for four flavours, normalized to the 2-loop approximation. Right panel: The resulting scale dependence of α_s for a value of 0.2 at 40 GeV^2, also normalized to the 2-loop result in order to show the small higher-order effects more clearly, for the scale range 1 GeV$^2 \leq \mu^2 \leq 10^4$ GeV^2.

this problem. Since the result for the gluonic Higgs boson decay rate is new let us present here the variation of the renormalisation scale to emphasise its perturbative convergence in two different renormalisation schemes in figure 4. Although different patterns are observed at lower orders both schemes converge to the same numerical values at this high perturbative order providing further confidence for the reliability of perturbation theory in QCD.

4. Summary

In this talk we presented results for the five-loop beta function and discussed its mild impact on the evolution of the strong coupling, which appears now to be under complete control. We also briefly discussed the R^*-methods which were used to obtain this result. Finally we briefly presented a new result for the Higgs boson decay rate into gluons at N^{4}LO in perturbative QCD, which was obtained using the same methods.

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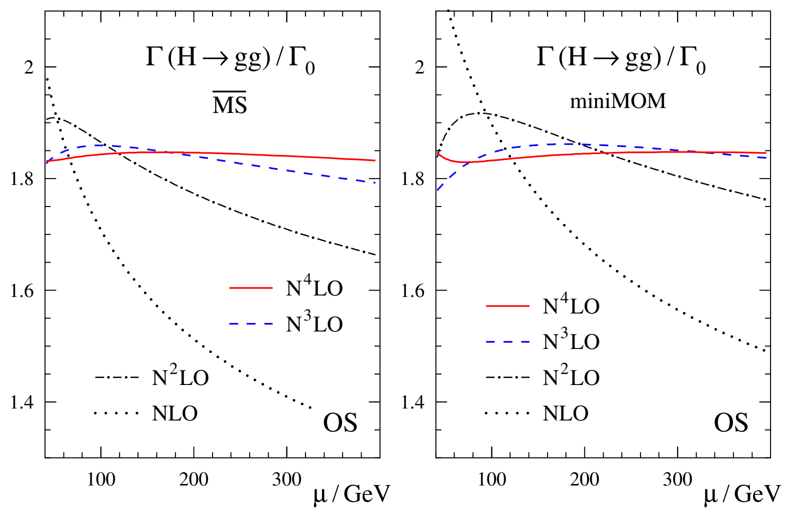

Figure 4: The renormalization scale dependence of the decay width $\Gamma_{H\to gg}$, for an on-shell top mass of 173 GeV in $\overline{\text{MS}}$ and the miniMOM scheme.

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