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A recent dispersive study of $\pi\pi \to K\bar{K}$ scattering data up to 1.47 GeV.

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In this talk we briefly reviewed our recent progress on a dispersive study of $\pi\pi \to K\bar{K}$ scattering data. First, we optimized the applicability of hyperbolic dispersion relations to this process and we have then showed the existence of sizable inconsistencies on the existing data sets. However, by imposing these dispersion relations as constraints on data fits, we are able to provide a simple set of parameterizations for the S, P and D partial waves up to 2 GeV that describe the data while being consistent with dispersion relations up to their maximum applicability limit of 1.47 GeV.

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The motivation to study the interactions of pions and kaons is due to the fact that, being the lightest mesons, stable under strong interactions, they appear as final products of almost all hadronic processes. In addition, they are the Goldstone bosons of the spontaneous chiral symmetry breaking of Quantum Chromodynamics (QCD) and a precise description of their interactions provides a test of that symmetry breaking pattern. Moreover most of the other light meson resonances appear in the scattering of pions and kaons, particularly the most controversial ones like the $\sigma/f_0(500)$, $\kappa/K_0^*(700)$ and the rest of light scalar mesons. However, data on meson-meson scattering is obtained indirectly from meson-nucleon to meson-meson nucleon experiments performed in the 70's and 80's. The extraction of these indirect data often involves the use of models, approximations and extrapolations that produce large systematic uncertainties leading to the existence of different and often conflicting data sets. Nevertheless, a precise and consistent data amplitude description can be obtained by means of dispersion relations. These are integral relations, which are a consequence of causality and crossing symmetry, that provide rigorous and model independent constraints on amplitudes.

In this contribution we will briefly review our recent results [1] on a dispersive analysis of $\pi\pi \to K\bar{K}$ scattering data. This is relevant by itself, as it yields an important contribution to rescattering effects in many hadronic processes, but also because, being the crossed channel of πK scattering, it becomes a relevant input in the most rigorous dispersive determinations of πK scattering amplitudes [2, 3] and the controversial $\kappa/K_0^*(700)$ resonance. Dispersive studies of $\pi\pi \to K\bar{K}$ and its relation to $\pi K \to \pi K$ scattering were first performed in the seventies [4, 5, 6, 7]. It was soon realized that combining fixed-t and hyperbolic dispersion relations (HDR) for partial waves [8] was best suited to study the physical regions of both channels simultaneously [5, 7]. Unfortunately, data was scarce and these analyses only allowed for crude checks of low-energy scalar partial waves, mostly for threshold parameters and the non-physical region between the two-pion and the two-kaon thresholds. Better experimental results were obtained in the early eighties [9, 10]. Dispersive analysis were carried out for threshold parameters and the unphysical region [11], but no full dispersive analysis was carried out in the physical region until now, particularly due to the low applicability range of the simplest $su = b$ hyperbolae used in those works, which we have extended to the general $(s-a)(u-a) = b$ case for $\pi\pi \to K\bar{K}$.

Our work follows closely the same approach used in a series of works done by the Madrid-Krakow group for the analysis of $\pi\pi$ scattering data with Forward Dispersion Relations (FDR) [12, 13] or with Roy equations. The latter, derived in [14], are partial-wave dispersion relations for $\pi\pi$ scattering, recently applied to analyze data in [15, 16, 17]. The same approach was recently followed by the present two authors for a FDR analysis of πK scattering data [18] that will provide the necessary input for the crossed channel here. Alternatively one can also find solutions of Roy equations in the elastic region of S and P waves given data on other waves and higher energies [19, 20]. The results of both methods are quite consistent.

Thus, following the Madrid-Krakow approach we first obtain fits to scattering data without any dispersive constraint providing realistic estimates of systematic uncertainties. These are called Unconstrained Fits to Data (UFD) and we try to use particularly simple parameterizations so that they can be easily used later both by the theoretical and experimental communities. The parameterizations are written in terms of phases, modules and inelasticities up to a maximum energy from where we use Regge and Veneziano parameterizations of high energy data when they exist, or use factorization to describe channels where data does not exist. In a second step we check the consistency within uncertainties of these data parameterizations. In general we find that the agreement between the input and the dispersive output is not satisfactory and that sizable inconsistencies appear in certain partial waves or energy regions. We then use the dispersion relations as constraints for the fits, by minimizing simultaneously the χ^2 of the fits and the distance between the input and output in the integral equations. The result are sets of Constrained Fits to Data (CFD) that satisfy the dispersion relations within uncertainties while still describing the data. Actually, the deviations from the data are usually within errors or a couple of standard deviations in some regions. These CFD provide a rather simple but consistent parameterizations of meson-meson amplitudes.

Let us now comment briefly on the dispersion relations we use for $\pi\pi \to K\bar{K}$ and later on we will describe the UFD and CFD sets.

1. Partial-wave Hyperbolic Dispersion Relations

The $\pi\pi \to K\bar{K}$ scattering amplitude $G^I(t,s,u)$, where $I = 0,1$ is the isospin and s,t,u are the usual Mandelstamm variables for the $\pi K \to \pi K$ process, actually depends only on two variables due to the constraint $s+t+u = 2(m_K^2 + m_\pi^2)$. Dispersion relations are obtained from applying Cauchy Theorem using the analyticity structure of the amplitude when considered as a function of a single complex variable. Generically, amplitudes have a right cut from threshold to infinity and a left cut due to thresholds in crossed channels. Unfortunately, the simplest dispersion relations obtained by fixing t can be shown to have very limited applicability range that does not reach the physical region of $\pi\pi \to K\bar{K}$. The applicability region is larger when using Hyperbolic Dispersion Relations along the hyperbolas $(s-a)(u-a) = b$ [8, 21]. One of the relevant aspects of our recent work is that we have derived the dispersion relations for $\pi\pi \to K\bar{K}$ in the $a \neq 0$ case and selected the value of a which maximizes the applicability region for this reaction.

In contrast to the fixed-t dispersion relations, the use of crossing on hyperbolic dispersion relations couple the $\pi\pi \to K\bar{K}$ amplitudes to those of πK scattering. Actually it is convenient to define the $s \to u$ symmetric and antisymmetric πK amplitudes $F^+(s,t,u) = G^0(t,s,u)/\sqrt{6}$ and $F^-(s,t,u) = G^1(t,s,u)/2$. Then HDR couple F^+ with G^0 and F^- with G^1. For brevity we do not provide the integral expressions since they can be found in [1] explained in full detail. It is also worth noticing that in order to ensure the convergence of the integrals we have to make one subtraction to the F^+ dispersion relations, which therefore depends on a subtraction constant that basically corresponds to the a_0^+ scattering length of πK scattering.

Once these hyperbolic dispersion relations are obtained it is relevant to project them into partial waves, since these are the ones we directly fit to data. For this one has to expand the amplitudes in partial waves inside the integrals and then project the global partial wave of each particular dispersion relation. As an example we provide here the so-called Roy-Steiner dispersion relation for $\pi\pi \to K\bar{K}$ in the isospin $I = 0$ angular momentum 0 partial wave:

$$\begin{aligned} g_0^0(t) &= \frac{\sqrt{3}}{2} m_+ a_0^+ + \frac{t}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\operatorname{Im} g_0^0(t')}{t'(t'-t)} dt' \\ &+ \frac{t}{\pi}\sum_{\ell\geq 2}\int_{4m_\pi^2}^{\infty} \frac{dt'}{t'} G_{0,2\ell-2}^0(t,t') \operatorname{Im} g_{2\ell-2}^0(t') + \frac{1}{\pi}\sum_{\ell}\int_{m_+^2}^{\infty} ds' G_{0,\ell}^+(t,s') \operatorname{Im} f_\ell^+(s') \end{aligned} \tag{1.1}$$

$$\equiv \Delta_0^0(t) + \frac{t}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dt'}{t'}\frac{\operatorname{Im} g_0^0(t)}{t'-t}, \tag{1.2}$$

where $g_\ell^I(t)$ and $f_\ell^I(t)$ are the partial waves of $\pi\pi \to K\bar{K}$ and $\pi K \to \pi K$, respectively. In addition the $G_{\ell,\ell'}^{I,\pm}$ are integral kernels that contain factors from kinematics, crossing, Legendre polynomials, subtractions, etc... whose expressions can be found in [1]. Note they carry a dependence in a that we will use to maximize the applicability region. In the final step above we have defined a Δ_0^0 function that contains the whole left hand cut contribution, the subtraction terms and right-hand cuts from higher partial waves, but does not depend on g_0^0 itself.

Note that some of these integrals extend to the "unphysical" region of $\pi\pi \to K\bar{K}$, i.e. between the two-pion and the two-kaon thresholds, where data do not exist and therefore we cannot get the amplitude there from fits. However, Watson's Theorem implies that the phase of the amplitude in that region is the same as the phase shift for elastic $\pi\pi$ scattering with the same quantum numbers. Thus we write $\phi_\ell^{I_t}(t) = \delta_{\ell,\pi\pi\to\pi\pi}^{I_t}(t)$ and we take the latter from our previous dispersive study in [16]. Then, the whole amplitude in that region can be determined by using the Mushkelishvili-Omnés approach [22, 23], by defining and Omnés function

$$\Omega_0^0(t) = \exp\left(\frac{t}{\pi}\int_{4m_\pi^2}^{t_m}\frac{\phi_0^0(t')dt'}{t'(t'-t)}\right), \tag{1.3}$$

where t_M is a matching point above the $K\bar{K}$ threshold. The Omnés function can be used to remove the phase (and the cut) from the two-pion threshold up to t_m. Then one can write another dispersion relation for $F_0^0(t) = [g_0^0(t) - \Delta_0^0(t)]/\Omega_0^0(t)$ and arrive at:

$$g_0^0(t) = \Delta_0^0(t) + \frac{t\Omega_0^0(t)}{t_m - t}\left[\alpha + \frac{t}{\pi}\int_{4m_\pi^2}^{t_m} dt' \frac{(t_m - t')\Delta_0^0(t')\sin\phi_0^0(t')}{\Omega_{0,R}^0(t')t'^2(t'-t)} + \frac{t}{\pi}\int_{t_m}^{\infty} dt' \frac{(t_m - t')|g_0^0(t')|\sin\phi_0^0(t')}{\Omega_{0,R}^0(t')t'^2(t'-t)}\right].$$

This is the kind of hyperbolic dispersion relations that we use in our work. This example has one subtraction for the Mushkelishvili-Omnés method, but for the P and D waves no subtractions are needed. For a more detailed derivation for all waves we refer the reader to [1].

The applicability region of these partial-wave hyperbolic dispersion relation is limited by the requirement that the arguments of the amplitudes F and G should not enter the so-called double spectral region where the imaginary parts would become imaginary. In addition, the convergence of the partial-wave expansion requires that the variables should lie inside the so-called Lehmann ellipse [24]. In [1] we have explicitly shown that by taking $a = -10.9m_\pi^2$, we can apply these relations to $\pi\pi \to K\bar{K}$ up to $t \leq 2.19\,\text{GeV}^2$. Namely, we can study the physical region from $K\bar{K}$ threshold $\simeq 0.992\,$GeV up to $\simeq 1.47\,$GeV. In contrast, the usual HDR with $a = 0$ projected into partial waves are only valid up to $\simeq 1.3,$GeV. Thus, with our choice of a, the applicability of the dispersive approach in the physical region has been extended by 67% in terms of t.

2. Checks of Unconstrained Fits to Data

As already emphasized, we have provided a set of Unconstrained Fits to Data in terms of simple parameterizations that can be found in [1]. These include a realistic estimate of systematic uncertainties. We have performed fits to the $(I,\ell) = (0,0), (1,1)$ and $(0,2)$ partial-wave data on phases and moduli obtained indirectly from kaon-nucleon to kaon-pion-nucleon. These experiments were performed at the Brookhaven National Laboratory, whose three works [10, 25, 26] we will call Brookhaven-I, Brookhaven-II and Brookhaven-III, respectively, as well as at the Argonne National Laboratory [9]. Higher wave contributions, although negligible, have been estimated by parameterizing the $\ell = 3,4$ Breit-Wigner resonances that appear below 2 GeV. Above 2 GeV we have used Regge Veneziano parameterizations obtained from factorization of KN, $\pi\pi$, πN and NN high energy scattering data (see [27, 18, 1] for details), the contribution is typically small.

In order to quantify the consistency of the fits, we first define a "distance-square"

$$d^2 = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{d_i}{\Delta d_i}\right)^2, \tag{2.1}$$

for each dispersion relation. Here d_i is the difference between the "input" and "output" of each dispersion relation at the energy $\sqrt{t_i}$. We evaluate this distances at around 30 points over the applicability region. Note the similarity with a χ^2 function, so that $d^2 \leq 1$ signals a good fulfillment of the dispersive constraints.

The partial wave that comes out more consistent already for the UFD set is the $I,\ell = 1,1$. The UFD fits are shown in the top row of Fig.1. Note that below $K\bar{K}$ threshold we only have data for the phase from $\pi\pi$ scattering and we use its parameterization in [16]. The resulting $d^2 = 1.1$ for the UFD fit and this consistency is pretty homogeneous throughout the applicability region, except maybe very close to threshold where the input and dispersive output are separated by slightly more than one standard deviation.

The UFD fits for the isospin 0 D-wave, are shown in Figure 2. The most relevant remark is that our parameterization has the correct behavior near $K\bar{K}$ threshold dictated by Watson's Theorem, and also displays a dip due to the presence of the $f_2(1810)$ in contrast to the model used by the Brookhaven-II analysis [25]. The overall agreement $d^2 = 1.6$, shows some inconsistency coming mostly from the threshold region. Therefore there is room for improvement by using the dispersion relation as constraints.

The most interesting fits are those of the scalar-isoscalar wave, which are show in Fig.3, since there are clearly two inconsistent sets of data for the modulus. We have fit them both, calling them UFD$_B$ and UFD$_C$ (see [1] for details). In addition there are some data for the phase at low energies that we discard because they do not satisfy Watson's Theorem. We thus use the same phase parameterization for both UFD sets. Interestingly, both sets show large inconsistencies with the dispersive representation, $d^2 = 5.6$ and $d^2 = 2.7$, respectively. In this case that inconsistency is not only due to the threshold region, particularly in the UFD$_B$ case. Thus, here there is not just room, but a need for improvement. This is achieved by using the dispersion relations as constraints for our fits.

3. Constrained Fits to Data

In order to impose the dispersion relations as constraints we have minimized the following function:

$$W_1^2 d^2_{g^I_\ell} + \frac{W_2^2}{N}\sum_k^N\left(\frac{|g^I_\ell|_{exp,k} - |g^I_\ell(s_k)|}{\delta|g^I_\ell|_{exp,k}}\right)^2 + \frac{W_3^2}{N'}\sum_k^{N'}\left(\frac{(\phi^I_\ell)_{exp,k} - \phi^I_\ell(s_k)}{\delta(\phi^I_\ell)_{exp,k}}\right)^2, \tag{3.1}$$

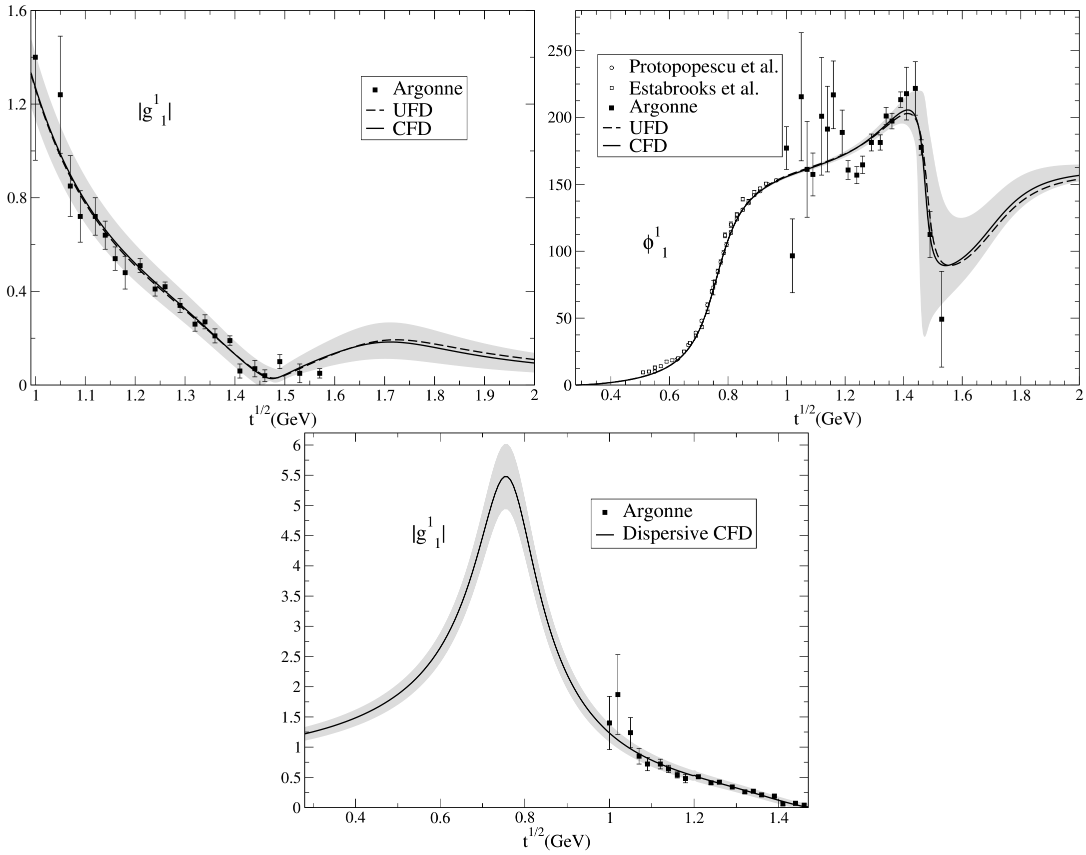

Figure 1: The continuous line and the uncertainty band corresponds to the CFD, the dashed line to the UFD. **Top:** Modulus and phase of the $g_1^1(t)$ $\pi\pi \to K\bar{K}$ partial wave. The white circles and squares come from the $\pi\pi$ scattering experiments of Protopopescu et al. [28] and Estabrooks et al.[29], respectively. **Bottom** Dispersive output for the modulus of the $g_1^1(t)$ $\pi\pi \to K\bar{K}$ partial wave obtained from the CFD set, including the unphysical region below $K\bar{K}$ threshold.

where $|g_\ell^I|_{exp,k}, (\phi_\ell^I)_{exp,k}$ is the k-th data point for the modulus and the phase, respectively, whereas $\delta|g_\ell^I|_{exp,k}, \delta(\phi_\ell^I)_{exp,k}$ stand for their corresponding errors. The weights W_i^2 roughly take into account the degrees of freedom needed to parameterize the modulus and the phase. In other words, we are minimizing the χ^2 of the data together with some penalty functions for the distance between the input and the output of the dispersion relations. The aim is to get both the χ^2 and the d^2 close or smaller than one uniformly over the dispersion relation applicability region. Hence, the resulting Constrained Fits to Data (CFD) are consistent with the dispersive representation while describing fairly well the existing data.

In the top row of Fig.1 we show the CFD g_1^1 versus the UFD. The CFD curves are barely distinguishable from the the UFD, since the latter was already very consistent with the dispersion relations. In the Bottom panel we show the CFD dispersive prediction for the unphysical region, where the $\rho(770)$ peak is clearly visible. Recall that the phase in that region is input from $\pi\pi$ scattering. As seen in the top left panel of Fig. 4 the consistency of the hyperbolic dispersion relation for this channel is remarkable throughout the applicability region, with a total $d^2 = 0.6$.

In Fig.2 we show the modulus and phase of the g_2^0 wave in the physical region compared to

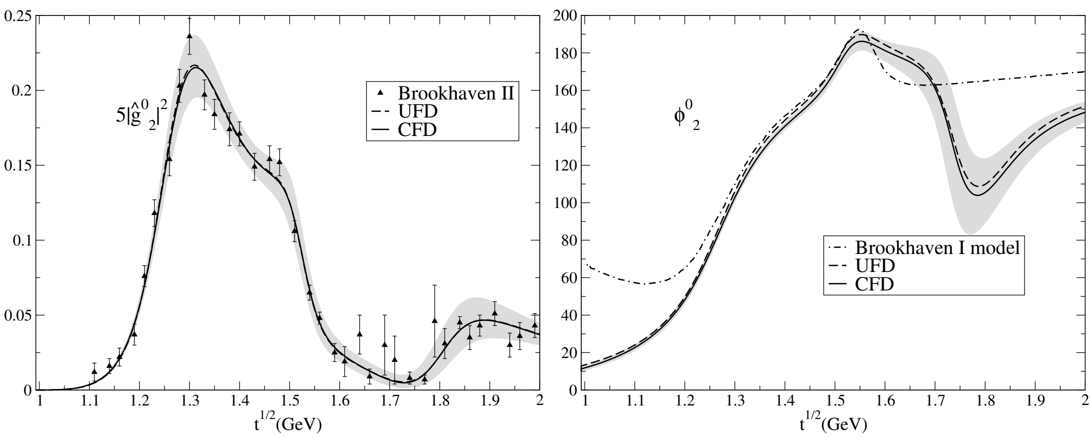

Figure 2: **Left:** Comparison between the UFD and CFD g_2^0 phases obtained with a model including a $f_2(1810)$ resonance and the one obtained with the Brookhaven model without it, using a flat background. **Right:** The continuous line is our final CFD parameterization of the data on the modulus of $\hat{g}_2^0(t)$ from the Brookhaven-II analysis [25]. The gray band stands for the uncertainty from the CFD parameters.The dashed line is the UFD parameterization. The difference between the UFD and CFD parameterization near threshold is imperceptible due to the q^5 barrier factor of the angular momentum.

the UFD. The difference between them is almost unnoticeable to the eye, except maybe on the phase. There is however, a sizable improvement on the consistency of the g_2^0 dispersion relation, as shown in the top right panel of Fig.4, which is rather good with a $d^2 = 1.1$ down from 1.6 from the UFD. Most of the improvement in d^2 comes from threshold. Unfortunately, this relevant difference between CFD and UFD at threshold is masked by the kinematic q^5 factor in Fig.4.

Finally, in Fig.3 we compare the UFD and CFD parameterizations of the g_0^0 data. Recall that there were two incompatible fits UFD$_B$ and UFD$_C$ to the modulus with a common UFD phase. Their constrained counterparts suffer some deviations from the data at the 1-σ level for the phase around 1.1 to 1.2 GeV. In addition, the CFD$_B$, which was the one whose UFD was less consistent with dispersion theory, now reaches some 2-σ deviations with respect to the UFD$_B$ in the modulus in the 1.05 to 1.5 GeV region. However, it can be noticed that once the dispersion relations are used as a constraint both the UFD$_B$ and UFD$_C$ are equally consistent, with a fairly good consistency within uncertainties, except in the threshold region. Both them have a similar $d^2 = 1.4$. We attribute the inconsistency in the threshold region (which also occurs for other waves, although to a lesser extent) mainly to the existence of isospin violation, since our formalism is isospin invariant. This effect is probably enhanced by the presence of the $f_0(980)$ resonance near threshold in the scalar-isoscalar wave. The UFD$_B$ fit is also more consistent with the favored "dip" solution of the dispersive analyses of $\pi\pi$ scattering [16]. For this reason we think the CFD$_B$ is slightly favored and should not be discarded as it usually is in the literature.

4. Summary

In this contribution we have presented a brief account of our recent work [1]. First we have reviewed the derivation of partial-wave hyperbolic dispersion relations and how to maximize their

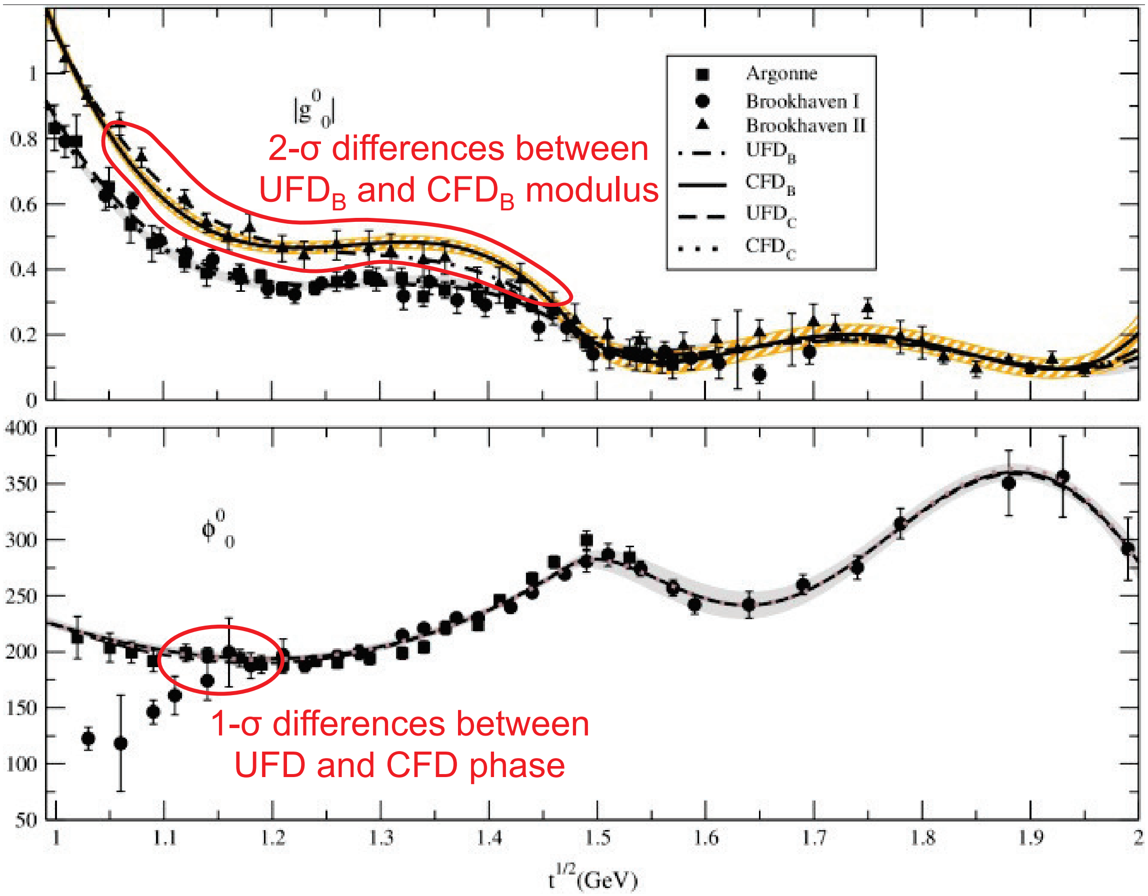

Figure 3: Comparison between the UFD and CFD parameterizations for $g_0^0(t)$. The bands cover the uncertainties of the CFD solutions. Upper panel: Modulus of the scalar-isoscalar $\pi\pi \to K\bar{K}$ scattering. The dotted line represents the CFD combined fit while the continuous line represents the CFD fit to the Brookhaven-II data only. The only significant change is in the 1.25 to 1.45 GeV between UFD$_B$ to CFD$_B$. Lower panel: Scalar-isoscalar phase for $\pi\pi \to K\bar{K}$ scattering. Note that the UFD, CFD$_B$ and CFD$_C$ phases are almost indistinguishable.

applicability region, which then extends up to 1.47 GeV. Next we have obtained fits to $\pi\pi \to K\bar{K}$ data up to 2 GeV by means of simple parameterizations including careful estimates of their systematic uncertainties. Thus, we have shown how these data show sizable inconsistencies with the constraints arising from hyperbolic dispersion relations. In a final step we have used the dispersive representation as a constraint on the data fits. Our main result is a description of data by means of simple parameterizations, consistent with dispersion relations up to 1.47 GeV, that we hope could be useful for further experimental and theoretical studies.

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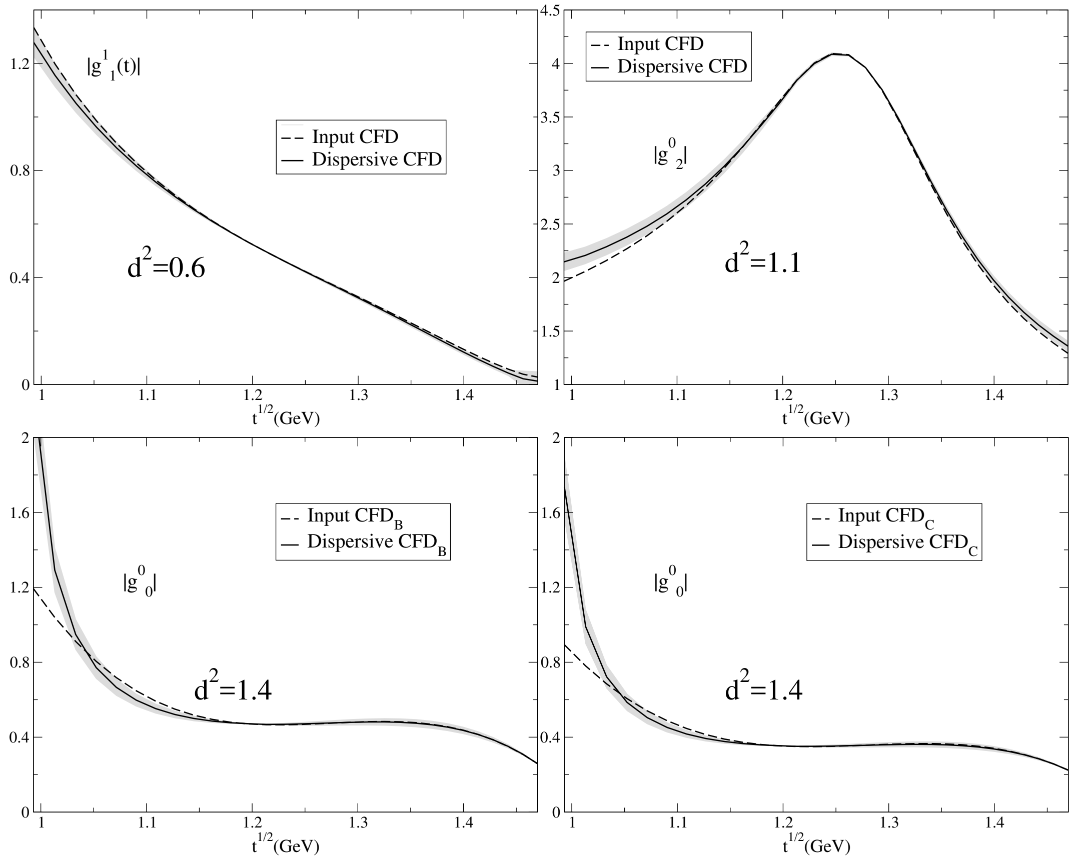

Figure 4: Comparison of the modulus and the dispersion relation after the minimization procedure. The gray band covers the uncertainty of the difference between the input and dispersive results.

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