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Forward virtual Compton scattering, Cottingham formula, and nucleon polarizabilities

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We review the derivation of dispersion relations for forward Compton scattering off the nucleon, including the Lorentz decomposition of the Compton tensor $T^{\mu\nu}$. We then discuss the dispersion relations for the resulting scalar functions T_1 and T_2 and present the consequences in terms of sum rules for the nucleon polarizabilities. Finally, we revisit the Cottingham formula and summarize the present status of its phenomenological evaluation. In particular, we stress that regardless of a fixed pole in T_1, the elastic contribution is unambiguous, providing a dispersive definition for the notion of the Born terms.

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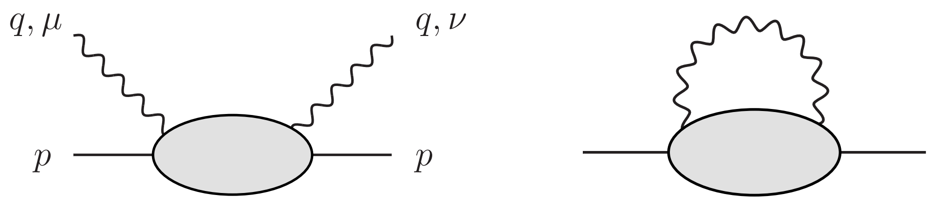

Figure 1: Kinematics for forward Compton scattering (left) and nucleon electromagnetic self energy (right). Solid and wiggly lines denote nucleons and photons, respectively, the gray blobs the hadronic four-point function.

1. Introduction

Forward Compton scattering off the nucleon, see Fig. 1, is related to the nucleon electromagnetic self energy by closing the photon loop. This naive relation can be made precise in the context of the Cottingham formula [1], which has been used in [2, 3] to extract the strong proton–neutron mass difference and provide the first estimates for the ratios of the light-quark masses, prior to Weinberg's estimate [4] based on the Dashen theorem [5]. A key assumption in the analysis concerns the asymptotic behavior of the Compton tensor, Reggeon dominance or the absence of fixed poles, in which case an otherwise necessary subtraction function can be calculated unambiguously from space-like scattering data [6, 7]. In recent years, doubts have been raised [8] concerning the treatment of this subtraction function in the analysis of [2, 3], claiming that a hadronic model was in fact unavoidable [8–10]. Here, we summarize the reasons why this criticism is unfounded [11], concentrating on several key technical points in the analysis.

2. Forward virtual Compton scattering

2.1 Lorentz decomposition

We consider the amplitude for the process

$$N(p)+\gamma^*(q,\mu)\to N(p)+\gamma^*(q,\nu), \tag{2.1}$$

the forward scattering of a photon with virtuality q off a nucleon with momentum p, see Fig. 1. Averaging over spins, this defines the Compton tensor

$$T^{\mu\nu}(p,q)=\frac{i}{2}\int d^4x\, e^{iq\cdot x}\,\frac{1}{2}\sum_s\langle N(p,s)|T\{j^\mu(x)j^\nu(0)\}|N(p,s)\rangle, \tag{2.2}$$

where $j^\mu(x)$ denotes the electromagnetic current. The kinematics are expressed in terms of the virtuality $q^2=-Q^2$ and $\nu=p\cdot q/m$, with nucleon mass m. First, we need a decomposition of $T^{\mu\nu}$ into Lorentz structures, in such a way that the coefficient functions are free of kinematic singularities and zeros. The general recipe for the derivation of such a decomposition proceeds as follows [12, 13]:

1. Write down all possible structures $\{T_i^{\mu\nu}\}=\{g^{\mu\nu},q^\mu q^\nu,p^\mu p^\nu,p^\mu q^\nu+p^\nu q^\mu\}$.

2. Apply projectors $I^{\mu\mu'}I^{\nu\nu'}T_{i\mu'\nu'}$ with $I^{\mu\nu}=g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}$,

$$\{\bar{T}_i^{\mu\nu}\}=I^{\mu\mu'}I^{\nu\nu'}\{T_{i\mu'\nu'}\}=\Big\{g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2},0,\frac{(q^2p^\mu-q^\mu p\cdot q)(q^2p^\nu-q^\nu p\cdot q)}{q^4},0\Big\}. \tag{2.3}$$

3. Take linear combinations to remove singularities in q^2 as far as possible (here $1/q^4$),

$$\begin{aligned}\bar{\bar{T}}_1^{\mu\nu}&=\bar{T}_1^{\mu\nu}=g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2},\\ \bar{\bar{T}}_2^{\mu\nu}&=\bar{T}_3^{\mu\nu}+\frac{(p\cdot q)^2}{q^2}\bar{T}_1^{\mu\nu}=\frac{1}{q^2}\Big\{g^{\mu\nu}(p\cdot q)^2+p^\mu p^\nu q^2-p\cdot q(p^\mu q^\nu+p^\nu q^\mu)\Big\}.\end{aligned} \tag{2.4}$$

4. Multiply by q^2 to remove the remaining singularities.

This recipe reproduces the Lorentz decomposition from [2]

$$\begin{aligned}T^{\mu\nu}(p,q)&=T_1(\nu,q^2)K_1^{\mu\nu}+T_2(\nu,q^2)K_2^{\mu\nu},\\ K_1^{\mu\nu}&=q^\mu q^\nu-g^{\mu\nu}q^2,\qquad K_2^{\mu\nu}=\frac{1}{m^2}\Big\{(p^\mu q^\nu+p^\nu q^\mu)p\cdot q-g^{\mu\nu}(p\cdot q)^2-p^\mu p^\nu q^2\Big\}.\end{aligned} \tag{2.5}$$

A careful choice of the Lorentz decomposition is critical, e.g., the alternative decomposition

$$\begin{aligned}T^{\mu\nu}(p,q)&=\tilde{T}_1(\nu,q^2)\tilde{K}_1^{\mu\nu}+\tilde{T}_2(\nu,q^2)\tilde{K}_2^{\mu\nu},\\ \tilde{K}_1^{\mu\nu}&=\frac{1}{2}\Big(\frac{q^\mu q^\nu}{q^2}-g^{\mu\nu}\Big),\qquad \tilde{K}_2^{\mu\nu}=\frac{1}{2m^2}\Big(p^\mu-\frac{p\cdot q}{q^2}q^\mu\Big)\Big(p^\nu-\frac{p\cdot q}{q^2}q^\nu\Big),\end{aligned} \tag{2.6}$$

often employed in the literature is related by

$$\tilde{T}_1(\nu,q^2)=2q^2T_1(\nu,q^2)+2\nu^2T_2(\nu,q^2),\qquad \tilde{T}_2(\nu,q^2)=-2q^2T_2(\nu,q^2), \tag{2.7}$$

where $\tilde{T}_1(\nu,q^2)$ and $\tilde{T}_2(\nu,q^2)$ now have kinematic zeros. This implies that subtractions already become required just to get the kinematic zeros right, which is the reason why the calculation of the elastic contribution to the Cottingham formula in [8] based on unsubtracted dispersion relations for the $\tilde{T}_i$ produced an incorrect result. With only two scalar functions the identification of a proper Lorentz basis is relatively straightforward, but for more complicated kinematics [13, 14] or processes [15] the derivation of a set of scalar functions amenable to a dispersive treatment becomes all but impossible without a systematic approach.

2.2 Dispersion relations

In general, the necessity of subtractions in the dispersion relations for the T_i depends on the behavior of the imaginary parts. To formulate these conditions, we consider the structure functions $V_i(\nu,q^2)$, which fulfill $\operatorname{Im}T_i(\nu,q^2)=\pi V_i(\nu,q^2)$ for $\nu\geq 0$, $q^2\leq 0$, and for $q^2<0$ are fully determined by eN cross sections. Regge theory predicts

$$V_1(\nu,q^2)\sim\nu^\alpha,\qquad V_2(\nu,q^2)\sim\nu^{\alpha-2}, \tag{2.8}$$

with $\alpha_P\sim 1$ for the Pomeron and $\alpha_R\sim 0.5$ for the f and a_2 trajectories. Since the analytic continuation is unique up to a polynomial $\Delta V_i(\nu,q^2)=\epsilon(\nu)\sum_{n=0}^N\sigma_n^{(i)}(q^2)\nu^{2n}$ [7], where $\sigma_n^{(i)}(q^2)=0$ for

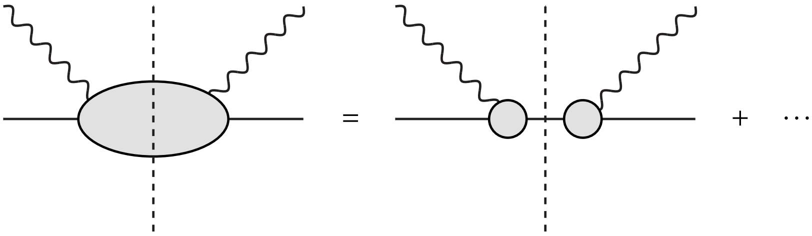

Figure 2: Unitarity relation for nucleon Compton scattering. The short-dashed line indicates that intermediate states are to be taken on-shell. The expansion starts with the nucleon pole, which defines the elastic contribution, while the ellipsis denotes inelastic corrections.

$q^2 \leq 0$, one has $\Delta V_2 = 0$ according to Regge theory, but ΔV_1 could be non-zero. Thus, unsubtracted dispersion relations for both $T_i(\nu, q^2)$ are possible if $\Delta V_1 = 0$, which is called the Reggeon dominance hypothesis, or, equivalently, referred to as the absence of fixed poles. We may then write the dispersion relations as

$$T_1(\nu,q^2) = S_1(q^2) + 2\nu^2 \int_0^\infty \frac{d\nu'}{\nu'} \frac{V_1(\nu',q^2)}{\nu'^2-\nu^2-i\epsilon}, \qquad T_2(\nu,q^2) = 2\int_0^\infty d\nu'\,\nu' \frac{V_2(\nu',q^2)}{\nu'^2-\nu^2-i\epsilon}, \tag{2.9}$$

valid in either case, while in the absence of fixed poles the subtraction function $S_1(q^2)$ can be calculated from the sum rule

$$S_1(q^2) = T_1(0,q^2) = T_1^R(0,q^2) + 2\int_0^\infty \frac{d\nu}{\nu}\Big[V_1(\nu,q^2) - V_1^R(\nu,q^2)\Big], \tag{2.10}$$

where the Regge amplitude $T_1^R(0,q^2)$ has been separated due to its known singularity structure.

The dominant contribution to the unitarity relation is given by elastic states, see Fig. 2, whose spectral functions are determined by the electromagnetic form factors of the nucleon, the Sachs form factors $G_E(q^2)$ and $G_M(q^2)$,

$$V_i^{\text{el}}(\nu,q^2) = v_i^{\text{el}}(q^2)\Big[\delta(q^2+2m\nu) - \delta(q^2-2m\nu)\Big], \tag{2.11}$$

$$v_1^{\text{el}}(q^2) = \frac{2m^2}{4m^2-q^2}\Big[G_E^2(q^2) - G_M^2(q^2)\Big], \qquad v_2^{\text{el}}(q^2) = \frac{2m^2}{(-q^2)(4m^2-q^2)})\Big[4m^2G_E^2(q^2) - q^2G_M^2(q^2)\Big],$$

providing a unique dispersive definition for the elastic contribution to $T_i(\nu,q^2)$

$$\begin{aligned} T_1^{\text{el}}(\nu,q^2) &= \frac{4m^2q^2}{(4m^2\nu^2-q^4)(4m^2-q^2)}\Big[G_E^2(q^2) - G_M^2(q^2)\Big], \\ T_2^{\text{el}}(\nu,q^2) &= -\frac{4m^2}{(4m^2\nu^2-q^4)(4m^2-q^2)}\Big[4m^2G_E^2(q^2) - q^2G_M^2(q^2)\Big]. \end{aligned} \tag{2.12}$$

The corresponding causal representation is constructed explicitly in App. D of [11].

In consequence, the relations (2.12) yield a possible rigorous definition of the Born contribution, which remains ambiguous in a diagrammatic derivation [16–18], but can be reproduced order-by-order in a effective field theory. For instance, this dispersive definition of the Born contribution differs from [18] by a numerically small term $\Delta T_1(\nu,q^2) = (F_2(q^2))^2/(4m^2)$ (with Pauli form factor $F_2(q^2)$), reflecting the fact that the nucleon polarizabilities as usually defined involve a small

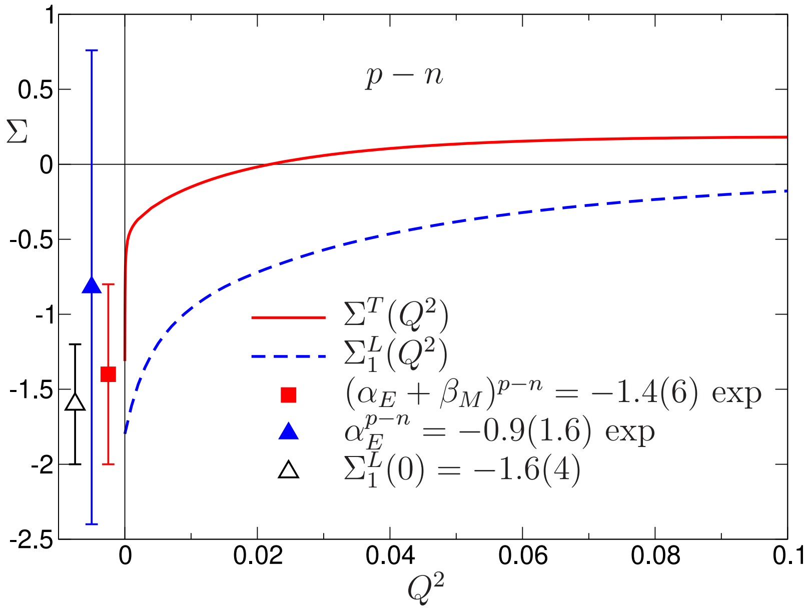

Figure 3: Contributions to the transverse (red solid) and longitudinal (blue dashed) sum rules, generalized to finite Q^2. The values at $Q^2 = 0$ are shown for comparison (all in units of 10^{-4}fm^3), see main text for details. Figure taken from [11].

elastic component. Moreover, only the truly elastic piece (2.12) can be effectively resummed in terms of nucleon form factors in the Cottingham formula, for the additional piece $\Delta T_1(\nu,q^2)$ that emerges diagrammatically such a dispersive justification does not exist. Here, a full treatment of the subtraction function is unavoidable.

3. Sum rules for the nucleon polarizabilities

The electric and magnetic nucleon polarizabilities α_E and β_M are related to the inelastic contribution to the subtraction functions $S_i^{\text{inel}}(q^2)$ by the low-energy theorems

$$S_1^{\text{inel}}(0) = -\frac{\kappa^2}{4m^2} - \frac{m}{\alpha_{\text{em}}}\beta_M, \qquad S_2^{\text{inel}}(0) = \frac{m}{\alpha_{\text{em}}}(\alpha_E+\beta_M), \tag{3.1}$$

with fine-structure constant $\alpha_{\text{em}} = e^2/(4\pi)$, and the elastic piece involving the magnetic moment κ arises because conventionally the polarizabilities are defined including $\Delta T_1(\nu,q^2)$ in the notion of the Born terms. The dispersion relation for $S_2^{\text{inel}}(0)$ simply reproduces the Baldin sum rule [19]

$$\alpha_E+\beta_M = \frac{1}{\pi}\int_{\nu_{\text{th}}}^{\infty}\frac{d\nu}{\nu^2}\sigma_{\text{tot}}(\nu), \tag{3.2}$$

with the total, transverse, cross section $\sigma_{\text{tot}}(\nu,0)$ and ν_{th} the threshold for pion production. For β_M separately or, equivalently, for α_E, a new sum rule emerges [11]

$$\alpha_E = \Sigma_1^L(0) + \frac{\alpha_{\text{em}}\kappa^2}{4m^3}, \qquad \Sigma_1^L(0) = \frac{\alpha_{\text{em}}}{m}T_1^R(0,0) + \frac{1}{2\pi^2}\int_{\nu_{\text{th}}}^{\infty} d\nu\, \frac{\sigma_L(\nu,Q^2)-\sigma_L^R(\nu,Q^2)}{Q^2}\bigg|_{Q^2=0}, \tag{3.3}$$

which is valid precisely in the absence of fixed poles. For the numerical evaluation one needs a parameterization of the Regge amplitudes T_1^R as well as the longitudinal cross sections $\sigma_L(\nu, Q^2)$.

The numerical evaluation from [11] is illustrated in Fig. 3, using input from photoproduction multipoles [20–24] for low energies, from [25, 26] for the resonance region, and from [27–35] for estimates of the Regge contribution. We reproduce well the analysis of the Baldin sum rule from [36, 37], shown by the red square. The figure includes both our result for $\Sigma_1^L(0)$ as well as the experimental result for α_E^{p-n} [38, 39]. Including the elastic κ^2 piece, we find that for the proton–neutron difference

$$\alpha_E^{p-n}\big|_{\text{Reggeon dominance}} = -1.7(4)\times 10^{-4}\text{fm}^3, \qquad \alpha_E^{p-n}\big|_{\text{exp}} = -0.9(1.6)\times 10^{-4}\text{fm}^3, \tag{3.4}$$

so the result of the dispersive analysis assuming Reggeon dominance agrees with experiment within uncertainties. If there is a fixed-pole contribution, its coefficient has to be small. In principle, the same analysis could be repeated for proton and neutron separately, but for this purpose a better understanding of the Pomeron Regge trajectory would be required.

4. Cottingham formula

The Cottingham formula expresses the electromagnetic contribution to the mass m_γ as a loop integral

$$m_\gamma = \frac{ie^2}{2m(2\pi)^4}\int d^4q D_\Lambda(q^2)\Big[3q^2T_1 + (2\nu^2+q^2)T_2\Big] + \text{counter terms}, \tag{4.1}$$

where $D_\Lambda(q^2)$ is the regularized photon propagator and the counter terms are needed for renormalization [40, 41]. For the elastic contribution, however, the regulator can be removed and one finds

$$m_\gamma^{\text{el}} = \frac{\alpha_{\text{em}}}{8\pi m^3}\int_0^\infty dQ^2 Q^2\Big[f_1\, v_1^{\text{el}}(-Q^2) + f_2\, v_2^{\text{el}}(-Q^2)\Big],$$
$$f_1 = 3\Big[\sqrt{1+\frac{1}{y}}-1\Big], \qquad f_2 = (1-2y)\sqrt{1+\frac{1}{y}}+2y, \tag{4.2}$$

which evaluates to

$$(m_\gamma^{\text{el}})^p = 0.63\,\text{MeV}, \qquad (m_\gamma^{\text{el}})^n = -0.13\,\text{MeV}, \qquad (m_\gamma^{\text{el}})^{p-n} = 0.76\,\text{MeV}. \tag{4.3}$$

With inelastic contributions estimated as $\pm 0.30\,\text{MeV}$ [2], this result is consistent with lattice QCD, e.g., $(m_\gamma)^{p-n} = 1.00(16)\,\text{MeV}$ [42] and $(m_\gamma)^{p-n} = 1.03(17)\,\text{MeV}$ [43], so at this level there is no evidence for a fixed pole. The prediction from the Cottingham formula could be sharpened with improved input for the nucleon structure functions given that [2] still relied on scaling assumptions, but the required detailed analysis has not been carried out so far. On the one hand, the parameterization for the structure functions needs to properly implement the Regge limit, in such a way that the subtraction of the Regge amplitude in (2.10) indeed reliably removes the Reggeon singularity, but at the same time the q^2-dependence of the Regge residues needs to be matched smoothly to deep-inelastic scattering because otherwise the renormalization of (4.1) as regards the photon propagator will not work. Indeed, the model evaluations [8, 10] display a strong sensitivity to the regulator varied within the narrow window $1.5\,\text{GeV}^2 < \Lambda^2 < 2.5\,\text{GeV}^2$.

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5. Conclusions

In summary, we reviewed the derivation of dispersion relations for forward virtual Compton scattering, including the Bardeen–Tung–Tarrach Lorentz decomposition to avoid kinematic singularities and zeros in the scalar amplitudes. With such a decomposition, in principle, the subtraction function in T_1 can be calculated if fixed poles are absent, but in either case the dispersive approach gives a unique definition of the elastic contributions. As a first test of the Reggeon dominance hypothesis we presented the result of a numerical analysis of a sum rule for the electric nucleon polarizability α_E that holds under the same assumption, with the result that if there is a fixed pole in the proton–neutron difference at $Q^2 = 0$, its coefficient has to be small. Finally, we stressed that within the considerable uncertainties from the inelastic contributions, the evaluation of the Cottingham formula based on elastic intermediate states only is consistent with lattice QCD. A more refined analysis of the inelastic contributions would require a detailed study of the role of the nucleon structure functions, in particular the interplay between deep inelastic scattering and the Regge limit, to obtain a parameterization that both ensures that the sum rule for the subtraction function for T_1 converges and that the Cottingham formula is fully renormalized.

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