PROCEEDINGS OF SCIENCE

Neutral pion transition form factor in coordinate space

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We calculate the $\pi^0 \to \gamma\gamma$ transition form factor which two photons located in coordinate space, using lattice QCD simulation with two flavors of quarks. The motivation is that the coordinate space form factor can be measured directly on a lattice and we can also study how the distance between two photons contributes the form factor. We set up a formula of the coordinate space form factor that shows the property of two photons' behaviour, and we compare the form factor in between momentum space and coordinate space.

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1. Introduction

The pion transition form factor place an important role in dispersive approach to compute hadronic light-by-light (HLbL) scattering [1, 2] in the muon $g-2$, which is one of the most precise tests of the Standard Model of particle physics [3, 4, 5]. The dispersive framework relates the, presumably, numerically dominant pion-pole contribution, which requires as hadronic input the pion transition form factor $F(q_1^2, q_2^2)$, and the HLbL contribution can be obtained by integrating some weight functions times the product of a single-virtual and a double-virtual transition form factor [3]. We start from the Euclidean space time pion transition form factor, which describe the interaction between a neutral pion and two off-shell photons in momentum space. It can be defined as the following matrix element [6, 7]:

$$
\begin{aligned}
M_{\mu\nu}(q_1, q_2) &= \int d^4u e^{-iq_1\cdot u - iq_2\cdot v}\langle 0|T\{iJ_\mu(u)iJ_\nu(v)\}|\pi^0(\vec{p})\rangle \\
&= \frac{i}{4\pi^2 F_\pi}\varepsilon_{\mu\nu\rho\sigma}q_{1,\rho}q_{2,\sigma}F(q_1^2, q_2^2)
\end{aligned} \tag{1.1}
$$

where q_1 and q_2 are the photon momenta, and $p = q_1 + q_2$ is the on-shell pion momentum with $p^2 = -m_\pi^2$, $J_\mu = \sum_f Q_f \overline{\psi}_f \gamma_\mu \psi_f$ is the hadronic electromagnetic current, $F_\pi = 92.4$ MeV is the pion decay width at the physical pion mass. The lattice calculation of the pion transition form factor in momentum space was well developed in Ref. [8]. What we study here is the three point correlation matrix element before the Fourier transform, which is $\langle 0|T\{iJ_\mu(u)iJ_\nu(v)\}|\pi^0(\vec{p})\rangle$, where the two photons locate in coordinate space. This matrix element can be computed directly on a lattice and we can also examine the contributions to coordinate space form factor from different distances of two currents. In this proceeding, we will discuss the formula of coordinate space form factor, the relation between momentum and coordinate space form factor, how we parametrize it and the lattice results.

2. Methodology

2.1 The coordinate space pion form factor

To construct coordinate space form factor, we firstly define a function

$$
F'(p\cdot(u-v), (u-v)^2)e^{ip\cdot v}, \tag{2.1}
$$

where u and v denote the positions of two currents and p is the momentum of pion. By writing momentum q_ρ to it's correspond coordinate operator $-i\partial_\rho$ from Eq.(1.1), the matrix element with two photons in coordinate space becomes:

$$
\langle 0|T\{iJ_\mu(u)iJ_\nu(v)\}|\pi^0(\vec{p})\rangle = \frac{i}{4\pi^2 F_\pi}\varepsilon_{\mu\nu\rho\sigma}(-i\partial_\rho^u)(-i\partial_\sigma^v)\left[F'(p\cdot(u-v), (u-v)^2)e^{ip\cdot v}\right]. \tag{2.2}
$$

To get a formula in terms of pure coordinate spaces u and v, by applying a Fourier transform for the component $p\cdot(u-v)$:

$$
F'(p\cdot(u-v),(u-v)^2) = \int_{-\infty}^{\infty} dx F_c(x, (u-v)^2)e^{ixp\cdot(u-v)}, \tag{2.3}
$$

we can construct what we call the pion form factor in coordinate space, F_c, which depend on the distance between the two currents and a factor x. We can prove that $F_c(x,(u-v)^2)$ will be vanish when $x<0$ or $x>1$, and the final matrix element with the pion operator can be expressed as:

$$\langle 0|T\{iJ_\mu(u)iJ_\nu(v)\}|\pi^0(\vec{p})\rangle = \langle 0|\frac{i}{4\pi^2 F_\pi}\varepsilon_{\mu\nu\rho\sigma}\int_0^1 dx\Big[-\partial_\rho^u F_c(x,(u-v)^2)\Big]\cdot\Big[\partial_\sigma\pi^0(xu+(1-x)v)\Big]|\pi^0(\vec{p})\rangle, \tag{2.4}$$

where the properly normalized pion operator is defined as:

$$\langle 0|\pi^0(x)|\pi^0(\vec{p})\rangle = e^{ip\cdot x}. \tag{2.5}$$

Eq.(2.4) imply that the behavior of two currents in coordinate spaces u and v can be thought as a pion living along the line between the two positions.

2.2 Conversion to momentum space form factor

Since matrix element in coordinate space and in momentum space are connected by a Fourier transform, we can also build a bridge from the coordinate space form factor to the momentum space by insert Eq.(2.2) and Eq.(2.3) into Eq.(1.1), and it will show:

$$F(q_1^2,q_2^2) = \int d^4u e^{-iq_1\cdot u}\int_0^1 dx F_c(x,u^2)e^{ixp\cdot u}. \tag{2.6}$$

It is also known that when two photons are on-shell, the pion decay width will lead $F(q_1^2\to 0, q_2^2\to 0)=1$. Because of this constraint, the coordinate form factor must satisfy the integral property:

$$\int d^4u\int_0^1 dx F_c(x,\mu^2)dx = 1. \tag{2.7}$$

3. Lattice calculation setup and results

3.1 Parametrization

We parametrize the pion form factor in coordinate space with the help of operator product expansion (OPE) [8], which refers to the expression of products of local operators at short distance. For our case, when u and v are very close, we have such approximation:

$$\begin{aligned} T\{\psi(u)\bar{\psi}(v)\} &\approx \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(u-v)}}{ip_\rho\gamma_\rho+m} \\ &= \frac{(u-v)_\rho\gamma_\rho}{2\pi^2((u-v)^2)^2} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} & T\Big\{\big[i\bar{\psi}(u)\gamma_\mu\psi(u)\big]\big[i\bar{\psi}(v)\gamma_\nu\psi(v)\big]\Big\} \\ \approx & -\frac{(u-v)_\rho}{2\pi^2((u-v)^2)^2}\bar{\psi}(u)\gamma_\mu\gamma_\rho\gamma_\nu\psi(v) - \frac{(v-u)_\rho}{2\pi^2((v-u)^2)^2}\bar{\psi}(v)\gamma_\nu\gamma_\rho\gamma_\mu\psi(u) \\ = & -\frac{\varepsilon_{\mu\nu\rho\sigma}(u-v)_\rho}{2\pi^2((u-v)^2)^2}\big[\bar{\psi}(u)\gamma_\sigma\gamma_5\psi(v)+\bar{\psi}(v)\gamma_\sigma\gamma_5\psi(u)\big] \\ \approx & \frac{i}{4\pi^2F_\pi}\varepsilon_{\mu\nu\rho\sigma}2(u-v)_\rho\Big[\frac{2F_\pi^2}{3}\frac{1}{((u-v)^2)^2}\Big]\partial_\sigma\pi^0\Big(\frac{u+v}{2}\Big). \end{aligned} \tag{3.2}$$

In the third line of Eq.(3.2), we use the definition of the pion form factor $\langle 0|\bar{u}(u)\gamma_\sigma\gamma_5 u(u)|\pi^0(\vec{p})\rangle = F_\pi p_\sigma e^{ip\cdot u} = \langle 0|-iF_\pi\partial_\sigma\pi^0(u)|\pi^0(\vec{p})\rangle$. For now, comparing Eq.(2.4) and Eq.(3.2) and doing Fourier series expension of $F_c(x,(u-v)^2)$ with the fact $x\in[0,1]$, the parametrized coordinate space pion form factor can be written as:

$$
\begin{aligned}
-\partial^u_\rho F_c(x,(u-v)^2) =& 2(u-v)_\rho\left[\frac{2F_\pi^2}{3}\frac{1}{((u-v)^2)^2}\right] \\
&\times\sum_{n=0}^{\infty} f_n(|u-v|)\frac{(2n+1)\pi}{2}\sin((2n+1)\pi x)
\end{aligned}
\tag{3.3}
$$

and the realtive matrix element that will be computed in the lattice can also be parametrized:

$$
\begin{aligned}
&\langle 0|T\{iJ_\mu(u)iJ_\nu(v)\}|\pi^0(\vec{p})\rangle \\
=&\frac{i}{4\pi^2F_\pi}\varepsilon_{\mu\nu\rho\sigma}2(u-v)_\rho\left[\frac{2F_\pi^2}{3}\frac{1}{((u-v)^2)^2}\right] \\
&\times\sum_{n=0}^{\infty}\left\{f_n(|u-v|)\frac{(2n+1)}{2}\int_0^1\sin((2n+1)\pi x)e^{ip\cdot(xu+(1-x)v)}dx\right\}.
\end{aligned}
$$

3.2 Study r Dependence of the Coordinate Space Form Factor

The coordinate space form factor $F_c(x,(u-v)^2)$ depend on the distance between two photons and a factor x, which denote the pion position between the them. In this proceedings, we only focus on the r dependence, where $r=u-v$. Let $u=r/2$, $v=-r/2$ and $r_t=0$, $\vec{p}=0$, and then we have such matrix element which is in terms of r only:

$$
\begin{aligned}
&\langle 0|T\left\{iJ_\mu(0,\vec{r}/2)iJ_\nu(0,-\vec{r}/2)\right\}|\pi^0(\vec{p}=0)\rangle \\
=&\frac{i}{4\pi^2F_\pi}\varepsilon_{\mu\nu\rho\sigma}2r_\rho ip_\sigma\left[\frac{2F_\pi^2}{3}\frac{1}{(|r|^2)^2}\right]f(|r|),
\end{aligned}
\tag{3.4}
$$

where $f(|r|)$ is defined as

$$
\int_0^1 dx\left[-\partial^u_\rho F_c(x,r^2)\right] = 2r_\rho\left[\frac{2F_\pi^2}{3}\frac{1}{(|r|^2)^2}\right]f(|r|)
\tag{3.5}
$$

and that is

$$
f(|r|)=\sum_{n=0}^{\infty}f_n(|r|).
\tag{3.6}
$$

$f(|r|)$ must also follow the same constraint as in Eq.(2.7):

$$
\frac{\pi^2}{2}\int_0^\infty\frac{2F_\pi^2}{3}f(|r|)2rdr=1,
\tag{3.7}
$$

raised from the pion decay width.

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3.3 Lattice Setup

We only consider the connected contribution to the three-point correlation function since the Ref.[8] shows that the disconnected contribution is below 1% of the total contribution. The connected contribution to the three-point correlation function reads:

$$C^{\rm conn}_{\mu\nu}(r,\vec{p},t_\pi) = \sum_{|\vec{r}'|=r,\vec{x}} \left(\langle 0|J_\mu(0,\vec{r}'/2)J_\nu(0,-\vec{r}'/2)P^+(t_\pi,\vec{x})|0\rangle \times e^{i\vec{p}\cdot\vec{x}} \right), \tag{3.8}$$

where $\vec{r}'$ denotes the spacial distance between two currents, t_π is a fix distance between two currents and pion in time direction and $\vec{p} = 0$ to just study r dependence.

The lattice calculation is based on $24^3 \times 64$ ensemble with lattice spacing 1.015 GeV^{-1}. The lattice parameters are shown in Table 1. There are 1024 point source propagators are chosen randomly in each configuration (256 random area groups are chosen from the lattice and 4 random points in each group). To compute three-point correlation function, we find all the pairs of point source propagators as the locations of two currents, which are chosen in the same time slices. Then we sum over the spacial coordinates of pion with time separations $t_{\rm sep} = 6$ and 10 lattice spacing between two currents and pion since we think these distances are long enough to create ground state pion.

Table 1: Parameters of the $24^3 \times 64$ ensembles

Observable	Fit	% err.
am_π	0.13975(10)	0.07
am_K	0.504154(89)	0.02
af_π	0.13055(11)	0.09
af_K	0.15815(13)	0.09
Z_A	0.73457(11)	0.02
Z_V^π	0.72672(35)	0.05
Z_V^K	0.7390(11)	0.15

3.4 Results

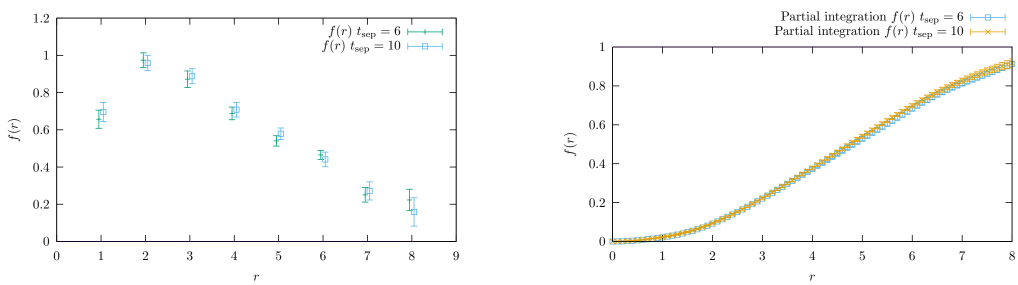

Figure 1: Left: r dependence of the coordinate space formulation $f(r)$, and r is in lattice spacing. $t_{\rm sep}$ denotes the distance between two currents and pion. Right: Partial integration of $f(r)$. The extrapolation in long distance must be close to 1 to satisfy pion decay constraint.

The lattice results of the r dependence of the coordinate space formulation $f(r)$ are shown in left figure in Fig.(1). Remember in Eq.(3.7) , $f(r)$ must satisfy pion decay width when two photons are on-shell. In right figure of Fig.(1), we can see the extrapolation of the partial integration of $\frac{\pi^2}{2}\frac{2F_\pi^2}{3}f(r)2r$ is around 1. To presume long distance behaviour of $f(r)$, we use a formula to fit the lattice result. The fitting formula reads:

$$f_{\text{fit}}(r) = (c_0 + c_1 r + c_2 r^2)e^{-0.77r}, \tag{3.9}$$

which is shown in Fig.(2). The partial integration of the fitting formula is shown in Fig.(2), in which the constraint has been well satisfied.

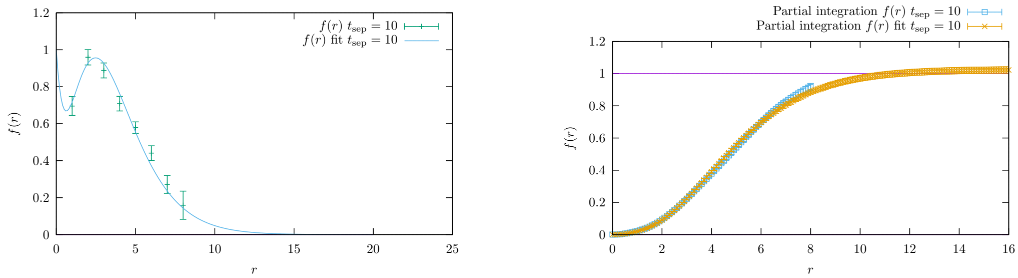

Figure 2: Left: The fitting plot of $f(r)$ when $t_{\text{sep}} = 10$. Right: The partial integration of the fitting formula.

3.5 Conclusions and acknowledgements

We have performed a calculation of r dependence of coordinate space pion form factor. We developed a formula of coordinate space form factor, which shows that the behavior of two currents in coordinate space can be thought as a pion living along the line between these two currents. The lattice result also satisfy the pion decay width constraint.

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