Non-Linear Invariance of Black Hole Entropy

Alessio Marrani

Alessio Marrani

Freudenthal duality is an anti-involutive, non-linear map acting on symplectic spaces. It generally holds in four-dimensional Maxwell-Einstein theories coupled to a non-linear sigma model of scalar fields. It is here reviewed, with some emphasis on its relation to the $U$-duality Lie groups of type $E_7$ in extended supergravity theories.

The European Physical Society Conference on High Energy Physics
5-12 July
Venice, Italy

*Speaker.
1. Freudenthal Duality

We start and consider the following Lagrangian density in four dimensions (cfr. e.g. [1]):
\[
\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\phi) \partial_{\mu} \phi^i \partial^{\mu} \phi^j + \frac{1}{4} I_{\Lambda \Sigma}(\phi) F_{\mu \nu}^{\Lambda} F^{\Sigma}_{\mu \nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda \Sigma}(\phi) \varepsilon^{\mu \nu \rho \sigma} F_{\mu \nu}^{\Lambda} F_{\rho \sigma}^{\Sigma},
\]  
(1.1)
describing Einstein gravity coupled to Maxwell (Abelian) vector fields and to a non-linear sigma model of scalar fields (with no potential); note that \( \mathcal{L} \) may -but does not necessarily need to- be conceived as the bosonic sector of \( D = 4 \) (ungauged) supergravity theory. Out of the Abelian two-form field strengths \( F^{\Lambda} \)'s, one can define their duals \( *G^{\Lambda} \), and construct a symplectic vector:
\[
H := (F^{\Lambda}, G^{\Lambda})^T, \quad *G^{\Lambda \mu \nu} := \frac{2}{\sqrt{G}} \frac{\delta \mathcal{L}}{\delta F^{\Lambda \mu \nu}}.
\]  
(1.2)
We then consider the simplest solution of the equations of motion deriving from \( \mathcal{L} \), namely a static, spherically symmetric, asymptotically flat, dyonic extremal black hole with metric
\[
ds^2 = -e^{2U(\tau)} d\tau^2 + e^{-2U(\tau)} \left[ \frac{d\tau^2}{\tau^2} + \frac{1}{\tau^2} (d\theta^2 + \sin \theta d\psi^2) \right],
\]  
(1.3)
where \( \tau := -1/r \). Thus, the two-form field strengths and their duals can be fluxed on the two-sphere at infinity \( S^2_\infty \) in such a background, respectively yielding the electric and magnetic charges of the black hole itself, which can be arranged in a symplectic vector \( \mathcal{Q} \):
\[
p^{\Lambda} := \frac{1}{4\pi} \int_{S^2_\infty} F^{\Lambda}, \quad q^{\Lambda} := \frac{1}{4\pi} \int_{S^2_\infty} G^{\Lambda},
\]  
(1.4)
\[
\mathcal{Q} := (p^{\Lambda}, q^{\Lambda})^T.
\]  
(1.5)
Then, by exploiting the symmetries of the background (1.3), the Lagrangian (1.1) can be dimensionally reduced from \( D = 4 \) to \( D = 1 \), obtaining a 1-dimensional effective Lagrangian \( \mathcal{L}_{D=1} := d/d\tau \) [3]:
\[
\mathcal{L}_{D=1} = (U')^2 + g_{ij}(\phi) \phi^i \phi^j + e^{2U} V_{BH}(\phi, \mathcal{Q})
\]  
(1.6)
along with the Hamiltonian constraint [3]
\[
(U')^2 + g_{ij}(\phi) \phi^i \phi^j - e^{2U} V_{BH}(\phi, \mathcal{Q}) = 0.
\]  
(1.7)
The so-called “effective black hole potential” \( V_{BH} \) appearing in (1.6) and (1.7) is defined as [3]
\[
V_{BH}(\phi, \mathcal{Q}) := -\frac{1}{2} \mathcal{Q}^T \mathcal{M}(\phi) \mathcal{Q},
\]  
(1.8)
in terms of the symplectic and symmetric matrix [1]
\[
\mathcal{M} := \begin{pmatrix} I & -R \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -R & I \end{pmatrix} = \begin{pmatrix} I + R I^{-1} R & -R I^{-1} \\ -R I^{-1} R & I^{-1} \end{pmatrix},
\]  
(1.9)
\[
\mathcal{M}^T = \mathcal{M}; \quad \mathcal{M} \Omega \mathcal{M} = \Omega,
\]  
(1.10)
where \( \mathbb{I} \) denotes the identity, and \( R(\varphi) \) and \( I(\varphi) \) are the scalar-dependent matrices occurring in (1.1); moreover, \( \Omega \) stands for the symplectic metric \( (\Omega^2 = -\mathbb{I}) \). Note that, regardless of the invertibility of \( R(\varphi) \) and as a consequence of the physical consistence of the kinetic vector matrix \( I(\varphi) \), \( \mathcal{M} \) is negative-definite; thus, the effective black hole potential (1.8) is positive-definite.

By virtue of the matrix \( \mathcal{M} \), one can introduce a (scalar-dependent) anti-involution \( \mathcal{S} \) in any Maxwell-Einstein-scalar theory described by (1.1) with a symplectic structure \( \Omega \), as follows:

\[
 \mathcal{S}(\varphi) := \Omega \mathcal{M}(\varphi); \\
 \mathcal{S}^2(\varphi) = \Omega \mathcal{M}(\varphi)\Omega \mathcal{M}(\varphi) = \Omega^2 = -\mathbb{I}; 
\]

in turn, this allows to define an anti-involution on the dyonic charge vector \( \mathcal{Q} \), which has been called (scalar-dependent) Freudenthal duality [4, 5, 6]:

\[
 \mathcal{S}(\mathcal{Q};\varphi) := -\mathcal{S}(\varphi)\mathcal{Q}; \\
 \mathcal{S}^2 = -\mathbb{I}, \ \ (\forall \{\varphi\}). 
\]

By recalling (1.8) and (1.11), the action of \( \mathcal{S} \) on \( \mathcal{Q} \), defining the so-called (\( \varphi \)-dependent) Freudenthal dual of \( \mathcal{Q} \) itself, can be related to the symplectic gradient of the effective black hole potential \( V_{BH} \):

\[
 \mathcal{S}(\mathcal{Q};\varphi) = \Omega \frac{\partial V_{BH}(\varphi,\mathcal{Q})}{\partial \mathcal{Q}}. 
\]

Through the attractor mechanism [7], all this enjoys an interesting physical interpretation when evaluated at the (unique) event horizon of the extremal black hole (1.3) (denoted below by the subscript “H”); indeed

\[
 \partial_\varphi V_{BH} = 0 \Leftrightarrow \lim_{\tau \to -\infty} \varphi'(\tau) = \varphi'_H(\mathcal{Q}); \\
 S_{BH}(\mathcal{Q}) = \frac{A_H}{4} = \pi V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2} \mathcal{Q}^T \mathcal{M}_H(\mathcal{Q}) \mathcal{Q}, 
\]

where \( S_{BH} \) and \( A_H \) respectively denote the Bekenstein-Hawking entropy [8] and the area of the horizon of the extremal black hole, and the matrix horizon value \( \mathcal{M}_H \) is defined as

\[
 \mathcal{M}_H(\mathcal{Q}) := \lim_{\tau \to -\infty} \mathcal{M}(\varphi(\tau)). 
\]

Correspondingly, one can define the (scalar-independent) horizon Freudenthal duality \( \mathcal{S}_H \) as the horizon limit of (1.13):

\[
 \mathcal{D} \equiv \mathcal{S}_H(\mathcal{Q}) := \lim_{\tau \to -\infty} \mathcal{S}(\mathcal{Q};\varphi(\tau)) = -\Omega \mathcal{M}_H(\mathcal{Q}) \mathcal{Q} = \frac{1}{\pi} \Omega \frac{\partial S_{BH}(\mathcal{Q})}{\partial \mathcal{Q}}. 
\]

Remarkably, the (horizon) Freudenthal dual of \( \mathcal{Q} \) is nothing but \( (1/\pi) \) times the symplectic gradient of the Bekenstein-Hawking black hole entropy \( S_{BH} \); this latter, from dimensional considerations, is only constrained to be an homogeneous function of degree two in \( \mathcal{Q} \). As a result, \( \mathcal{D} = \mathcal{D}(\mathcal{Q}) \) is generally a complicated (non-linear) function, homogeneous of degree one in \( \mathcal{Q} \).

It can be proved that the entropy \( S_{BH} \) itself is invariant along the flow in the charge space \( \mathcal{Q} \) defined by the symplectic gradient (or, equivalently, by the horizon Freudenthal dual) of \( \mathcal{Q} \) itself:

\[
 S_{BH}(\mathcal{Q}) = S_{BH}(\mathcal{S}_H(\mathcal{Q})) = S_{BH}\left(\frac{1}{\pi} \Omega \frac{\partial S_{BH}(\mathcal{Q})}{\partial \mathcal{Q}}\right) = S_{BH}(\mathcal{D}). 
\]
It is here worth pointing out that this invariance is pretty remarkable: the (semi-classical) Bekenstein-Hawking entropy of an extremal black hole turns out to be invariant under a generally non-linear map acting on the black hole charges themselves, and corresponding to a symplectic gradient flow in their corresponding vector space.

For other applications and instances of Freudenthal duality, see [9, 10, 11, 12].

2. Groups of Type $E_7$

The concept of Lie groups of type $E_7$ as introduced in the 60s by Brown [13], and then later developed e.g. by [14, 15, 16, 17, 18]. Starting from a pair $(G, R)$ made of a Lie group $G$ and its faithful representation $R$, the three axioms defining $(G, R)$ as a group of type $E_7$ read as follows:

1. Existence of a (unique) symplectic invariant structure $\Omega$ in $R$:
   \[ \exists! \Omega \equiv \mathbf{1} \in \mathbb{R} \times_a \mathbb{R}, \quad (2.1) \]
   which then allows to define a symplectic product $\langle \cdot , \cdot \rangle$ among two vectors in the representation space $R$ itself:
   \[ \langle Q_1, Q_2 \rangle := Q_1^M Q_2^N \Omega_{MN} = - \langle Q_2, Q_1 \rangle. \quad (2.2) \]

2. Existence of (unique) rank-4 completely symmetric invariant tensor ($K$-tensor) in $R$:
   \[ \exists! K \equiv \mathbf{1} \in (\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}), \quad (2.3) \]
   which then allows to define a degree-4 invariant polynomial $I_4$ in $R$ itself:
   \[ I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q. \quad (2.4) \]

3. Defining a triple map $T$ in $R$ as
   \[ T : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}; \quad (2.5) \]
   \[ \langle T(Q_1, Q_2, Q_3), Q_4 \rangle := K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q, \quad (2.6) \]
   it holds that
   \[ \langle T(Q_1, Q_2), T(Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q. \quad (2.7) \]

This property makes a group of type $E_7$ amenable to a description as an automorphism group of a Freudenthal triple system (or, equivalently, as the conformal groups of the underlying Jordan triple system - whose a Jordan algebra is a particular case -).

All electric-magnetic duality ($U$-duality\footnote{Here $U$-duality is referred to as the “continuous” symmetries of [19]. Their discrete versions are the $U$-duality non-perturbative string theory symmetries introduced by Hull and Townsend [20].}) groups of $\mathcal{N} \geq 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are of type $E_7$. Among these, degenerate groups of type $E_7$ are those in which the $K$-tensor is actually reducible, and thus $I_4$ is the square of a quadratic...
invariant polynomial $I_2$. In fact, in general, in theories with electric-magnetic duality groups of type $E_7$ holds that

$$S_{BH} = \pi \sqrt{|I_4(\mathcal{D})|} = \pi \sqrt{|K_{MNPQ} \mathcal{D}^M \mathcal{D}^N \mathcal{D}^P \mathcal{D}^Q|},$$  \hspace{1cm} (2.8)$$

whereas in the case of degenerate groups of type $E_7$ it holds that $I_4(\mathcal{D}) = (I_2(\mathcal{D}))^2$, and therefore the latter formula simplifies to

$$S_{BH} = \pi \sqrt{|I_4(\mathcal{D})|} = \pi |I_2(\mathcal{D})|.$$  \hspace{1cm} (2.9)$$

Simple, non-degenerate groups of type $E_7$ relevant to $\mathcal{N} \geq 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are listed e.g. in Table 1 of [21].

Semi-simple, non-degenerate groups of type $E_7$ of the same kind are given by $G = SL(2, \mathbb{R}) \times SO(2, n)$ and $G = SL(2, \mathbb{R}) \times SO(6, n)$, with $R = (2, 2 + n)$ and $R = (2, 6 + n)$, respectively relevant for $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supergravity.

Moreover, degenerate (simple) groups of type $E_7$ relevant to the same class of theories are $G = U(1, n)$ and $G = U(3, n)$, with complex fundamental representations $R = n + 1$ and $R = 3 + n$, respectively relevant for $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supergravity [17].

The classification of groups of type $E_7$ is still an open problem, even if some progress have been recently made e.g. in [22] (in particular, cfr. Table D therein).

References


Non-Linear Invariance of Black Hole Entropy

Alessio Marrani


