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Low-energy limit of the $O(4)$ quark-meson model

Jürgen Eser* **and Florian Divotgey**
Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität,
Max-von-Laue-Straße 1, D-60438 Frankfurt am Main, Germany
E-mail: eser@th.physik.uni-frankfurt.de,
fdivotgey@th.physik.uni-frankfurt.de

Mario Mitter
Department of Physics, Brookhaven National Laboratory,
Upton, NY 11973, USA
E-mail: mitter@bnl.gov

We study the generation of low-energy couplings induced by quantum fluctuations within the $O(4)$-symmetric quark-meson model. To this end, we compute the functional renormalization group flow of the linearly realized quark-meson model including higher-derivative interactions and subsequently transform the resulting effective action into a nonlinear effective pion action. The latter is referred to as the low-energy limit of the $O(4)$ quark-meson model. The present study may be considered as a preparatory work for the dynamical generation of low-energy couplings from functional QCD fluctuations in order to determine meaningful renormalization scales for purely pionic models.

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*Speaker.

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1. Introduction

The predominant feature of the strong interaction at low energies is the chiral $SU(N_f) \times SU(N_f)$ symmetry, where N_f represents the number of dynamical quark flavors. From the spontaneous breakdown of this symmetry arise $N_f^2 - 1$ (pseudo) Nambu-Goldstone bosons (pNGBs), which, in the case of $N_f = 2$, are identified with the three pion fields. The chiral symmetry and its breaking is therefore the main guiding principle for the construction of effective field theories (EFTs) describing the hadronic low-energy regime of the strong interaction.

The most prominent EFT is given by chiral perturbation theory (ChPT) [1, 2], which is based on a nonlinear realization of chiral symmetry. It relies on a simultaneous expansion of the generating functional of QCD in terms of pion momenta and quark masses. The associated Lagrangian of ChPT involves (momentum-dependent) pion self-interactions parametrized by low-energy constants.

So-called linear sigma models (LSMs) constitute another interesting family of EFTs for the strong interaction. In contrast to ChPT, they are constructed from a linear realization of chiral symmetry, treating the pNGB fields and their chiral partners on equal footing. These LSMs also have a longstanding tradition and are frequently used, e.g., for nonzero-temperature and nonzero-chemical potential applications. The $O(4)$ LSM, emerging from the relation $SU(2) \times SU(2) \sim O(4)$, is one of the simplest examples in this family, which considers the three pion fields $\vec{\pi}$ and their chiral partner, the σ meson. Taking also constituent quarks into account, it is extended to the $O(4)$ quark-meson model (QMM), the primary subject of investigations in the following.

The presented work is motivated by the question whether quantum fluctuations of the (hadronic) degrees of freedom within the $O(4)$ QMM and related EFTs, such as extended LSMs [3, 4], are able to reproduce the low-energy couplings of QCD? To address this question, we compute fluctuation-induced higher-derivative interactions in the linearly realized QMM and transform the infrared effective action into a nonlinearly realized effective pion action by restricting the dynamics to the respective vacuum manifold.

In a first (exploratory) approach [5], we studied the scaling behavior of higher-derivative pion self-interactions within the $O(4)$ QMM using the functional renormalization group (FRG) formalism. This investigation was initialized by the tree-level computation of the low-energy couplings in an extended LSM [6]. The subsequent study in Ref. [7] elaborated on this first analysis by including momentum-dependent $\sigma\pi$ as well as σ self-interactions into the FRG flow. This second work also allows for a first conclusion regarding appropriate renormalization scales for purely pionic EFTs as obtained from the FRG. Here, we briefly review latest results of these studies.

2. Methods

In Secs. 2.1 and 2.2, we introduce the $O(4)$ QMM and the related truncation within the FRG formalism including higher-derivative interactions. The next section 2.3 shows the corresponding low-energy limit, i.e., the nonlinearly realized effective pion action derived from the linear QMM. This action contains (momentum-dependent) pion self-interactions, which are parametrized by the low-energy couplings of the QMM.

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2.1 Quark-meson model: Lagrangian

The $O(4)$ QMM is constructed from the four-dimensional vector

$$\varphi = \begin{pmatrix} \vec{\pi} \\ \sigma \end{pmatrix}, \tag{2.1}$$

which contains the three pion fields $\vec{\pi}$ and the isoscalar σ field. The Lagrangian of the QMM (on Minkowski spacetime) reads

$$\mathscr{L}_{\mathrm{QMM}} = \frac{1}{2}\left(\partial_\mu \varphi\right) \cdot \partial^\mu \varphi - \frac{m_0^2}{2} \varphi \cdot \varphi - \frac{\lambda}{4}\left(\varphi \cdot \varphi\right)^2 + h_{\mathrm{ESB}} \sigma + \bar{\psi}\left(i\gamma^\mu \partial_\mu - y\Phi_5\right)\psi, \tag{2.2}$$

with

$$\Phi_5 = \sigma t_0 + i\gamma_5 \vec{\pi} \cdot \vec{t}. \tag{2.3}$$

In Eq. (2.2), we introduced the Yukawa coupling y of the mesonic fields to the quarks ($\bar{\psi}$, ψ) as well as the parameter h_{ESB}, which leads to an explicit breaking of the $O(4)$ invariance of the model (besides the spontaneous breaking). The generators in Eq. (2.3) are defined as $t_0 = \mathbb{1}/2$ and $\vec{t} = \vec{\tau}/2$, where $\vec{\tau}$ are the three Pauli matrices.

2.2 Quark-meson model: Truncation with higher-derivative interactions

The FRG implements the integration over quantum fluctuations in terms of momentum shells. It works with the scale-dependent effective average action Γ_k, where the subscript k denotes the flowing infrared (IR) cutoff in momentum space. The action Γ_k interpolates between the renormalized classical action at a specific ultraviolet (UV) cutoff Λ, $k \to \Lambda$, and the quantum effective action Γ in the IR, $k \to 0$.

The scaling behavior of the effective average action Γ_k is determined by the Wetterich flow equation [8]

$$\partial_k \Gamma_k = \frac{1}{2} \operatorname{tr}\left[\partial_k R_k \left(\Gamma_k^{(2)} + R_k\right)^{-1}\right]. \tag{2.4}$$

The function R_k in the above equation denotes the IR regulator and $\Gamma_k^{(2)}$ is the scale-dependent two-point function, i.e., the second functional derivative of Γ_k with respect to the (classical) fields.

We solve Eq. (2.4) by choosing a truncation for the Euclidean QMM, cf. the Lagrangian (2.2) given in the previous section, which involves complete sets of higher-derivative couplings of order $\mathscr{O}(\varphi^4, \partial^2)$ and $\mathscr{O}(\varphi^4, \partial^4)$,

$$\begin{aligned} \Gamma_k = \int_x \Big\{ & \frac{Z_k}{2}\left(\partial_\mu \varphi\right) \cdot \partial_\mu \varphi + U_k(\rho) - h_{\mathrm{ESB}} \sigma + \bar{\psi}\left(Z_k^\psi \gamma_\mu \partial_\mu + y_k \Phi_5\right)\psi \\ & + C_{2,k}\left(\varphi \cdot \partial_\mu \varphi\right)^2 + Z_{2,k}\, \varphi^2 \left(\partial_\mu \varphi\right) \cdot \partial_\mu \varphi \\ & - C_{3,k}\left[\left(\partial_\mu \varphi\right) \cdot \partial_\mu \varphi\right]^2 - C_{4,k}\left[\left(\partial_\mu \varphi\right) \cdot \partial_\nu \varphi\right]^2 - C_{5,k}\, \varphi \cdot \left(\partial_\mu \partial_\mu \varphi\right)\left(\partial_\nu \varphi\right) \cdot \partial_\nu \varphi \\ & - C_{6,k}\, \varphi^2 \left(\partial_\mu \partial_\nu \varphi\right) \cdot \partial_\mu \partial_\nu \varphi - C_{7,k}\left(\varphi \cdot \partial_\mu \partial_\mu \varphi\right)^2 - C_{8,k}\, \varphi^2 \left(\partial_\mu \partial_\mu \varphi\right)^2 \Big\}. \end{aligned} \tag{2.5}$$

This truncation was developed in our first work [5] on this topic. Apart from the derivative couplings $C_{2,k}$ and $Z_{2,k}$ as well as $C_{i,k}$, $i = 3,\ldots,8$, this ansatz also includes an $O(4)$-symmetric scale-dependent effective potential $U_k(\rho)$, where $\rho = \varphi\cdot\varphi = \vec{\pi}^2+\sigma^2$, and wave-function renormalization factors for the bosonic and fermionic fields, Z_k and Z_k^ψ, respectively. Furthermore, a scale dependence of the Yukawa coupling y_k is taken into account.

The flow equations for the scale-dependent constituents of truncation (2.5) are obtained from Eq. (2.4) and its functional derivatives, see Refs. [5] and [7] for details.

2.3 Low-energy limit of the quark-meson model: Effective pion action

Restricting the dynamics of the original four-dimensional Euclidean field space (spanned by the fields $\vec{\pi}$ and σ) to the vacuum manifold associated with the spontaneous symmetry breaking, i.e., the three-sphere S^3, yields the low-energy limit of the $O(4)$ QMM (the quark fields are dropped as they do not influence vacuum-to-vacuum amplitudes computed from the effective action at tree level; the fermionic quantum fluctuations are already integrated into the higher-derivative couplings).

The transition to the nonlinear low-energy limit is formally denoted as

$$\Gamma_k\left[\tilde{\sigma},\tilde{\vec{\pi}},\tilde{\bar{\psi}},\tilde{\psi}\right]\to\Gamma_k\left[\tilde{\Pi}\right], \tag{2.6}$$

where $\tilde{\cdot}$ are the respective renormalized fields,

$$\tilde{\sigma}=\sqrt{Z_k^\pi}\sigma,\qquad \tilde{\vec{\pi}}=\sqrt{Z_k^\pi}\vec{\pi},\qquad \tilde{\bar{\psi}}=\sqrt{Z_k^\psi}\bar{\psi},\qquad \tilde{\psi}=\sqrt{Z_k^\psi}\psi, \tag{2.7}$$

with Z_k^π as the effective pion wave-function renormalization in the two-point function,

$$Z_k^\pi = Z_k+2\sigma^2 Z_{2,k}-2\sigma^2p^2\left(C_{6,k}+C_{8,k}\right). \tag{2.8}$$

This factor Z_k^π obviously differs from Z_k as soon as the σ field (and the external momentum p) acquires a nonzero value. The corresponding renormalized higher-derivative couplings in truncation (2.5) read

$$\tilde{C}_{i,k}=\frac{C_{i,k}}{(Z_k^\pi)^2},\quad i=1,\ldots,8,\qquad \tilde{Z}_{2,k}=\frac{Z_{2,k}}{(Z_k^\pi)^2}, \tag{2.9}$$

where $C_{1,k}$ is identified with the momentum-independent quartic interaction in the effective potential U_k. In Eq. (2.6), the fields $\tilde{\Pi}^a$, $a = 1,2,3$, represent the properly renormalized nonlinear pNGBs and we kept the symbol Γ_k for the resulting effective pion action.

After choosing stereographic coordinates on the three-sphere, the effective pion action for the $O(4)$ QMM turns out to be (see again Ref. [7] for technical details)

$$\begin{aligned}\Gamma_k=\int_x\bigg\{&\frac{1}{2}\left(\partial_\mu\tilde{\Pi}_a\right)\partial_\mu\tilde{\Pi}^a+\frac{1}{2}\tilde{\mathscr{M}}^2_{\Pi,k}\tilde{\Pi}_a\tilde{\Pi}^a-\tilde{\mathscr{C}}_{1,k}\left(\tilde{\Pi}_a\tilde{\Pi}^a\right)^2+\tilde{\mathscr{Z}}_{2,k}\tilde{\Pi}_a\tilde{\Pi}^a\left(\partial_\mu\tilde{\Pi}_b\right)\partial_\mu\tilde{\Pi}^b\\&-\tilde{\mathscr{C}}_{3,k}\left[\left(\partial_\mu\tilde{\Pi}_a\right)\partial_\mu\tilde{\Pi}^a\right]^2-\tilde{\mathscr{C}}_{4,k}\left[\left(\partial_\mu\tilde{\Pi}_a\right)\partial_\nu\tilde{\Pi}^a\right]^2-\tilde{\mathscr{C}}_{5,k}\tilde{\Pi}_a\left(\partial_\mu\partial_\mu\tilde{\Pi}^a\right)\left(\partial_\nu\tilde{\Pi}_b\right)\partial_\nu\tilde{\Pi}^b\\&-\tilde{\mathscr{C}}_{6,k}\tilde{\Pi}_a\tilde{\Pi}^a\left(\partial_\mu\partial_\nu\tilde{\Pi}_b\right)\partial_\mu\partial_\nu\tilde{\Pi}^b-\tilde{\mathscr{C}}_{8,k}\tilde{\Pi}_a\tilde{\Pi}^a\left(\partial_\mu\partial_\mu\tilde{\Pi}_b\right)\partial_\nu\partial_\nu\tilde{\Pi}^b\bigg\}.\end{aligned} \tag{2.10}$$

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The low-energy couplings in the nonlinear model (2.10) are functions of the corresponding couplings in the linear QMM,

$$\tilde{\mathscr{C}}_{1,k} = \frac{\tilde{\mathscr{M}}^2_{\Pi,k}}{8f_\pi^2}, \qquad \tilde{\mathscr{Z}}_{2,k} = -\frac{1}{4f_\pi^2},$$
$$\tilde{\mathscr{C}}_{3,k} = \tilde{C}_{3,k} - \tilde{C}_{5,k} + \tilde{C}_{7,k} + 2\big(\tilde{C}_{6,k} + \tilde{C}_{8,k}\big), \qquad \tilde{\mathscr{C}}_{4,k} = \tilde{C}_{4,k},$$
$$\tilde{\mathscr{C}}_{5,k} = 2\big(\tilde{C}_{6,k} + \tilde{C}_{8,k}\big), \qquad \tilde{\mathscr{C}}_{6,k} = -\tilde{C}_{6,k} - \tilde{C}_{8,k}, \qquad \tilde{\mathscr{C}}_{8,k} = \frac{1}{2}\big(\tilde{C}_{6,k} + \tilde{C}_{8,k}\big). \tag{2.11}$$

To be more precise, the couplings $\tilde{\mathscr{C}}_{1,k}$ and $\tilde{\mathscr{Z}}_{2,k}$ are geometrical constants, which means that they are exclusive functions of the pion decay constant f_π (the radius of the three-sphere) and the (squared) pion mass $\tilde{\mathscr{M}}^2_{\Pi,k}$,

$$\tilde{\mathscr{M}}^2_{\Pi,k} = \frac{\tilde{h}_{\mathrm{ESB}}}{f_\pi} = \frac{h_{\mathrm{ESB}}}{\sqrt{Z^\pi_k}}\frac{1}{f_\pi}. \tag{2.12}$$

The couplings $\tilde{\mathscr{C}}_{5,k}$, $\tilde{\mathscr{C}}_{6,k}$, and $\tilde{\mathscr{C}}_{8,k}$ in the nonlinear model appear to be linearly dependent. The couplings $\tilde{\mathscr{C}}_{2,k}$ and $\tilde{\mathscr{C}}_{7,k}$ vanish.

3. Numerical results

The numerical values of the low-energy couplings of the effective pion action (2.10) are computed by solving the FRG flow of the $O(4)$ QMM according to truncation (2.5) and subsequently using the relations (2.11) to transform the result into the nonlinear model.

The flow of the QMM is initialized at a hadronic cutoff scale of $\Lambda = 500$ MeV. At this scale, the higher-derivative couplings $C_{2,k}$ and $Z_{2,k}$ as well as $C_{i,k}$, $i = 3,\ldots,8$, are set to zero [Ref. [5] confirmed that this is a reasonable assumption for the derivative couplings of order $\mathscr{O}(\partial^4)$]. Hence, these interactions are solely dynamically generated during the integration process. All other concrete starting values, e.g., for the effective potential and the Yukawa interaction, can be found

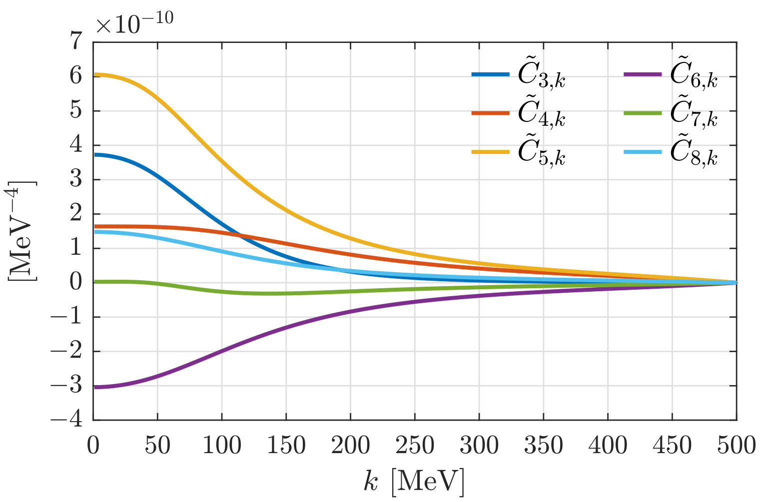

Figure 1: Higher-derivative couplings. Scale evolution of the renormalized couplings $\tilde{C}_{i,k}$, $i = 3,\ldots,8$, of the $O(4)$ QMM (2.5), cf. Ref. [7]; $k_{\mathrm{IR}} = 1$ MeV.

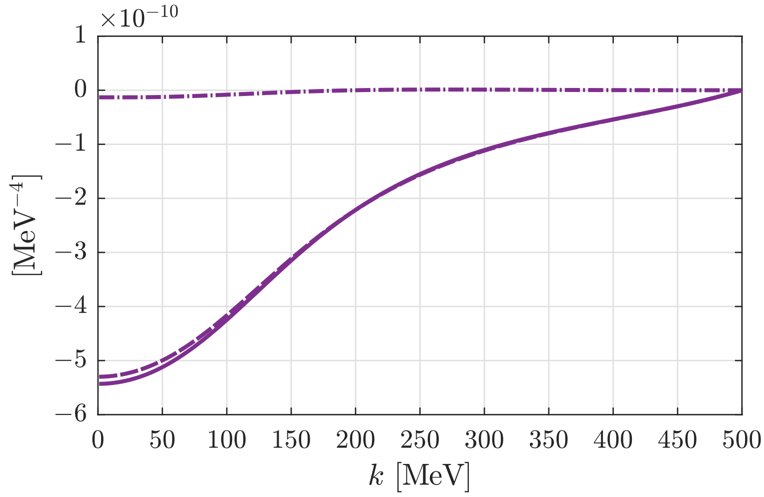

Figure 2: Low-energy coupling $\tilde{\mathscr{C}}_{3,k}$. Scale evolution of the renormalized low-energy coupling $\tilde{\mathscr{C}}_{3,k}$, which is qualitatively representative for the derivative couplings in the effective pion action (2.10), cf. Ref. [7]. The total evolution (solid line) is decomposed into bosonic (dash-dotted line) and fermionic (dashed line) loop contributions; $k_{\mathrm{IR}} = 1$ MeV.

in Ref. [7]. These values are tuned such that, in the IR, the pions, the σ meson, and the quarks have a mass of 138.5 MeV, 509.4 MeV, and 296.8 MeV, respectively. The pion decay constant f_π amounts to 93.8 MeV in the IR limit.

Figure 1 shows the generation of the derivative couplings of order $\mathscr{O}(\partial^4)$ during the flow of the linear QMM [we do not show the derivative couplings $\tilde{C}_{2,k}$ and $\tilde{Z}_{2,k}$, which are irrelevant for the low-energy couplings, cf. relations (2.11)]. The IR values of the higher-derivative couplings are listed in the first part of Table 1 ("Linear model"). They differ from the ones quoted in our first calculation [5] due to the influence of the momentum-dependent $\sigma\pi$ and σ self-interactions on the FRG flow.

It is clear that the transition to the nonlinear effective pion action is only meaningful at energy-momentum scales at which the non-pNGB degrees of freedom have decoupled from the flow, i.e.,

Linear model		**Nonlinear model**	
$\tilde{C}_2\ [1/f_\pi^2] \times 10$	-0.88	...	
$\tilde{Z}_2\ [1/f_\pi^2] \times 10$	-2.30	$\tilde{\mathscr{Z}}_2\ [1/f_\pi^2] \times 10$	-2.50
$\tilde{C}_3\ [1/f_\pi^4] \times 10^2$	2.88	$\tilde{\mathscr{C}}_3\ [1/f_\pi^4] \times 10^2$	-4.20
$\tilde{C}_4\ [1/f_\pi^4] \times 10^2$	1.27	$\tilde{\mathscr{C}}_4\ [1/f_\pi^4] \times 10^2$	1.27
$\tilde{C}_5\ [1/f_\pi^4] \times 10^2$	4.69	$\tilde{\mathscr{C}}_5\ [1/f_\pi^4] \times 10^2$	-2.41
$\tilde{C}_6\ [1/f_\pi^4] \times 10^2$	-2.35	$\tilde{\mathscr{C}}_6\ [1/f_\pi^4] \times 10^2$	1.21
$\tilde{C}_7\ [1/f_\pi^4] \times 10^2$	0.02	...	
$\tilde{C}_8\ [1/f_\pi^4] \times 10^2$	1.14	$\tilde{\mathscr{C}}_8\ [1/f_\pi^4] \times 10^2$	-0.60

Table 1: Higher-derivative couplings and the related low-energy (derivative) couplings. Table from Ref. [7].

they have become too massive. Nevertheless, we can use Eq. (2.11) to determine the low-energy couplings of the nonlinear model (2.10) at all scales.

Thus, as an example for the low-energy couplings $\tilde{\mathscr{C}}_{i,k}$, $i \in \{3,4,5,6,8\}$, all of which exhibit the same qualitative behavior [7], Fig. 2 presents the scale evolution of the coupling $\tilde{\mathscr{C}}_{3,k}$. Additionally, the evolution of this coupling is decomposed into bosonic and fermionic loop contributions. Apparently, the latter dominate the FRG flow and the bosonic loop contributions are only important at scales roughly below 150 MeV.

Finally, the IR-limit values of the low-energy couplings of the action (2.10) are also collected in Table 1; see the second part (named "Nonlinear model"). The vanishing couplings in the nonlinear case are indicated by three dots.

4. Summary and outlook

We have computed the low-energy limit of the $O(4)$ QMM within the FRG framework. This low-energy limit is given by a nonlinearly realized effective pion action, whose interaction terms are parametrized by low-energy couplings. These couplings were determined from the flow of the corresponding higher-derivative interactions in the linear QMM.

We found that the FRG flow is drastically dominated by the fermionic fluctuations [7], even for relatively small scales below 100 MeV, cf. Fig. 2. Moreover, the pNGB fluctuations are only relevant at scales roughly below 150 MeV; a result, which is consistent with recent functional QCD publications, see, e.g., Refs. [9, 10, 11, 12, 13]. These findings suggest renormalization scales of around 50 – 100 MeV for the effective pion action as obtained from the FRG.

The present studies are understood as a preparatory work for the derivation of low-energy couplings from functional QCD, i.e., from quark-gluon fluctuations initialized at scales of order 10 GeV. Such an analysis will follow in the near future.

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