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Degenerate metrics on a dual geometry of spherically symmetric space-time

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We consider a massive spinless particle in a particular curved space-time described by a phase space formalism. For this space-time, we determine nontrivial conserved quantities and geometrical invariants by analytical and numerical methods. By analytic calculations, we revisit (perform) a method to relate two kinds of geometries associated with conserved quantities, whose dual metrics are constructed. The method relies on the calculation of the Stackel-Killing tensors (SKT) presented in a spherically symmetric space-time. We have a system of partial differential equations (PDE) for the symmetric components of the SKT obtained for a given space-time metric. We note that the PDE system is highly non-trivial according to the isometry structure of the original metric. A degenerate metric solution appears in the dual structure, and it is compared with recent works involving space-time bridge solutions in vacuum gravity and nontrivial extensions of the Schwarzschild space-time.

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1. Introduction

General relativity has some solutions involving impulsive gravitational waves whose space-time entails theoretic models of bursts of gravitational radiation [1].

Using models from these space-times one has null hypersurfaces with a singularity of delta function type. Degenerate metrics also describe these hypersurfaces with applications in dark matter theories of gravity [2]. Degenerate metrics also appear in problems of topology and signature change space-times [3].

A problem with the degenerate metric is that there is no usual inverse since the determinant of the metric is zero. Several forms have already been considered in order to circumvent this situation, for example, the use of pseudo-inverse [4]. However, remarkable applications of the degenerate metrics recently involved an extension of the Schwarzschild metric [5] and space-time-bridge solutions [6]. In [5], an extension of the Schwarzschild metric was used in order to replace the black hole horizon and its inner region with a new structure in a classical setting. A connection between invertible and non-invertible tetrads along space-time-bridge solutions was done in [6].

Another extension of Schwarzschild space-time was considered in [7] where a new kind of Kretschmann scalar invariant was obtained, whose expression was compared recently with a dual version of a spherically symmetric space-time [8]. In the last case, was performed an analysis of duality from the geodesic perspective.

In this paper, we consider a neutral particle in a curved space-time with spherically symmetric geometry. For these space-times, the symmetries are fundamental to characterize their geodesics. With the Killing vectors, these symmetries are related with conserved quantities defined along the geodesic motion [9]. In the Hamiltonian formalism the conserved quantities consist of multiples of particle momenta contracted with symmetric Killing tensors of higher rank [10]. According to space-time considered, the task to obtain these Killing tensors is not a quick calculation. This task since involves systems of non-linear partial differential equations which are proper, implicating in an irreducible Killing tensor, or improper, which implies in a degenerate Killing tensor [8].

With these tensors, there is a new kind of symmetry known as geometric duality, related with the rank two case [11]. This kind of duality implies in an entirely non-trivial new metric associated to the space-time which has, however, the similar dynamics from the phase space Hamiltonian. In section 2, we consider the particle in curved space-time and obtain a dual metric. In the following section, the geometrical invariants and the behaviour around singularity are analyzed. The degenerate metric extension, from the dual aspect, is considered in section 4. In the final section, we conclude and indicate new outcomes.

2. Symmetry and Killing Tensors

When we consider the Hamiltonian of a neutral particle in a space-time with metric $g_{\mu\nu}$,

$$H = \frac{1}{2m} g^{\mu\nu} P_\mu P_\nu, \tag{2.1}$$

where

$$P_\mu = m g_{\mu\nu} \dot{x}^\nu \tag{2.2}$$

is the canonical momenta component, the usual dynamic is manifested in the nondegenerate metric region.

This formulation is useful to find invariants in phase-space since the standard Hamiltonian (using canonical momenta) is constructed with the appropriated inverse of metric. In an extended metric version, including its degenerate aspect, the pseudo-inverse can be used in the place of $g^{\mu\nu}$.

To describe non-evident symmetries in the model, we consider the Killing tensors associated to $g_{\mu\nu}$. The Killing tensors, for a particular rank, preserve the metric in the isometry transformation. The resulting isometry, through a geodesic analysis, reveals hidden symmetries in space-time. The Lie derivative of the metric generate the usual isometries that are useful in our problem. The first order Killing tensor components corresponds to ξ^μ and appears in the isometry condition given by

$$\mathscr{L}_\xi g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\mu\lambda} = 0, \tag{2.3}$$

where $\mathscr{L}_\xi g_{\mu\nu}$ denotes de Lie derivative of the metric with respect to a vector field. Considering a symmetric connection, the above expression can be written as

$$\overset{\circ}{\nabla}_\mu \xi_\nu + \overset{\circ}{\nabla}_\nu \xi_\mu = \overset{\circ}{\nabla}_{(\mu} \xi_{\nu)} = 0, \tag{2.4}$$

where $\overset{\circ}{\nabla}_\mu$ denotes a covariant derivative written using a symmetric connection. It also indicates that the usual isometry may be re-obtained in its general form (2.3), so that includes the possibility of a degenerate metric extension.

When we consider an isometry, we know that the usual equation for a Killing vector ξ^μ, given by (2.4), may admit a generalization to a higher rank tensor [10]. In the following, we can consider a generalization of (2.4) for objects of higher rank, i.e., for tensorial fields defined in the space-time. In this case, we have

$$\overset{\circ}{\nabla}_{(\mu} \xi_{\nu_1)\nu_2\cdots\nu_n} = 0, \tag{2.5}$$

$$\overset{\circ}{\nabla}_{(\mu} \xi_{\nu_1\nu_2\cdots\nu_n)} = 0, \tag{2.6}$$

where in (2.6) we have a definition of a symmetric Stackel-Killing tensor (SKT) of order $n > 1$ $\xi_{\mu_1\mu_2\cdots\mu_n}$, with $\xi_{[\nu_1\nu_2\cdots\nu_n]} = 0$ and $\xi_{(\nu_1\nu_2\cdots\nu_n)} = \xi_{\nu_1\cdots\nu_n}$, which represents a rank-n totally symmetric tensor. Analogously, for an antisymmetric tensor, we define in (2.5) a Killing-Yano Tensor (KYT).

An important property of the SKT object (2.6) appears when we use a standard Hamiltonian for a neutral particle in curved space $H = \frac{1}{2} p_\mu p_\nu g^{\mu\nu}(x)$, which implies in a constant of motion given by $\xi_{(2)} = \frac{1}{2}\xi^{\mu\nu} p_\mu p_\nu$, since its time evolution is given by

$$\{\xi_{(2)}, H\} = \{p_\mu p_\nu \xi^{\mu\nu}, p_\lambda p_\rho g^{\lambda\rho}\} \tag{2.7}$$

$$= \frac{2}{3} p^\alpha p^\beta p^\lambda \overset{\circ}{\nabla}_{(\lambda} \xi_{\alpha\beta)} = 0, \tag{2.8}$$

where $\xi_{\alpha\beta}$ is a rank-2 SKT. We can interpret from the equation (2.7) that we can have a Hamiltonian in a space-time with metric $\xi^{\mu\nu}$ and a Killing tensor $g^{\alpha\beta}$, or even, in the original form, a space-time with metric $g^{\mu\nu}$ and Killing tensor $\xi^{\lambda\rho}$. This dual aspect is known as geometric duality [11]. At this point, we observe that a SKT of rank-2 could be obtained from a rank-2 KYT.

Considering the mentioned approach, we will consider in the next section a particular solution of 2.8 related to geometric duality.

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3. Dual Schwarzschild space-time

Assuming the Schwarzschild metric,

$$ds^2 = (1 - \frac{2m}{r})^{-1} dr^2 + \frac{r^2}{1-u^2} du^2 + r^2(1-u^2) d\phi^2 + (-1 + \frac{2m}{r}) dt^2, \tag{3.1}$$

where $u = cos\theta$ from the usual metric coordinates. From solutions of (2.8), whose details were discussed in [8], we obtain the following dual metric:

$$\begin{aligned}\widetilde{ds}^2 \;=\;& \frac{C_2 r}{r-2m} dr^2 + \frac{(-C_2 + C_4 r^2) r^2}{(u^2-1)} du^2 + (u^2-1) r^2 ((C_5(u^2-1) + C_4) r^2 - C_2) d\phi^2 \\ &+ 2C_1 (r-2m) r (u^2-1) d\phi dt + \frac{(2m-r)(2(C_2+C_3)m - C_3 r)}{r^2} dt^2,\end{aligned} \tag{3.2}$$

where we have free parameters C_i with $i = 1...5$. If all parameters are nonzero, the metric acquires a non trivial aspect.

This aspect can be noted by the metric determinant from (3.2):

$$\begin{aligned}\widetilde{g} \;=\;& -(-2r^2 C_5 u^2 m C_2 - 2r^2 C_5 u^2 m C_3 + r^3 C_5 u^2 C_3 + 2r^2 C_5 m C_2 + 2r^2 C_5 m C_3 \\ &- r^3 C_5 C_3 - 2C_4 r^2 m C_2 - 2C_4 r^2 m C_3 + C_4 r^3 C_3 + 2mC_2^2 + 2C_2 m C_3 - C_2 C_3 r \\ &+ 2r^2 C_1^2 m u^2 - 2r^2 C_1^2 m - r^3 C_1^2 u^2 + r^3 C_1^2)(C_2 - C_4 r^2) r^3 C_2.\end{aligned} \tag{3.3}$$

The dual metric $\tilde{g}_{\mu\nu}$ is degenerate if the determinant $\tilde{g}$ is equal to zero when $C_2 = 0$. The last condition is an evident one, and it can have other possibilities to generate a degenerate metric with non-trivial solution. For example, C_2 values can give a signature changing region [3], as we can see in the Figure 1.

In the graph of Figure 1, an analysis was carried out in a range of $r = 0.1...1$ and $C_2 = 0.1...0.9$. It is indicating that C_2 implies in a signature change region, as the values of $\tilde{g}$ may be negative, positive or zero on that region.

3.1 The dual Kretschmann scalar

The Kretschmann scalar K is an invariant geometric, built from the Riemann tensor, written as $K = R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}$, and its dual given by $\tilde{K} = \tilde{R}_{\mu\nu\lambda\rho} \tilde{R}^{\mu\nu\lambda\rho}$. The Kretschmann scalar is an object that can check if a dual system can keep some singular properties of the original system [12]. It is well known that for the original Schwarzschild metric, we have for the Kretschmann scalar $K = \frac{48m^2}{r^6}$. However, the behavior of the $\tilde{K}$, for the dual metric, has a surprising behavior.

For non-degenerate metric, at least we must have the parameter $C_2 \neq 0$. In this way, we can get the Kretschmann scalar dual discussed in [8]. In this case, the Kretschmann scalar has a very large number of elements that is very difficult to exhibit. However, to get the display of K, we set the values $(C_1 = 1 = C_2, C_3 = C_4 = C_5 = 0)$, and $m = 1$:

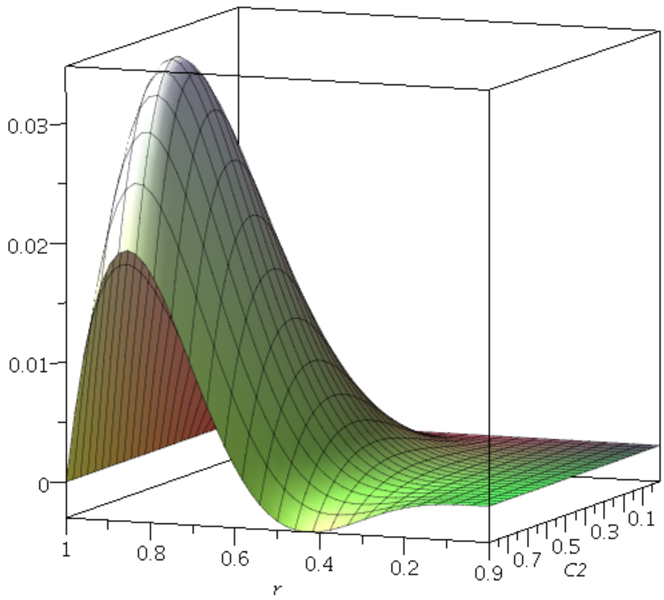

Figure 1: Metric determinant as function of r and C_2.

$$
\begin{aligned}
\tilde{K} =\ & \frac{4}{r^6(2+2u^2r^2-2r^2-r^3u^2+r^3)^4} \\
\times\ & \big(624-184r-8r^{13}u^4+68r^7u^6+840r^9u^2-960r^9u^4+r^{12}u^8-568u^2r^{10}+628r^4u^4 \\
-\ & 750r^4u^2+84r^8u^8-10r^{12}u^6+1401r^6u^4-670r^7u^4-2r^{1}4u^2-70r^{1}2u^2-16r^{11}u^8 \\
+\ & 674r^8u^4-238r^{11}u^4-96r^6u^6+r^{14}u^4+100r^{11}u^6-1536u^4r^5+678u^4r^{10}-282r^8u^6 \\
-\ & 132r^9u^8+48r^{12}u^4+1368u^2r^2+520r^9u^6+236r^{11}u^2+336r^8-474r^7-1132r^5 \\
-\ & 1351r^2-82r^{11}-8r^{13}+881r^6+169r^{10}+218r^4+31r^{12}-268r^9+73r^{10}u^8-2154r^6u^2 \\
+\ & 16r^{13}u^2+1044r^7u^2-804r^8u^2+2652u^2r^5+1376r^3-1600r^3u^2+r^{14}-352u^6r^{10}\big). \quad (3.4)
\end{aligned}
$$

The above equation contains a reduction in the number of terms, due to the simplification in the choice of parameters. This setting of values was motivated in a recent work [7], and we performed a comparision with a singularity structure of dual metrics in [8]. In a less restrictive situation, in the case where C_2 and C_3 are nonzero and arbitrary, the number of terms increases strongly so that we have to express a plethora of terms by a combination of factors such as the one given below:

$$\tilde{K}(r,u) = \frac{4}{C_2}[(2+2r^2(u^2-1)-r^3(u^2-1)^4)r^6]^{-1}\sum_{j=0}^{14} f^j(\theta)r^j g_j(C_2,C_3) \quad (3.5)$$

where $j(j=1...n)$,

$$f^{j}(\theta)=\sum_{l=0}^{4}d_{2l}^{(j)}u^{2l}, \tag{3.6}$$

and $g_j(C_2,C_3)$ depends arbitrarily of two parameters and its complete expression will depend from fixed values of C_1,C_4,C_5. However, to analyze the behavior of a degenerate metric region, we need to consider at least one arbitrary parameter $C_2 \to 0$. With a convenient choice of parameters, such as $C_1=1,-C_3=C_4=C_5=C_2$, it is possible to perform an analyses of singularity structure on the degenerate metric limit. In this way, we can present an expression of $\tilde{K}$ for expansion until $1/r^2$

$$\begin{aligned}
\widetilde{K} \;=\; & \frac{156}{C_2^2}r^{-6}-\frac{46}{C_2^2}r^{-5}+\left(-\frac{624}{C_2^4}(u^2-1)+\frac{17C_2^{12}+1368C_2^{10}u^2-1368C_2^{10}}{4C_2^{14}}\right)r^{-4}\\
+ & \left(\frac{496}{C_2^4}(u^2-1)+\frac{-1600C_2^{10}u^2+1376C_2^{10}}{4C_2^{14}}\right)r^{-3}\\
+ & \left(\frac{1560}{C_2^6}(u^2-1)^2-\frac{92}{C_2^4}(u^2-1)-\frac{17C_2^{12}+1368C_2^{10}u^2-1368C_2^{10}}{C_2^{16}}(u^2-1)\right.\\
+ & \left.\frac{506C_2^{10}u^2+628C_2^8u^4-410C_2^{10}-1256C_2^8u^2+628C_2^8}{4C_2^{14}}\right)r^{-2}+O(r^{-1}).
\end{aligned} \tag{3.7}$$

In the expression (3.7), we can see the influence of C_2 on the singularity of K. It is an indication of a degenerate metric behaviour, as $C_2 \to 0$, contributing to a physical singularity. To observe the singular aspect qualitatively, we describe a set of values with $u=0$ and display them in the Figure 2.

From the Figure 2 we observe that the immediate condition, $C_2 \to 0$, of the degenerate metric region, does not alter de usual singularity of Kretschmann scalar in the dual aspect.

This particular case does not exclude other possibilities to find other regions with degenerate metrics in the dual region. New analyses of the Kretschmann scalar dual behavior, including a broader range of C_2 and other parameters, will be the subject of future investigations.

4. Conclusion

As we can see in this brief paper, many properties associated with dual metrics, for the particular case of the Schwarzschild metric, are presented and interspersed with degenerate behavior. For the case of a simple parameter of a dual metric, the intricate structure of a geometric invariant presents a non-trivial aspect when observing the near-singular behavior of the origin of the coordinate system. Not enough of this default region, we can note that there are other points not expected, such as the location of areas with a change of signature of the metric. The curious thing about this work was the fact that such behavior appeared in a simple extension of the Schwarzschild metric. As recent references have obtained surprising results with other forms of extension of this metric, we can expect that future results explore the behavior of conserved quantities in new types of space-time. Work in this direction, to other kinds of metrics, will be presented in the future [13].

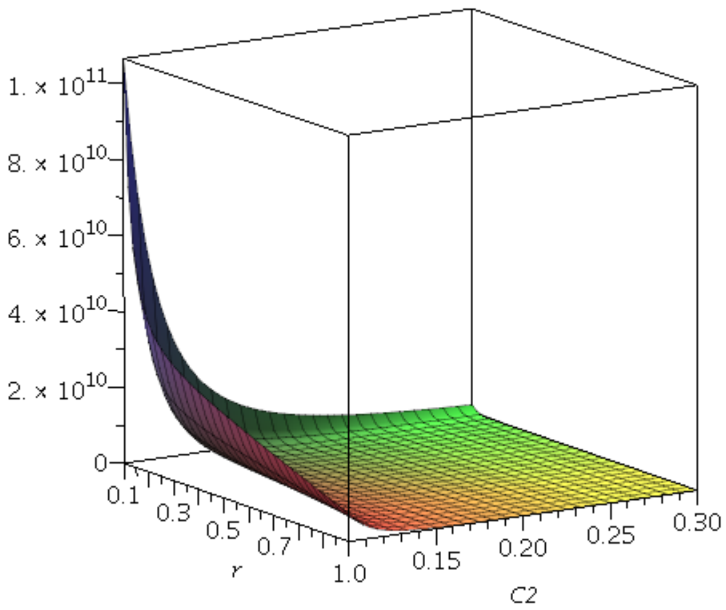

Figure 2: $\tilde{K}(r,C_2)$ for $(r,C_2) \to (0,0)$

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