Polarization effects in the reactions
\[ p + ^3He \rightarrow \pi^+ + ^4He, \quad \pi^+ + ^4He \rightarrow p + ^3He \]
and quantum character of spin correlations
in the final \((p, ^3He)\) system

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The general consequences of \(T\) invariance for the direct and inverse binary reactions \(a + b \rightarrow c + d,\)
\(c + d \rightarrow a + b\) with spin-1/2 particles \(a, b\) and unpolarized particles \(c, d\) are considered. Using the
formalism of helicity amplitudes, the polarization effects are studied in the reaction \(p + ^3He \rightarrow \pi^+ + ^4He\)
and in the inverse process \(\pi^+ + ^4He \rightarrow p + ^3He\). It is shown that in the reaction
\(\pi^+ + ^4He \rightarrow p + ^3He\) the spins of the final proton and \(^3He\) nucleus are strongly correlated. A
structural expression through helicity amplitudes, corresponding to arbitrary emission angles, is
obtained for the correlation tensor. It is established that in the reaction \(\pi^+ + ^4He \rightarrow p + ^3He\)
one of the “classical” incoherence inequalities of the Bell type for diagonal components of the
correlation tensor is necessarily violated.
1. Consequences of $T$ invariance for binary reactions

Let us consider the reaction $a + b \rightarrow c + d$, where $a$ and $b$ are the spin-1/2 particles and the particles $c$, $d$ have arbitrary spins. The structure of the effective cross-section $\sigma_{a+b\rightarrow c+d}$ in the c.m. frame of the particles $a$ and $b$, summed over spin projections of the final particles $c$ and $d$, is as follows [1]:

$$\sigma_{a+b\rightarrow c+d}(\mathbf{k}_a, \mathbf{P}^{(a)}; \mathbf{k}_c) = \sigma_0(E, \theta) L(\mathbf{k}_a, \mathbf{P}^{(a)}; \mathbf{k}_c),$$

(1.1)

where $\sigma_0(E, \theta)$ is the respective cross-section for unpolarized particles $a$, $b$ and $L$ is the linear function of the polarization vectors $\mathbf{P}^{(a)}$ and $\mathbf{P}^{(b)}$:

$$L(\mathbf{k}_a, \mathbf{P}^{(a)}, \mathbf{P}^{(b)}; \mathbf{k}_c) = 1 + A(E, \theta)(\mathbf{P}^{(a)} \cdot \mathbf{n}) + B(E, \theta)(\mathbf{P}^{(b)} \cdot \mathbf{n}) + C(E, \theta)(\mathbf{P}^{(a)} \cdot \mathbf{P}^{(b)}) +$$

$$+ D(E, \theta)(\mathbf{P}^{(a)} \cdot \mathbf{l})(\mathbf{P}^{(b)} \cdot \mathbf{l}) + F(E, \theta)(\mathbf{P}^{(a)} \cdot \mathbf{m})(\mathbf{P}^{(b)} \cdot \mathbf{m}) +$$

$$+ G(E, \theta)(\mathbf{P}^{(a)} \cdot \mathbf{m})(\mathbf{P}^{(b)} \cdot \mathbf{m}) + H(E, \theta)(\mathbf{P}^{(a)} \cdot \mathbf{m})(\mathbf{P}^{(b)} \cdot \mathbf{l}).$$

(1.2)

Here $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ are the mutually orthogonal unit vectors, defined as:

$$\mathbf{l} = \mathbf{k}_a / k_a; \quad \mathbf{m} = \frac{\mathbf{Y} - \mathbf{l}}{\sin \theta}; \quad \mathbf{n} = \frac{\mathbf{l} \times \mathbf{Y}}{\sin \theta}$$

(1.3)

($\mathbf{Y} = \mathbf{k}_c / k_c$); $\mathbf{k}_a$, $\mathbf{k}_c$ are the respective momenta of the particles $a$ and $c$, $E$ is the total energy in the c.m. frame and $\theta = \arccos(\mathbf{Y})$ is the emission angle.

Meantime, for the inverse reaction $c + d \rightarrow a + b$ with the unpolarized particles $c$, $d$ and the fixed polarization vectors of the final particles $\zeta^{(a)}$, $\zeta^{(b)}$ the effective cross-section takes, due to $T$ invariance and the principle of detailed balance, the following form [1]:

$$\sigma_{c+d\rightarrow a+b}(\mathbf{k}_c, \mathbf{P}^{(a)}; \mathbf{k}_c, \mathbf{P}^{(b)}) = \frac{1}{4} \tilde{\sigma}_0(E, \theta) L(-\mathbf{k}_a, -\zeta^{(a)}; -\zeta^{(b)}; -\mathbf{k}_c),$$

(1.4)

where

$$\tilde{\sigma}_0(E, \theta) = \frac{4k_a^2}{k_c^2(2j_a+1)(2j_d+1)} \sigma_0(E, \theta)$$

(1.5)

is the cross-section of the inverse reaction, summed over the spin projections of the final particles $a$, $b$. Further, the two-particle spin density matrix $\hat{\rho}^{(a,b)}$ for the final particles $a$, $b$ can also be expressed through the same function $L$, replacing the polarization vectors by the vector Pauli operators:

$$\hat{\rho}^{(a,b)} = \frac{1}{4} \left[ \hat{J}^{(a)} \otimes \hat{J}^{(b)} + (\mathbf{P}^{(a)}(E, \theta) \hat{\sigma}^{(a)}(E, \theta) \otimes \hat{J}^{(b)} + \hat{J}^{(a)} \otimes (\mathbf{P}^{(b)}(E, \theta) \hat{\sigma}^{(b)}(E, \theta) +$$

$$+ \sum_{i=1}^{3} \sum_{k=1}^{3} T_{ik}(E, \theta) \hat{\sigma}_{i}^{(a)} \otimes \hat{\sigma}_{k}^{(b)} \right] = \frac{1}{4} \hat{L}(\mathbf{P}^{(a)}(E, \theta) \otimes \mathbf{P}^{(b)}(E, \theta) \otimes \mathbf{n}).$$

(1.6)

In Eq. (1.6) $\hat{J}^{(a)}$, $\hat{J}^{(b)}$ are two-row unit matrices, $\mathbf{P}^{(a)}(E, \theta) = -A(E, \theta)\mathbf{n}$, $\mathbf{P}^{(b)}(E, \theta) = -B(E, \theta)\mathbf{n}$ are the polarization vectors of the particles $a$ and $b$,

$$T_{ik}(E, \theta) = C(E, \theta) \delta_{ik} + D(E, \theta) l_i l_k + F(E, \theta) m_i m_k + G(E, \theta) l_i m_k + H(E, \theta) m_i l_k$$

(1.7)
are components of the correlation tensor describing spin correlations in the final two-particle system \((a, b)\). In doing so, all the functions \(A, B, C, D, F, G, H, \sigma_0\) and the unit vectors \(\mathbf{l}, \mathbf{m}, \mathbf{n}\) in Eqs. (1.4)–(1.7) are the same as for the direct reaction \(a + b \rightarrow c + d\).

Thus, due to \(T\) invariance, the dependence of the effective cross-section of the direct reaction \(a + b \rightarrow c + d\) upon the polarizations of initial particles completely determines the polarization vectors and spin correlations for the same particles \(a, b\) produced in the inverse reaction \(c + d \rightarrow a + b\) with unpolarized primary particles.

2. Polarization effects in the reaction \(p + ^3He \rightarrow \pi^+ + ^4He\)

This reaction belongs to the type \(1/2 + 1/2 \rightarrow 0 + 0\) (the proton and \(^3He\) nucleus have spin \(1/2, \pi^+\) and \(^4He\) have zero spin). Thus, on account of the negative internal parity of the \(\pi^+\) meson, this reaction can proceed only from triplet states of the system \((p, ^3He)\) [2,3] (as follows from the parity and angular momentum conservation).

Let us choose the axis of the total spin quantization \(z\) along the vector \(\mathbf{l} = k_p/k_p\). There exist three possible triplet states of the \((p, ^3He)\) system with the spin projections \(+1, -1\) and 0 onto \(z\):

\[
|+1, 1\rangle = |+1/2, 1\rangle^{(p)} \otimes |+1/2, 1\rangle^{(^3He)}, \quad |-1, 1\rangle = |-1/2, 1\rangle^{(p)} \otimes |-1/2, 1\rangle^{(^3He)},
\]

\[
|0, 1\rangle = \frac{1}{\sqrt{2}} \left(|+1/2, 1\rangle^{(p)} \otimes |-1/2, 1\rangle^{(^3He)} + |-1/2, 1\rangle^{(p)} \otimes |+1/2, 1\rangle^{(^3He)} \right).
\]

(2.1)

The two-particle spin density matrix for the \((p, ^3He)\) system is:

\[
\hat{\rho}^{(p, ^3He)} = \frac{1}{4} \left( \hat{\rho}^{(p)} + \hat{\rho}^{(^3He)} \right) \otimes \left( \hat{\rho}^{(^3He)} + \hat{\rho}^{(p)} \right).
\]

(2.2)

\((\hat{\rho}^{(p)}\) and \(\hat{\rho}^{(^3He)}\) are the independent polarization vectors). Using the technique of helicity amplitudes (the helicity amplitude \(R_\lambda(E, \theta), \lambda = \pm 1, 0\), is the amplitude of the reaction \(p + ^3He \rightarrow \pi^+ + ^4He\) proceeding from the state \(|\lambda, 1\rangle\) (2.1)), we may write [1]:

\[
\sigma_{p + ^3He \rightarrow \pi^+ + ^4He} = \langle \psi | \hat{\rho}^{(p, ^3He)} | \psi \rangle = \sum_\lambda \sum_{\lambda'} R_\lambda(E, \theta) \langle \lambda, 1 | \hat{\rho}^{(p, ^3He)} | \lambda', 1 \rangle R_{\lambda'}^*(E, \theta);
\]

(2.3)

\[
| \psi \rangle = \sum_{\lambda = \pm 1, 0} R_{\lambda}^*(E, \theta) | \lambda, 1 \rangle =
\]

\[
= R^*_0(E, \theta) \left( |+1/2, z\rangle^{(p)} \otimes |-1/2, z\rangle^{(^3He)} + |+1/2, z\rangle^{(p)} \otimes |-1/2, z\rangle^{(^3He)} \right) +
\]

\[
+ \frac{1}{\sqrt{2}} R^*_0(E, \theta) \left( |+1/2, z\rangle^{(p)} \otimes |-1/2, z\rangle^{(^3He)} + |+1/2, z\rangle^{(p)} \otimes |-1/2, z\rangle^{(^3He)} \right)
\]

(2.4)

is the non-normalized initial two-particle spin state selected by the reaction (due to the parity conservation, \(R_{+1}(E, \theta) = -R_{-1}(E, \theta) \equiv R_{1}(E, \theta)\)).

Finally, using Eq. (2.4) and the formula (2.2) for the spin density matrix, we find that the cross-section \(\sigma_{p + ^3He \rightarrow \pi^+ + ^4He}\) (2.3) is described by the general structural formula for \(\sigma_{a + b \rightarrow c + d}\) (1.1)–(1.2), where the functions \(\sigma_0, A, B, C, D, F, G, H\) are bilinear combinations of the helicity amplitudes \(R_1, R_0\) [1]:

\[
\sigma_0(E, \theta) = \frac{1}{4} \langle \psi \rangle \langle \psi \rangle = \frac{1}{4} \left( |R_0(E, \theta)|^2 + 2 |R_1(E, \theta)|^2 \right),
\]

(2.5)
\begin{align}
A(E, \theta) &= B(E, \theta) = \frac{1}{\sqrt{2\sigma_0(E, \theta)}} \text{Im} (R_1(E, \theta) R_0^*(E, \theta)), \\
C(E, \theta) &= 1, \quad D(E, \theta) = -\frac{|R_0(E, \theta)|^2}{2\sigma_0(E, \theta)}, \quad F(E, \theta) = -\frac{|R_1(E, \theta)|^2}{2\sigma_0(E, \theta)}, \\
G(E, \theta) &= H(E, \theta) = \frac{1}{\sqrt{2\sigma_0(E, \theta)}} \text{Re} (R_1(E, \theta) R_0^*(E, \theta)).
\end{align}

For the particular cases \( \theta = 0 \) and \( \theta = \pi \), when \( R_1(E, \theta) = 0 \), the expression for cross-section
\( \sigma_{p+^3He \rightarrow \pi^+ + ^4He} \) takes a considerably simpler form:
\[
\sigma_{p+^3He \rightarrow \pi^+ + ^4He} = \frac{1}{4} |R_0|^2 \left( 1 + \mathbf{P}^{(p)} \mathbf{P}^{(He)} - 2(\mathbf{P}^{(p)} \mathbf{P}^{(He)}) \right).
\]

3. Spin effects in the inverse reaction \( \pi^+ + ^4He \rightarrow p + ^3He \)

In the reaction \( \pi^+ + ^4He \rightarrow p + ^3He \) the \( (p, ^3He) \) system is produced in the triplet state only. This state, normalized to unity, is as follows [1]:
\[
\frac{1}{(2l_1+1)(2l_2+1)} \left[ R_0(1, \theta) \left( |1/2,1\rangle^{(p)} \otimes |1/2,1\rangle^{(He)} - | -1/2,1\rangle^{(p)} \otimes | -1/2,1\rangle^{(He)} \right) + \right. \\
+ \frac{1}{\sqrt{2}} R_0(1, \theta) \left( |1/2,1\rangle^{(p)} \otimes | -1/2,1\rangle^{(He)} + | -1/2,1\rangle^{(p)} \otimes |1/2,1\rangle^{(He)} \right),
\]
and it is symmetric under the interchange of spin quantum numbers of the proton and \( ^3He \). The state \( \langle \tilde{\psi} | \langle \psi \rangle \) is similar in structure to the initial triplet state \( | \psi \rangle \) selected by the direct reaction \( p + ^3He \rightarrow \pi^+ + ^4He \) (see Section 2), but differs by complex conjugation of helicity amplitudes.

Basing on the \( T \) invariance (see Section 1), we obtain [1]:

1) For the effective cross-section of the reaction \( \pi^+ + ^4He \rightarrow p + ^3He \) in the c.m. frame, summed over the spin projections in the final state:
\[
\tilde{\sigma}_0(E, \theta) = (k_p/k_e)^2 \left[ |R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2 \right].
\]

2) For the polarization vectors of the proton and \( ^3He \) in the final system:
\[
\mathbf{P}^{(p)}(E, \theta) = \langle \tilde{\psi} | \hat{\sigma}^{(p)} | \tilde{\psi} \rangle = \mathbf{P}^{(He)}(E, \theta) = \langle \tilde{\psi} | \hat{\sigma}^{(He)} | \tilde{\psi} \rangle = -A(E, \theta) \mathbf{n} = -2\sqrt{2} \frac{\text{Im} (R_1(E, \theta) R_0^*(E, \theta))}{|R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2} \mathbf{n};
\]

3) For the correlation tensor of the \( (p, ^3He) \) system:
\[
T_{ik}(E, \theta) = \langle \tilde{\psi} | \hat{\sigma}_i^{(p)} \hat{\sigma}_k^{(He)} | \tilde{\psi} \rangle = \delta_{ik} - \frac{2}{|R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2} \times \\
\times \left[ |R_0(E, \theta)|^2 l_i l_k + 2|R_1(E, \theta)|^2 m_i m_k - \sqrt{2} \text{Re} (R_1(E, \theta) R_0^*(E, \theta)) (i |m_k + m_i l_k) \right].
\]
In all the expressions (3.1) – (3.4), the helicity amplitudes $R_0(E, \theta)$, $R_1(E, \theta)$ and the unit vectors $\mathbf{l, m, n}$ are the same as for the previously considered direct process $p + ^3He \rightarrow \pi^+ + ^4He$.

In accordance with Eqs. (3.3), (3.4): 1) the polarization of the $^3He$ nucleus along the normal to the reaction plane is identical to that of the proton; 2) the correlation tensor $T_{ik}(E, \theta)$, describing the spin correlations in the $(p, ^3He)$ system, is symmetric. Thus, the spins of the proton and the $^3He$ nucleus in the reaction $\pi^+ + ^4He \rightarrow p + ^3He$ must be tightly correlated, which enables one, in principle, to prepare a beam of $^3He$ nuclei with controllable polarization without acting directly on these nuclei (see [1] for more details).

4. Violation of the incoherence inequalities for the correlation tensor

As it was established in the paper [4], in the case of incoherent mixtures of factorizable two-particle states of spin-1/2 fermions the following inequalities for the diagonal components of the correlation tensor should be satisfied:

$$|T_{11} + T_{22} + T_{33}| \leq 1; \quad |T_{11} + T_{22}| \leq 1; \quad |T_{11} + T_{33}| \leq 1; \quad |T_{22} + T_{33}| \leq 1. \quad (4.1)$$

However, for non-factorizable quantum-mechanical superpositions these inequalities may be violated. The triplet state $|\tilde{\psi}\rangle$ (3.1) of the final system in the reaction $\pi^+ + ^4He \rightarrow p + ^3He$ represents a characteristic example of such non-factorizable spin states (it is well seen that the state $|\tilde{\psi}\rangle$ cannot be reduced to the product of one-particle spin states).

Let us calculate the components of the correlation tensor $T_{ik}$ (3.4) for the system $(p, ^3He)$ in the coordinate frame with $z \parallel \mathbf{l}$, $x \parallel \mathbf{m}$, $y \parallel \mathbf{n}$. Finally, we obtain the following expressions:

$$T_{11} = \frac{|R_0(E, \theta)|^2 - 2|R_1(E, \theta)|^2}{|R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2}; \quad T_{22} = 1; \quad T_{33} = \frac{2|R_1(E, \theta)|^2 - |R_0(E, \theta)|^2}{|R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2} = -T_{11};$$

$$T_{13} = T_{31} = \frac{2\sqrt{2}}{2|R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2} \text{Re}(R_1 R_0^*) ; \quad T_{12} = T_{21} = T_{23} = T_{32} = 0; \quad (4.3)$$

(indexes: $1 \rightarrow x$, $2 \rightarrow y$, $3 \rightarrow z$); in doing so, $tr(T) = 1$.

Thus, as follows from Eq. (4.2), in the reaction $\pi^+ + ^4He \rightarrow p + ^3He$ one of the incoherence inequalities (4.1) for the diagonal components of the correlation tensor is necessarily violated, irrespective of the concrete mechanism of generation of the system $(p, ^3He)$. Indeed, if $|R_0|^2 > 2|R_1|^2$, we obtain that $|T_{11} + T_{22}| > 1$; if, on the contrary, $|R_0|^2 < 2|R_1|^2$, then $|T_{22} + T_{33}| > 1$. Meantime, in both the cases the other three incoherence inequalities (4.1) are satisfied.

References