PROCEEDINGS OF SCIENCE

Pion-pole contribution to hadronic light-by-light scattering

Bai-Long Hoid*
Helmholtz-Institut für Strahlen- und Kernphysik (Theory) and
Bethe Center for Theoretical Physics, University of Bonn, 53115 Bonn, Germany
E-mail: longbai@hiskp.uni-bonn.de

Martin Hoferichter
Institute for Nuclear Theory, University of Washington, Seattle, WA 98195-1550, USA
E-mail: mhofer@uw.edu

Bastian Kubis
Helmholtz-Institut für Strahlen- und Kernphysik (Theory) and
Bethe Center for Theoretical Physics, University of Bonn, 53115 Bonn, Germany
E-mail: kubis@hiskp.uni-bonn.de

Stefan Leupold
Institutionen för fysik och astronomi, Uppsala Universitet, Box 516, 75120 Uppsala, Sweden
E-mail: stefan.leupold@physics.uu.se

Sebastian P. Schneider
Helmholtz-Institut für Strahlen- und Kernphysik (Theory) and
Bethe Center for Theoretical Physics, University of Bonn, 53115 Bonn, Germany
E-mail: schneider@hiskp.uni-bonn.de

PoS(CD2018)052

We report on the first dispersive calculation of the pion-pole contribution to hadronic light-by-light scattering in the anomalous magnetic moment of the muon $(g-2)_\mu$. It is unambiguously defined in terms of the doubly-virtual pion transition form factor, which is in principle directly accessible in experiment. In the absence of a complete measurement covering all kinematic regions relevant for $(g-2)_\mu$, we derive a dispersive representation that accounts for all the low-lying singularities, reproduces the correct high- and low-energy limits, and proves suitable for the evaluation of the $(g-2)_\mu$ loop integral based on the available data for $\pi^0 \to \gamma\gamma$, $e^+e^- \to 3\pi$, and $e^+e^- \to e^+e^-\pi^0$. Our final result as a complete data-driven determination with fully controlled uncertainty estimates, $a_\mu^{\pi^0\text{-pole}} = 62.6^{+3.0}_{-2.5} \times 10^{-11}$, can be further improved in light of forthcoming singly-virtual measurements.

The 9th International Workshop on Chiral Dynamics
17–21 September 2018
Durham, NC, USA

*Speaker.

1. Introduction

The anomalous magnetic moment of the muon, $a_\mu = (g-2)_\mu/2$, is one of the prime physical quantities to monitor signals coming from physics beyond the Standard Model (SM). Its current value [1, 2] reveals a tantalizing deviation of about $(3\text{–}4)\sigma$ from the SM prediction. More ambitious upgraded experiments at Fermilab [3] and J-PARC [4] were planned for this reason, which demand simultaneous advances in the prediction of the SM contributions.

The dominant SM uncertainty is due to hadronic effects [5 – 7]. The dispersive evaluation of the first leading category, hadronic vacuum polarization (HVP) entering at $\mathcal{O}(\alpha^2)$, benefits from improved experimental measurements of $e^+e^- \to$ hadrons [8]. In contrast, a dispersive framework for the evaluation of the second[1] hadronic light-by-light (HLbL) scattering category at $\mathcal{O}(\alpha^3)$ based on the general principles of analyticity, unitarity, and crossing symmetry was only recently developed [12 – 17]. Such a model-independent framework reconstructs the HLbL tensor from its singularities and attributes the contributions to on-shell form factors and scattering amplitudes accessible from experiment, complementary to lattice QCD calculations of HLbL scattering [18 – 22].

The simplest such singularities of the HLbL tensor arise from light pseudoscalar-meson poles. The residue of the numerically dominant pion pole is determined by the doubly-virtual pion transition form factor (TFF) evaluated at all relevant space-like momenta. In the absence of double-tag measurements for $e^+e^- \to e^+e^-\pi^0$, we employ the existing data for the $\pi^0 \to \gamma\gamma$ decay width, the $e^+e^- \to 3\pi$ cross section, and the space-like singly-virtual form factor from $e^+e^- \to e^+e^-\pi^0$ to reconstruct the doubly-virtual pion TFF, owing to the constraints from analyticity and unitarity. The constructed form factor representation

$$F_{\pi^0\gamma^*\gamma^*} = F^{\text{disp}}_{\pi^0\gamma^*\gamma^*} + F^{\text{eff}}_{\pi^0\gamma^*\gamma^*} + F^{\text{asym}}_{\pi^0\gamma^*\gamma^*} \tag{1.1}$$

has the following remarkable properties: its first dispersive part takes into account all low-energy intermediate states; the second (small) contribution from higher intermediate states incorporates the normalization and space-like high-energy data; and the asymptotic constraints for arbitrary virtualities are implemented via the last term at $\mathcal{O}(1/Q^2)$. The calculation of the pion-pole contribution based on this comprehensive dispersive determination of the pion TFF completes a dedicated effort to obtain a fully data-driven determination of $a_\mu^{\pi^0\text{-pole}}$ [23 – 29].

2. Pion-pole contribution to a_μ

In order to evaluate the HLbL scattering contribution to a_μ, we define the full fourth-rank HLbL tensor $\Pi_{\mu\nu\lambda\sigma}$, the Green's function of four electromagnetic currents evaluated in pure QCD [15]:

$$\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = -i\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\, e^{-i(q_1\cdot x+q_2\cdot y+q_3\cdot z)}\langle 0|T\left\{j_\mu(x)j_\nu(y)j_\lambda(z)j_\sigma(0)\right\}|0\rangle. \tag{2.1}$$

The pion-pole contribution to a_μ is attributed to the single-pion one-particle reducible piece of $\Pi_{\mu\nu\lambda\sigma}$, as depicted in Fig. 1. It can be calculated with the help of projection techniques and simplifies to a three-dimensional integral representation [5] after performing Wick rotations for the two-

[1]Higher-order insertions of HVP and HLbL scattering were considered in [9 – 11].

PoS(CD2018)052

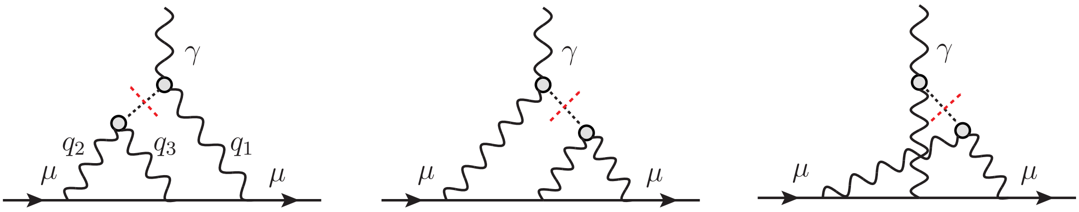

Figure 1: The pion-pole contribution to HLbL scattering of a_μ; figure taken from [29].

loop integrals [30] and carrying out five out of six angular integrations resorting to Gegenbauer-polynomials,

$$a_\mu^{\pi^0\text{-pole}} = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty \mathrm{d}Q_1 \int_0^\infty \mathrm{d}Q_2 \int_{-1}^1 \mathrm{d}\tau \Big[w_1(Q_1,Q_2,\tau)\, F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_3^2)\, F_{\pi^0\gamma^*\gamma^*}(-Q_2^2,0) + w_2(Q_1,Q_2,\tau)\, F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)\, F_{\pi^0\gamma^*\gamma^*}(-Q_3^2,0)\Big], \tag{2.2}$$

where $Q_3^2 = Q_1^2 + 2Q_1Q_2\tau + Q_2^2$, and $\tau = \cos\theta$, the remaining angle between Q_1 and Q_2. The dimensionless weight functions w_1 and w_2 are peaked at small momenta ranges such that $a_\mu^{\pi^0\text{-pole}}$ gets its most prevailing contribution from the low-energy region. Consequently, the dominant pion-pole contribution can be obtained with well-controlled uncertainties as the pion TFF $F_{\pi^0\gamma^*\gamma^*}$ can be precisely determined in our dispersive framework at low energy.

3. Pion transition form factor

The pion TFF is defined by the matrix element of two light-quark currents

$$i \int \mathrm{d}^4x\, e^{iq_1\cdot x} \langle 0|T\,\{j_\mu(x) j_\nu(0)\}|\pi^0(q_1+q_2)\rangle = -\varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\, F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2). \tag{3.1}$$

The normalization of the form factor is related to the pion decay constant $F_\pi = 92.28(9)\,\text{MeV}$ [31], dictated by a low-energy theorem [32 – 34],

$$F_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{1}{4\pi^2 F_\pi}. \tag{3.2}$$

A Primakoff measurement of the $\pi^0 \to \gamma\gamma$ decay tested it up to 1.4% [35]. The chiral tree-level prediction (3.2), along with its 1.4% uncertainty, is used as the central value and uncertainty estimate for the normalization of the TFF (for potential improvements, see [36]). Despite the lack of double-tag measurements of the TFF thus far, there is ample experimental information for the singly-virtual form factor [37 – 40]. These data sets cover primarily large virtualities, though the low-energy region is most relevant for a_μ. Recent theoretical activities on the pion TFF mainly concern lattice QCD [41], light-cone sum rules [42 – 44], Dyson–Schwinger equations [45, 46], and other frameworks [47 – 51]. We focus on the dispersive approach here.

3.1 Dispersion relations

We decompose the form factor into isovector and isoscalar virtualities

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = F_{vs}(q_1^2,q_2^2) + F_{vs}(q_2^2,q_1^2) \tag{3.3}$$

to study its dispersion relations. $F_{vs}(q_1^2,q_2^2)$ fulfills a dispersion relation under the assumption of maximum analyticity,

$$F_{vs}(q_1^2,q_2^2) = \frac{1}{12\pi^2}\int_{4M_\pi^2}^{s_{\rm iv}} \mathrm{d}x \frac{q_\pi^3(x)\big(F_\pi^V(x)\big)^* f_1(x,q_2^2)}{x^{1/2}(x-q_1^2)}, \tag{3.4}$$

where $s_{\rm iv}$ is a chosen isovector cutoff and $q_\pi(s) = \sqrt{s/4-M_\pi^2}$. The pion vector form factor $F_\pi^V(s)$ and the $\gamma_s^* \to 3\pi$ P-wave amplitude $f_1(s,q^2)$ are the crucial building blocks in the unitarity relation. The representation (3.4) in an unsubtracted form facilitates its smooth matching onto the asymptotic behavior from perturbative QCD and the evaluation of a convergent a_μ loop integral.

The building block $F_\pi^V(s)$ is described by different variants of the Omnès representation [52] fit to the data measured in τ decays [53]. We consider $\pi\pi$ P-wave phase shifts from [54, 55] and an extension of [54] including the $\rho'(1450)$ and $\rho''(1700)$ resonances in an elastic approximation [24]. In contrast, the analytic structure of $f_1(s,q^2)$ is much more complicated. The normalization $a(q^2)$ of its iterative solution [26] is determined by another low-energy theorem [56 – 60] at $q^2 = 0$, and is accessible in $e^+e^- \to 3\pi$ in the decay region $q^2 > 9M_\pi^2$. We have improved the parameterization derived in [26] by introducing a conformal polynomial to account for the effects from inelastic channels. This indeed substantially improves the fit results above the ϕ peak and allows us to describe the low-energy data [61 – 63] with a reduced $\chi^2/\mathrm{dof} \sim 1$, as shown in Fig. 2.

In this manner, the representation (3.4), in principle, already determines the general doubly-virtual form factor. But we derive a more compact form advantageous to the evaluation in the entire space-like region by applying yet another dispersion relation in the isoscalar variable

$$F_{vs}(-Q_1^2,q_2^2) = \frac{1}{\pi}\int_{s_{\rm thr}}^{s_{\rm is}} \mathrm{d}y \frac{\mathrm{Im}\, F_{vs}(-Q_1^2,y)}{y-q_2^2-i\varepsilon}, \tag{3.5}$$

with an isoscalar cutoff $s_{\rm is}$ and an integration threshold $s_{\rm thr}$. This leads to a double-spectral representation of the form factor

$$\begin{aligned} F^{\rm disp}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2) &= \frac{1}{\pi^2}\int_{4M_\pi^2}^{s_{\rm iv}} \mathrm{d}x \int_{s_{\rm thr}}^{s_{\rm is}} \mathrm{d}y \frac{\rho^{\rm disp}(x,y)}{(x+Q_1^2)(y+Q_2^2)} + (Q_1 \leftrightarrow Q_2), \\ \rho^{\rm disp}(x,y) &= \frac{q_\pi^3(x)}{12\pi\sqrt{x}} \mathrm{Im}\Big[\big(F_\pi^V(x)\big)^* f_1(x,y)\Big]. \end{aligned} \tag{3.6}$$

3.2 Asymptotic behavior

The expansion of the pion TFF along the light cone for large momenta reads [64 – 66]

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = -\frac{2F_\pi}{3}\int_0^1 \mathrm{d}u \frac{\phi_\pi(u)}{uq_1^2+(1-u)q_2^2} + \mathcal{O}(q_i^{-4}), \tag{3.7}$$

with an asymptotic pion distribution amplitude $\phi_\pi(u) = 6u(1-u)$. The Brodsky–Lepage (BL) limit $\lim_{Q^2\to\infty} Q^2 F_{\pi^0\gamma^*\gamma^*}(-Q^2,0) = 2F_\pi$ and the operator product expansion (OPE) limit [67, 68] $\lim_{Q^2\to\infty} Q^2 F_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2) = 2F_\pi/3$ follow from (3.7).

The implementation of the asymptotic constraints proceeds as follows. Firstly, it has been observed that (3.7) can be transformed into a dispersion relation by a simple change of variables [69].

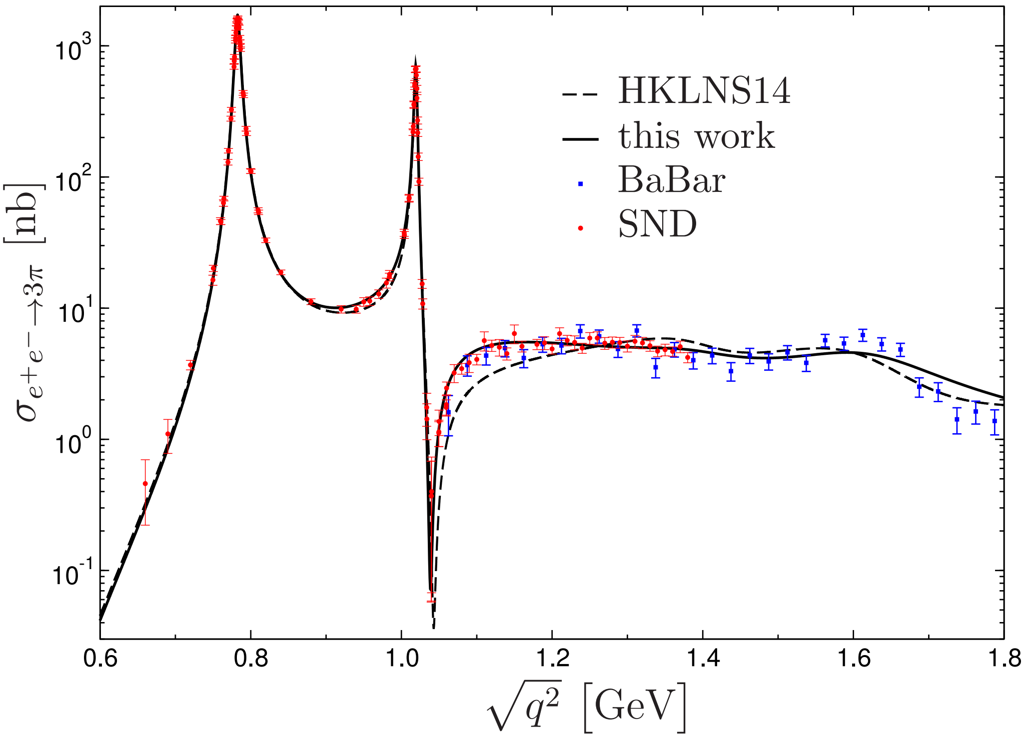

Figure 2: Fit to the $e^+e^- \to 3\pi$ cross section from SND [61, 62] and BaBar [63], in comparison to [26] (HKLNS14). Figure taken from [28].

Furthermore, we find that identifying the discontinuities in the second variable leads to a new double-spectral representation for the asymptotic expression:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi^2}\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \frac{\rho^{\text{asym}}(x,y)}{(x-q_1^2)(y-q_2^2)}, \tag{3.8}$$

where

$$\rho^{\text{asym}}(x,y) = -2\pi^2 F_\pi xy\delta''(x-y) \tag{3.9}$$

is a double-spectral density. This defines an asymptotic contribution after introducing a continuum threshold s_{m},

$$F^{\text{asym}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = 2F_\pi \int_{s_{\text{m}}}^\infty \mathrm{d}x \frac{q_1^2 q_2^2}{(x-q_1^2)^2(x-q_2^2)^2}, \tag{3.10}$$

which ensures consistency with (3.7) for non-vanishing virtualities.

As a final step, we present how to incorporate high-energy TFF data in our representation (1.1). This is achieved by introducing an effective pole term subsuming the contributions from higher intermediate states,

$$F^{\text{eff}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{g_{\text{eff}}}{4\pi^2 F_\pi}\frac{M_{\text{eff}}^4}{(M_{\text{eff}}^2-q_1^2)(M_{\text{eff}}^2-q_2^2)}, \tag{3.11}$$

where the coupling g_{eff} is determined by imposing the sum rule for the normalization (3.2) and the mass parameter M_{eff} is fit to the space-like singly-virtual data [37–40]. The resulting parameters g_{eff} and M_{eff} are found to be around 10% and (1.5–2) GeV respectively, consistent with the assumption of an effective pole.

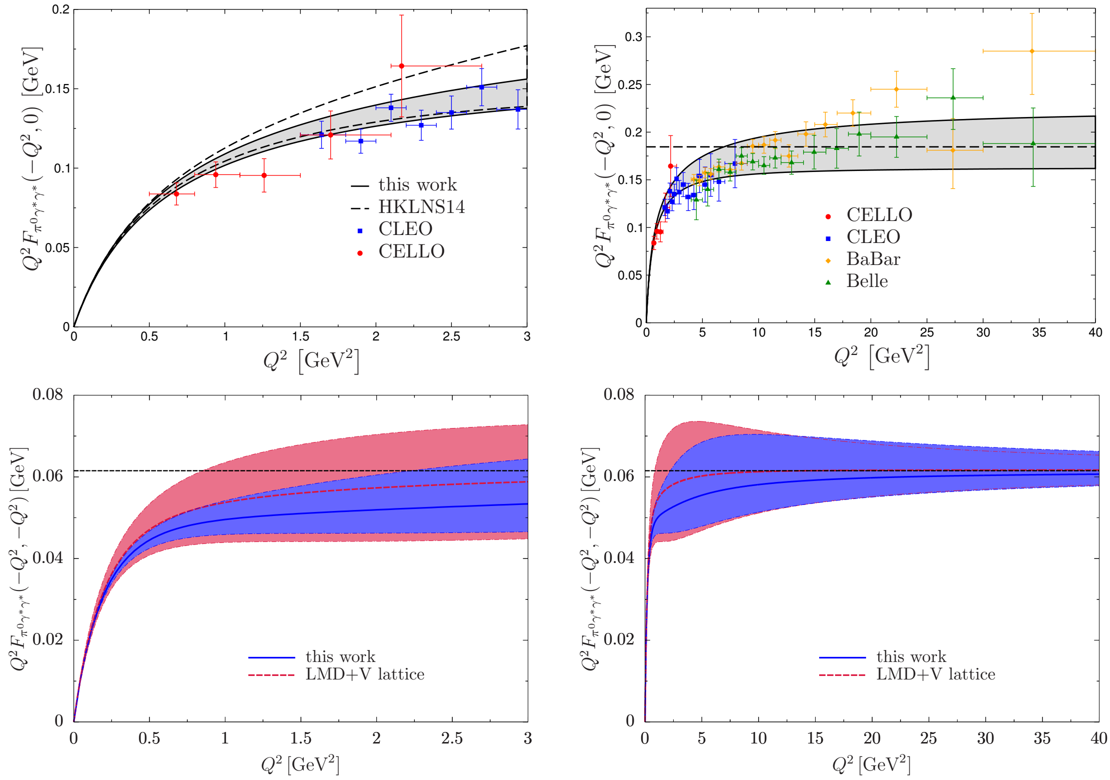

Figure 3: The singly-virtual form factor $Q^2F_{\pi^0\gamma^*\gamma^*}(-Q^2,0)$ and the diagonal form factor $Q^2F_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2)$ versus Q^2, in comparison to the experimental data [37 – 40] and the LMD+V (lowest meson dominance + vector) model fit to the lattice data [41]. The dashed horizontal lines indicate the BL and OPE limits. Figures taken from [29].

Concerning the error analysis, we vary g_{eff} within the 1.4% precision determined by the Primakoff measurement [35]. We define our central value as the fit to all data but BaBar [39] above $5\,\text{GeV}^2$ to retain the prediction of the low-energy region. A very generous uncertainty band $^{+20}_{-10}\%$ around the central value fully incorporates systematic effects from the tension of the BaBar data [39] both with the BL limit and the other data sets. In particular, the results of the TFF along the singly-virtual and diagonal directions are shown in Fig. 3.

4. Consequences for a_μ

Our final result for the pion-pole contribution reads

$$\begin{aligned} a_\mu^{\pi^0\text{-pole}} &= 62.6(1.7)_{F_{\pi\gamma\gamma}}(1.1)_{\text{disp}}\left({}^{2.2}_{1.4}\right)_{\text{BL}}(0.5)_{\text{asym}} \times 10^{-11} \\ &= 62.6^{+3.0}_{-2.5} \times 10^{-11}, \end{aligned} \tag{4.1}$$

where the numerical integration uncertainty is negligible. Therefore all uncertainties derive from the TFF: $\pm 1.4\%$ $F_{\pi\gamma\gamma}$ normalization uncertainty varying g_{eff}; the dispersive error defined as the maximum deviation from the central result of all variants of the dispersive formalism; the $^{+20}_{-10}\%$ BL uncertainty varying M_{eff}; and the asymptotic error from the variation of $s_{\text{m}} = 1.7(3)\,\text{GeV}^2$.

Our central value is in the same ballpark as most recent phenomenological evaluations [70–72], while it provides a rigorous uncertainty estimate in contrast to the model-based analyses. Our result is also consistent with lattice QCD employing a LMD+V model fit [41], of which upcoming high statistical data will allow for a model-independent direct comparison. Our determination is already safely below the level required for the upcoming experiments. These uncertainties can be controlled further by virtue of future more precise measurements both in low and high-energy regimes [73, 74].

5. Conclusions and outlook

We have presented a dispersive treatment of the doubly-virtual pion TFF, which is key for the pion-pole contribution to a_μ. The low-lying 2π and 3π singularities in the isovector and isoscalar virtualities were considered first. We have found the leading-order, leading-twist light-cone expansion (3.7) can be reformulated in terms of an asymptotic double-spectral density. The corresponding asymptotic contribution provides the correct high-energy behavior of the TFF for non-vanishing virtualities. Finally, an effective pole term was introduced to remedy the normalization of the form factor and account for constraints from space-like singly-virtual data.

This detailed study of the pion TFF accumulates to the first dispersive determination of the pion-pole contribution to a_μ. Our data-driven evaluation provides for the first time a determination which fulfills the constraints from analyticity, unitarity, crossing symmetry, and perturbative QCD.

The pion-pole contribution to a_μ is the largest indispensable piece of a complete data-driven evaluation of HLbL scattering. Furthermore, the conceptual advances in the incorporation of high-energy constraints will shed light on similar studies for the η and η' poles [75–78].

Acknowledgments

The speaker acknowledges Jacobo Ruiz de Elvira, the convener of the Goldstone-Boson Dynamics Working Group, for making remote participation possible. Financial support by the DFG (CRC 110, “Symmetries and the Emergence of Structure in QCD”), the Bonn–Cologne Graduate School of Physics and Astronomy (BCGS), and the DOE (Grant No. DE-FG02-00ER41132) is gratefully acknowledged.

References

[1] G. W. Bennett *et al.* [Muon $g-2$ Collaboration], Phys. Rev. D **73** (2006) 072003 [hep-ex/0602035].

[2] P. J. Mohr, D. B. Newell and B. N. Taylor, Rev. Mod. Phys. **88** (2016) 035009 [arXiv:1507.07956 [physics.atom-ph]].

[3] J. Grange *et al.* [Muon $g-2$ Collaboration], arXiv:1501.06858 [physics.ins-det].

[4] N. Saito [J-PARC $g-2$/EDM Collaboration], AIP Conf. Proc. **1467** (2012) 45.

[5] F. Jegerlehner and A. Nyffeler, Phys. Rept. **477** (2009) 1 [arXiv:0902.3360 [hep-ph]].

[6] J. Prades, E. de Rafael and A. Vainshtein, Adv. Ser. Direct. High Energy Phys. **20** (2009) 303 [arXiv:0901.0306 [hep-ph]].

PoS(CD2018)052

[7] C. Aubin *et al.*, arXiv:1407.4021 [hep-ph].

[8] T. Blum, A. Denig, I. Logashenko, E. de Rafael, B. L. Roberts, T. Teubner and G. Venanzoni, arXiv:1311.2198 [hep-ph].

[9] J. Calmet, S. Narison, M. Perrottet and E. de Rafael, Phys. Lett. **61B** (1976) 283.

[10] A. Kurz, T. Liu, P. Marquard and M. Steinhauser, Phys. Lett. B **734** (2014) 144 [arXiv:1403.6400 [hep-ph]].

[11] G. Colangelo, M. Hoferichter, A. Nyffeler, M. Passera and P. Stoffer, Phys. Lett. B **735** (2014) 90 [arXiv:1403.7512 [hep-ph]].

[12] M. Hoferichter, G. Colangelo, M. Procura and P. Stoffer, Int. J. Mod. Phys. Conf. Ser. **35** (2014) 1460400 [arXiv:1309.6877 [hep-ph]].

[13] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, JHEP **1409** (2014) 091 [arXiv:1402.7081 [hep-ph]].

[14] G. Colangelo, M. Hoferichter, B. Kubis, M. Procura and P. Stoffer, Phys. Lett. B **738** (2014) 6 [arXiv:1408.2517 [hep-ph]].

[15] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, JHEP **1509** (2015) 074 [arXiv:1506.01386 [hep-ph]].

[16] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, Phys. Rev. Lett. **118** (2017) 232001 [arXiv:1701.06554 [hep-ph]].

[17] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, JHEP **1704** (2017) 161 [arXiv:1702.07347 [hep-ph]].

[18] T. Blum, S. Chowdhury, M. Hayakawa and T. Izubuchi, Phys. Rev. Lett. **114** (2015) 012001 [arXiv:1407.2923 [hep-lat]].

[19] J. Green, O. Gryniuk, G. von Hippel, H. B. Meyer and V. Pascalutsa, Phys. Rev. Lett. **115** (2015) 222003 [arXiv:1507.01577 [hep-lat]].

[20] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin and C. Lehner, Phys. Rev. D **93** (2016) 014503 [arXiv:1510.07100 [hep-lat]].

[21] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung and C. Lehner, Phys. Rev. Lett. **118** (2017) 022005 [arXiv:1610.04603 [hep-lat]].

[22] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung and C. Lehner, Phys. Rev. D **96** (2017) 034515 [arXiv:1705.01067 [hep-lat]].

[23] F. Niecknig, B. Kubis and S. P. Schneider, Eur. Phys. J. C **72** (2012) 2014 [arXiv:1203.2501 [hep-ph]].

[24] S. P. Schneider, B. Kubis and F. Niecknig, Phys. Rev. D **86** (2012) 054013 [arXiv:1206.3098 [hep-ph]].

[25] M. Hoferichter, B. Kubis and D. Sakkas, Phys. Rev. D **86** (2012) 116009 [arXiv:1210.6793 [hep-ph]].

[26] M. Hoferichter, B. Kubis, S. Leupold, F. Niecknig and S. P. Schneider, Eur. Phys. J. C **74** (2014) 3180 [arXiv:1410.4691 [hep-ph]].

[27] M. Hoferichter, B. Kubis and M. Zanke, Phys. Rev. D **96** (2017) 114016 [arXiv:1710.00824 [hep-ph]].

[28] M. Hoferichter, B.-L. Hoid, B. Kubis, S. Leupold and S. P. Schneider, Phys. Rev. Lett. **121** (2018) 112002 [arXiv:1805.01471 [hep-ph]].

PoS(CD2018)052

[29] M. Hoferichter, B. L. Hoid, B. Kubis, S. Leupold and S. P. Schneider, JHEP **1810** (2018) 141 [arXiv:1808.04823 [hep-ph]].

[30] M. Knecht and A. Nyffeler, Phys. Rev. D **65** (2002) 073034 [hep-ph/0111058].

[31] C. Patrignani *et al.* [Particle Data Group Collaboration], Chin. Phys. C **40** (2016) 100001.

[32] S. L. Adler, Phys. Rev. **177** (1969) 2426.

[33] J. S. Bell and R. Jackiw, Nuovo Cim. A **60** (1969) 47.

[34] W. A. Bardeen, Phys. Rev. **184** (1969) 1848.

[35] I. Larin *et al.* [PrimEx Collaboration], Phys. Rev. Lett. **106** (2011) 162303 [arXiv:1009.1681 [nucl-ex]].

[36] A. H. Gasparian, PoS CD **15** (2016) 048.

[37] H. J. Behrend *et al.* [CELLO Collaboration], Z. Phys. C **49** (1991) 401.

[38] J. Gronberg *et al.* [CLEO Collaboration], Phys. Rev. D **57** (1998) 33 [hep-ex/9707031].

[39] B. Aubert *et al.* [BaBar Collaboration], Phys. Rev. D **80** (2009) 052002 [arXiv:0905.4778 [hep-ex]].

[40] S. Uehara *et al.* [Belle Collaboration], Phys. Rev. D **86** (2012) 092007 [arXiv:1205.3249 [hep-ex]].

[41] A. Gérardin, H. B. Meyer and A. Nyffeler, Phys. Rev. D **94** (2016) 074507 [arXiv:1607.08174 [hep-lat]].

[42] S. S. Agaev, V. M. Braun, N. Offen and F. A. Porkert, Phys. Rev. D **83** (2011) 054020 [arXiv:1012.4671 [hep-ph]].

[43] S. V. Mikhailov, A. V. Pimikov and N. G. Stefanis, Phys. Rev. D **93** (2016) 114018 [arXiv:1604.06391 [hep-ph]].

[44] Y. L. Shen, J. Gao, C. D. Lü and Y. Miao, arXiv:1901.10259 [hep-ph].

[45] K. Raya, L. Chang, A. Bashir, J. J. Cobos-Martinez, L. X. Gutiérrez-Guerrero, C. D. Roberts and P. C. Tandy, Phys. Rev. D **93** (2016) 074017 [arXiv:1510.02799 [nucl-th]].

[46] G. Eichmann, C. Fischer, E. Weil and R. Williams, Phys. Lett. B **774** (2017) 425 [arXiv:1704.05774 [hep-ph]].

[47] E. Ruiz Arriola and W. Broniowski, Phys. Rev. D **74** (2006) 034008 [hep-ph/0605318].

[48] E. Ruiz Arriola and W. Broniowski, Phys. Rev. D **81** (2010) 094021 [arXiv:1004.0837 [hep-ph]].

[49] M. Gorchtein, P. Guo and A. P. Szczepaniak, Phys. Rev. C **86** (2012) 015205 [arXiv:1102.5558 [nucl-th]].

[50] P. Masjuan, Phys. Rev. D **86** (2012) 094021 [arXiv:1206.2549 [hep-ph]].

[51] P. Roig, A. Guevara and G. López Castro, Phys. Rev. D **89** (2014) 073016 [arXiv:1401.4099 [hep-ph]].

[52] R. Omnès, Nuovo Cim. **8** (1958) 316.

[53] M. Fujikawa *et al.* [Belle Collaboration], Phys. Rev. D **78** (2008) 072006 [arXiv:0805.3773 [hep-ex]].

[54] I. Caprini, G. Colangelo and H. Leutwyler, Eur. Phys. J. C **72** (2012) 1860 [arXiv:1111.7160 [hep-ph]].

[55] R. García-Martín, R. Kamiński, J. R. Peláez, J. Ruiz de Elvira and F. J. Ynduráin, Phys. Rev. D **83** (2011) 074004 [arXiv:1102.2183 [hep-ph]].

[56] J. Wess and B. Zumino, Phys. Lett. **37B** (1971) 95.

[57] S. L. Adler, B. W. Lee, S. B. Treiman and A. Zee, Phys. Rev. D **4** (1971) 3497.

[58] M. V. Terent'ev, Phys. Lett. **38B** (1972) 419.

[59] R. Aviv and A. Zee, Phys. Rev. D **5** (1972) 2372.

[60] E. Witten, Nucl. Phys. B **223** (1983) 422.

[61] M. N. Achasov *et al.*, Phys. Rev. D **66** (2002) 032001 [hep-ex/0201040].

[62] M. N. Achasov *et al.*, Phys. Rev. D **68** (2003) 052006 [hep-ex/0305049].

[63] B. Aubert *et al.* [BaBar Collaboration], Phys. Rev. D **70** (2004) 072004 [hep-ex/0408078].

[64] G. P. Lepage and S. J. Brodsky, Phys. Lett. B **87** (1979) 359.

[65] G. P. Lepage and S. J. Brodsky, Phys. Rev. D **22** (1980) 2157.

[66] S. J. Brodsky and G. P. Lepage, Phys. Rev. D **24** (1981) 1808.

[67] V. A. Nesterenko and A. V. Radyushkin, Sov. J. Nucl. Phys. **38** (1983) 284 [Yad. Fiz. **38** (1983) 476].

[68] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, M. B. Voloshin and V. I. Zakharov, Nucl. Phys. B **237** (1984) 525.

[69] A. Khodjamirian, Eur. Phys. J. C **6** (1999) 477 [hep-ph/9712451].

[70] A. Nyffeler, Phys. Rev. D **94** (2016) 053006 [arXiv:1602.03398 [hep-ph]].

[71] P. Masjuan and P. Sanchez-Puertas, Phys. Rev. D **95** (2017) 054026 [arXiv:1701.05829 [hep-ph]].

[72] A. Guevara, P. Roig and J. J. Sanz-Cillero, JHEP **1806** (2018) 160 [arXiv:1803.08099 [hep-ph]].

[73] C. F. Redmer [BESIII Collaboration], EPJ Web Conf. **166** (2018) 00017.

[74] E. Kou *et al.*, arXiv:1808.10567 [hep-ex].

[75] F. Stollenwerk, C. Hanhart, A. Kupść, U.-G. Meißner and A. Wirzba, Phys. Lett. B **707** (2012) 184 [arXiv:1108.2419 [nucl-th]].

[76] C. Hanhart, A. Kupść, U.-G. Meißner, F. Stollenwerk and A. Wirzba, Eur. Phys. J. C **73** (2013) 2668 [Erratum: Eur. Phys. J. C **75** (2015) 242] [arXiv:1307.5654 [hep-ph]].

[77] B. Kubis and J. Plenter, Eur. Phys. J. C **75** (2015) 283 [arXiv:1504.02588 [hep-ph]].

[78] C. W. Xiao, T. Dato, C. Hanhart, B. Kubis, U.-G. Meißner and A. Wirzba, arXiv:1509.02194 [hep-ph].