PROCEEDINGS OF SCIENCE

Non-Linear Invariance of Black Hole Entropy

Alessio Marrani*
Centro Studi e Ricerche "Enrico Fermi",
Via Panisperna 89A, I-00184, Roma, IT
and Dipartimento di Fisica e Astronomia "Galileo Galilei", Università di Padova,
and INFN - sez. di Padova, Via Marzolo 8, I-35131 Padova, IT
E-mail: marrani@pd.infn.it

Freudenthal duality is an anti-involutive, non-linear map acting on symplectic spaces. It generally holds in four-dimensional Maxwell-Einstein theories coupled to a non-linear sigma model of scalar fields. It is here reviewed, with some emphasis on its relation to the U-duality Lie groups of type E_7 in extended supergravity theories.

The European Physical Society Conference on High Energy Physics
5-12 July
Venice, Italy

*Speaker.

PoS(EPS-HEP2017)543

1. Freudenthal Duality

We start and consider the following Lagrangian density in four dimensions (*cfr. e.g.* [1]):

$$\mathscr{L}=-\frac{R}{2}+\frac{1}{2}g_{ij}(\varphi)\partial_{\mu}\varphi^{i}\partial^{\mu}\varphi^{j}+\frac{1}{4}I_{\Lambda\Sigma}(\varphi)F_{\mu\nu}^{\Lambda}F^{\Sigma|\mu\nu}+\frac{1}{8\sqrt{-G}}R_{\Lambda\Sigma}(\varphi)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^{\Lambda}F_{\rho\sigma}^{\Sigma}, \quad (1.1)$$

describing Einstein gravity coupled to Maxwell (Abelian) vector fields and to a non-linear sigma model of scalar fields (with no potential); note that $\mathscr{L}$ may -but does not necessarily need to - be conceived as the bosonic sector of $D=4$ (*ungauged*) supergravity theory. Out of the Abelian two-form field strengths F^{Λ}'s, one can define their duals G_{Λ}, and construct a symplectic vector :

$$H:=\left(F^{\Lambda},G_{\Lambda}\right)^{T},\quad {}^{*}G_{\Lambda|\mu\nu}:=2\frac{\delta\mathscr{L}}{\delta F^{\Lambda|\mu\nu}}. \quad (1.2)$$

We then consider the simplest solution of the equations of motion deriving from $\mathscr{L}$, namely a static, spherically symmetric, asymptotically flat, dyonic extremal black hole with metric [2]

$$ds^{2}=-e^{2U(\tau)}dt^{2}+e^{-2U(\tau)}\left[\frac{d\tau^{2}}{\tau^{4}}+\frac{1}{\tau^{2}}\left(d\theta^{2}+\sin\theta d\psi^{2}\right)\right], \quad (1.3)$$

where $\tau:=-1/r$. Thus, the two-form field strengths and their duals can be fluxed on the two-sphere at infinity S_{∞}^{2} in such a background, respectively yielding the electric and magnetic charges of the black hole itself, which can be arranged in a symplectic vector $\mathscr{Q}$:

$$p^{\Lambda}:=\frac{1}{4\pi}\int_{S_{\infty}^{2}}F^{\Lambda},\quad q_{\Lambda}:=\frac{1}{4\pi}\int_{S_{\infty}^{2}}G_{\Lambda}, \quad (1.4)$$

$$\mathscr{Q}:=\left(p^{\Lambda},q_{\Lambda}\right)^{T}. \quad (1.5)$$

Then, by exploiting the symmetries of the background (1.3), the Lagrangian (1.1) can be dimensionally reduced from $D=4$ to $D=1$, obtaining a 1-dimensional effective Lagrangian ($'\,:=d/d\tau$) [3]:

$$\mathscr{L}_{D=1}=\left(U'\right)^{2}+g_{ij}(\varphi)\varphi^{i\prime}\varphi^{j\prime}+e^{2U}V_{BH}(\varphi,\mathscr{Q}) \quad (1.6)$$

along with the Hamiltonian constraint [3]

$$\left(U'\right)^{2}+g_{ij}(\varphi)\varphi^{i\prime}\varphi^{j\prime}-e^{2U}V_{BH}(\varphi,\mathscr{Q})=0. \quad (1.7)$$

The so-called "effective black hole potential" V_{BH} appearing in (1.6) and (1.7) is defined as [3]

$$V_{BH}(\varphi,\mathscr{Q}):=-\frac{1}{2}\mathscr{Q}^{T}\mathscr{M}(\varphi)\mathscr{Q}, \quad (1.8)$$

in terms of the symplectic and symmetric matrix [1]

$$\mathscr{M}:=\begin{pmatrix}\mathbb{I}&-R\\0&\mathbb{I}\end{pmatrix}\begin{pmatrix}I&0\\0&I^{-1}\end{pmatrix}\begin{pmatrix}\mathbb{I}&0\\-R&\mathbb{I}\end{pmatrix}=\begin{pmatrix}I+RI^{-1}R&-RI^{-1}\\-I^{-1}R&I^{-1}\end{pmatrix}, \quad (1.9)$$

$$\mathscr{M}^{T}=\mathscr{M};\quad \mathscr{M}\Omega\mathscr{M}=\Omega, \quad (1.10)$$

PoS(EPS-HEP2017)543

where $\mathbb{I}$ denotes the identity, and $R(\varphi)$ and $I(\varphi)$ are the scalar-dependent matrices occurring in (1.1); moreover, Ω stands for the symplectic metric ($\Omega^2 = -\mathbb{I}$). Note that, regardless of the invertibility of $R(\varphi)$ and as a consequence of the physical consistence of the kinetic vector matrix $I(\varphi)$, $\mathscr{M}$ is negative-definite; thus, the effective black hole potential (1.8) is positive-definite.

By virtue of the matrix $\mathscr{M}$, one can introduce a (scalar-dependent) *anti-involution* $\mathscr{S}$ in any Maxwell-Einstein-scalar theory described by (1.1) with a symplectic structure Ω, as follows :

$$\mathscr{S}(\varphi) \; : \; = \Omega\mathscr{M}(\varphi); \tag{1.11}$$

$$\mathscr{S}^2(\varphi) \; = \; \Omega\mathscr{M}(\varphi)\Omega\mathscr{M}(\varphi) = \Omega^2 = -\mathbb{I}; \tag{1.12}$$

in turn, this allows to define an anti-involution on the dyonic charge vector $\mathscr{Q}$, which has been called (scalar-dependent) *Freudenthal duality* [4, 5, 6]:

$$\mathfrak{F}(\mathscr{Q};\varphi) \; : \; = -\mathscr{S}(\varphi)\mathscr{Q}; \tag{1.13}$$

$$\mathfrak{F}^2 \; = \; -\mathbb{I}, \quad (\forall\{\varphi\}). \tag{1.14}$$

By recalling (1.8) and (1.11), the action of $\mathfrak{F}$ on $\mathscr{Q}$, defining the so-called (φ-dependent) Freudenthal dual of $\mathscr{Q}$ itself, can be related to the symplectic gradient of the effective black hole potential V_{BH} :

$$\mathfrak{F}(\mathscr{Q};\varphi) = \Omega\frac{\partial V_{BH}(\varphi,\mathscr{Q})}{\partial\mathscr{Q}}. \tag{1.15}$$

Through the attractor mechanism [7], all this enjoys an interesting physical interpretation when evaluated at the (unique) event horizon of the extremal black hole (1.3) (denoted below by the subscript "H"); indeed

$$\partial_\varphi V_{BH} \; = \; 0 \Leftrightarrow \lim_{\tau\to-\infty}\varphi^i(\tau) = \varphi^i_H(\mathscr{Q}); \tag{1.16}$$

$$S_{BH}(\mathscr{Q}) \; = \; \frac{A_H}{4} = \pi\, V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathscr{Q}^T\mathscr{M}_H(\mathscr{Q})\mathscr{Q}, \tag{1.17}$$

where S_{BH} and A_H respectively denote the Bekenstein-Hawking entropy [8] and the area of the horizon of the extremal black hole, and the matrix horizon value $\mathscr{M}_H$ is defined as

$$\mathscr{M}_H(\mathscr{Q}) := \lim_{\tau\to-\infty}\mathscr{M}(\varphi(\tau)). \tag{1.18}$$

Correspondingly, one can define the (scalar-independent) horizon Freudenthal duality $\mathfrak{F}_H$ as the horizon limit of (1.13) :

$$\widetilde{\mathscr{Q}} \equiv \mathfrak{F}_H(\mathscr{Q}) := \lim_{\tau\to-\infty}\mathfrak{F}(\mathscr{Q};\varphi(\tau)) = -\Omega\mathscr{M}_H(\mathscr{Q})\mathscr{Q} = \frac{1}{\pi}\Omega\frac{\partial S_{BH}(\mathscr{Q})}{\partial\mathscr{Q}}. \tag{1.19}$$

Remarkably, the (horizon) Freudenthal dual of $\mathscr{Q}$ is nothing but ($1/\pi$ times) the symplectic gradient of the Bekenstein-Hawking black hole entropy S_{BH}; this latter, from dimensional considerations, is only constrained to be an homogeneous function of degree two in $\mathscr{Q}$. As a result, $\widetilde{\mathscr{Q}} = \widetilde{\mathscr{Q}}(\mathscr{Q})$ is generally a complicated (non-linear) function, homogeneous of degree one in $\mathscr{Q}$.

It can be proved that the entropy S_{BH} itself is invariant along the flow in the charge space $\mathscr{Q}$ defined by the symplectic gradient (or, equivalently, by the horizon Freudenthal dual) of $\mathscr{Q}$ itself :

$$S_{BH}(\mathscr{Q}) = S_{BH}(\mathfrak{F}_H(\mathscr{Q})) = S_{BH}\left(\frac{1}{\pi}\Omega\frac{\partial S_{BH}(\mathscr{Q})}{\partial\mathscr{Q}}\right) = S_{BH}\left(\widetilde{\mathscr{Q}}\right). \tag{1.20}$$

PoS(EPS-HEP2017)543

It is here worth pointing out that this invariance is pretty remarkable : the (semi-classical) Bekenstein-Hawking entropy of an extremal black hole turns out to be invariant under a generally non-linear map acting on the black hole charges themselves, and corresponding to a symplectic gradient flow in their corresponding vector space.

For other applications and instances of Freudenthal duality, see [9, 10, 11, 12].

2. Groups of Type E_7

The concept of Lie groups *of type* E_7 as introduced in the 60s by Brown [13], and then later developed *e.g.* by [14, 15, 16, 17, 18]. Starting from a pair $(G, \mathbf{R})$ made of a Lie group G and its faithful representation $\mathbf{R}$, the three axioms defining $(G, \mathbf{R})$ as a group of type E_7 read as follows :

1. Existence of a (unique) symplectic invariant structure Ω in $\mathbf{R}$:

$$\exists!\Omega \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R}, \tag{2.1}$$

   which then allows to define a symplectic product $\langle \cdot, \cdot \rangle$ among two vectors in the representation space $\mathbf{R}$ itself :

$$\langle Q_1, Q_2 \rangle := Q_1^M Q_2^N \Omega_{MN} = -\langle Q_2, Q_1 \rangle . \tag{2.2}$$

2. Existence of (unique) rank-4 completely symmetric invariant tensor (K-tensor) in $\mathbf{R}$:

$$\exists!K \equiv \mathbf{1} \in (\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R})_s , \tag{2.3}$$

   which then allows to define a degree-4 invariant polynomial I_4 in $\mathbf{R}$ itself :

$$I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q. \tag{2.4}$$

3. Defining a triple map T in $\mathbf{R}$ as

$$T \,:\, \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}; \tag{2.5}$$

$$\langle T(Q_1, Q_2, Q_3), Q_4 \rangle \,:=\, K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q, \tag{2.6}$$

   it holds that

$$\langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q. \tag{2.7}$$

   This property makes a group of type E_7 amenable to a description as an automorphism group of a *Freudenthal triple system* (or, equivalently, as the conformal groups of the underlying Jordan triple system - whose a Jordan algebra is a particular case -).

All electric-magnetic duality (U-duality[1]) groups of $\mathcal{N} \geqslant 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are of type E_7. Among these, degenerate groups of type E_7 are those in which the K-tensor is actually reducible, and thus I_4 is the square of a quadratic

[1] Here U-duality is referred to as the "continuous" symmetries of [19]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced by Hull and Townsend [20].

invariant polynomial I_2. In fact, in general, in theories with electric-magnetic duality groups of type E_7 holds that

$$S_{BH} = \pi\sqrt{|I_4(\mathscr{Q})|} = \pi\sqrt{|K_{MNPQ}\mathscr{Q}^M\mathscr{Q}^N\mathscr{Q}^P\mathscr{Q}^Q|}, \tag{2.8}$$

whereas in the case of degenerate groups of type E_7 it holds that $I_4(\mathscr{Q}) = (I_2(\mathscr{Q}))^2$, and therefore the latter formula simplifies to

$$S_{BH} = \pi\sqrt{|I_4(\mathscr{Q})|} = \pi|I_2(\mathscr{Q})|. \tag{2.9}$$

Simple, non-degenerate groups of type E_7 relevant to $\mathscr{N} \geqslant 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are listed *e.g.* in Table 1 of [21].

Semi-simple, non-degenerate groups of type E_7 of the same kind are given by $G = SL(2,\mathbb{R}) \times SO(2,n)$ and $G = SL(2,\mathbb{R}) \times SO(6,n)$, with $\mathbf{R} = (\mathbf{2}, \mathbf{2}+\mathbf{n})$ and $\mathbf{R} = (\mathbf{2}, \mathbf{6}+\mathbf{n})$, respectively relevant for $\mathscr{N} = 2$ and $\mathscr{N} = 4$ supergravity.

Moreover, degenerate (simple) groups of type E_7 relevant to the same class of theories are $G = U(1,n)$ and $G = U(3,n)$, with complex fundamental representations $\mathbf{R} = \mathbf{n}+\mathbf{1}$ and $\mathbf{R} = \mathbf{3}+\mathbf{n}$, respectively relevant for $\mathscr{N} = 2$ and $\mathscr{N} = 3$ supergravity [17].

The classification of groups of type E_7 is still an open problem, even if some progress have been recently made *e.g.* in [22] (in particular, *cfr.* Table D therein).

References

[1] P. Breitenlohner, G. W. Gibbons, and D. Maison, *Four-Dimensional Black Holes from Kaluza-Klein Theories*, Commun. Math. Phys. **120**, 295 (1988).

[2] A. Papapetrou, *A static solution of the equations of the gravitational field for an arbitrary charge distribution*, Proc. R. Irish Acad. **A51**, 191 (1947). S. D. Majumdar, *A Class of Exact Solutions of Einstein's Field Equations*, Phys. Rev. **72**, 930 (1947).

[3] S. Ferrara, G. W. Gibbons, and R. Kallosh, *Black holes and critical points in moduli space*, Nucl. Phys. B500, 75 (1997), hep-th/9702103.

[4] L. Borsten, D. Dahanayake, M. J. Duff, and W. Rubens, *Black holes admitting a Freudenthal dual*, Phys. Rev. **D80** (2009) 026003, arXiv:0903.5517 [hep-th].

[5] S. Ferrara, A. Marrani, and A. Yeranyan, *Freudenthal Duality and Generalized Special Geometry*, Phys. Lett. **B701** (2011) 640, arXiv:1102.4857 [hep-th].

[6] L. Borsten, M. J. Duff, S. Ferrara, and A. Marrani, *Freudenthal Dual Lagrangians*, Class. Quant.Grav. **30** (2013) 235003, arXiv:1212.3254 [hep-th].

[7] S. Ferrara, R. Kallosh, and A. Strominger, *$\mathscr{N} = 2$ Extremal Black Holes*, Phys. Rev. **D52**, 5412 (1995), hep-th/9508072. A. Strominger, *Macroscopic Entropy of $\mathscr{N} = 2$ Extremal Black Holes*, Phys. Lett. **B383**, 39 (1996), hep-th/9602111. S. Ferrara and R. Kallosh, *Supersymmetry and Attractors*, Phys. Rev. **D54**, 1514 (1996), hep-th/9602136. S. Ferrara and R. Kallosh, *Universality of Supersymmetric Attractors*, Phys. Rev. **D54**, 1525 (1996), hep-th/9603090.

[8] S. W. Hawking: *Gravitational Radiation from Colliding Black Holes*, Phys. Rev. Lett. **26**, 1344 (1971). J. D. Bekenstein: *Black Holes and Entropy*, Phys. Rev. **D7**, 2333 (1973).

[9] P. Galli, P. Meessen, and T. Ortín, *The Freudenthal gauge symmetry of the black holes of $\mathscr{N} = 2, d = 4$ supergravity*, JHEP **1305** (2013) 011, arXiv:1211.7296 [hep-th].

[10] J.J. Fernandez-Melgarejo and E. Torrente-Lujan, *$\mathcal{N}=2$ Sugra BPS Multi-center solutions, quadratic prepotentials and Freudenthal transformations*, JHEP **1405** (2014) 081, arXiv:1310.4182 [hep-th].

[11] A. Marrani, C.-X. Qiu, S.-Y. D. Shih, A. Tagliaferro, and B. Zumino, *Freudenthal Gauge Theory*, JHEP **1303** (2013) 132, arXiv:1208.0013 [hep-th].

[12] A. Marrani, P. K. Tripathy, T. Mandal, *Supersymmetric Black Holes and Freudenthal Duality*, Int.J.Mod.Phys. **A32** (2017) no.19n20, 1750114, arXiv:1703.08669 [hep-th].

[13] R. B. Brown, *Groups of Type E_7*, J. Reine Angew. Math. **236**, 79 (1969).

[14] K. Meyberg, *Eine Theorie der Freudenthalschen Triplesysteme. I, II*, Nederl. Akad. Wetensch. Proc. Ser. **A71**, 162 (1968).

[15] R. S. Garibaldi, *Groups of type E_7 over Arbitrary Fields*, Commun. in Algebra **29**, 2689 (2001), math/9811056 [math.AG].

[16] S. Krutelevich, *Jordan algebras, exceptional groups, and higher composition laws*, arXiv:math/0411104. S. Krutelevich, *Jordan algebras, exceptional groups, and Bhargava composition*, J. of Algebra **314**, 924 (2007).

[17] S. Ferrara, R. Kallosh, and A. Marrani, *Degeneration of Groups of Type E_7 and Minimal Coupling in Supergravity*, JHEP **1206** (2012) 074, arXiv:1202.1290 [hep-th].

[18] A. Marrani, E. Orazi, and F. Riccioni, *Exceptional Reductions*, J. Phys. **A44**, 155207 (2011), arXiv:1012.5797 [hep-th].

[19] E. Cremmer and B. Julia, *The $\mathcal{N}=8$ Supergravity Theory. 1. The Lagrangian*, Phys. Lett. **B80**, 48 (1978). E. Cremmer and B. Julia, *The $SO(8)$ Supergravity*, Nucl. Phys. **B159**, 141 (1979).

[20] C. Hull and P. K. Townsend, *Unity of Superstring Dualities*, Nucl. Phys. **B438**, 109 (1995), hep-th/9410167.

[21] A. Marrani, *Freudenthal Duality in Gravity: from Groups of Type E_7 to Pre-Homogeneous Spaces*, p Adic Ultra.Anal.Appl. **7** (2015) 322-331, arXiv:1509.01031 [hep-th].

[22] S. Garibaldi and R. Guralnick, *Simple groups stabilizing polynomials*, Forum of Mathematics, Pi (2015), vol. 3, e3, arXiv:1309.6611 [math.GR].