PROCEEDINGS OF SCIENCE

On isospin breaking in τ decays for $(g-2)_\mu$ from Lattice QCD

Mattia Bruno*
Brookhaven National Laboratory
E-mail: mbruno@bnl.gov

Taku Izubuchi
Brookhaven National Laboratory

Christoph Lehner
Brookhaven National Laboratory

Aaron S. Meyer
Brookhaven National Laboratory

Hadronic spectral functions of τ decays have been used in the past to provide an alternative determination of the LO Hadronic Vacuum Polarization relevant for the (g-2) of the muon. Following recent developments and results in Lattice QCD+QED calculations, we explore the possibility of studying the isospin breaking corrections of τ spectral functions for this prediction. We present preliminary results at physical pion mass based on Domain Wall Fermion ensembles generated by the RBC/UKQCD collaboration.

The 36th Annual International Symposium on Lattice Field Theory - LATTICE2018
22-28 July, 2018
Michigan State University, East Lansing, Michigan, USA.

*Speaker.

PoS(LATTICE2018)135

1. Introduction

The discrepancy between the theoretical determination of the muon anomalous magnetic moment a_μ and the experimental measurement performed at the Brookhaven National Laboratory is a very interesting place to look for new physics beyond the Standard Model. A crucial piece in the theoretical prediction is the hadronic vacuum polarization (HVP) where data-driven approaches based on dispersive analysis are presently overpowering pure first-principles non-perturbative calculations from the lattice. Data-driven approaches are based on results from dedicated measurements of cross sections of $e^+e^- \to$ hadrons, whose accuracy must meet certain criteria to guarantee the desired final precision in the HVP. The fact that we are analyzing the anomalous moment of the muon has a direct impact on the kernel appearing in the dispersive integral, weighting each energy region differently: as a consequence approximately 70% of the total HVP contribution to a_μ comes from the $\pi^+\pi^-$ channel alone.

At a time when experimental decay rates of the τ lepton were more precise than electron-production experiments, the authors of Ref. [1] proposed to use the vector spectral functions measured in hadronic τ decays to compute the HVP with the standard dispersive methods. In the following we will mostly focus our attention on the $\pi\pi$ channel, given its importance and dominance in the total signal and error of a_μ. The decays of the τ leptons are mediated by weak interactions and as consequence the observed $\pi^-\pi^0$ state is charged and purely isospin 1. Instead the hadronic vacuum polarization originates from the electromagnetic current, thus producing intermediate $\pi^+\pi^-$ states which are neutral and predominantly isospin 1, with a small isospin-0 component as well. Therefore to relate the spectral functions obtained from τ decays, which we denote by v_-, to the calculation of the HVP, the proper isospin correction factor R_{IB} is required. The neutral spectral function v_0 can be obtained according to $v_0(s) = R_{\text{IB}}(s)v_-(s)$, s being the invariant mass of the hadronic system, and isospin breaking effects contributing to the correction factor $R_{\text{IB}}(s)$ can be split in different parts[1]: long-distance radiative corrections, where a soft photon is emitted from the τ^- or π^-, or exchanged between the two; these contributions are contained in the function G_{EM}, computed in Refs. [2, 3] in the framework of Chiral Perturbation Theory and in Refs. [4, 5] with Vector Meson Dominance models; final state radiation (FSR) of the $\pi^+\pi^-$ state, computed assuming point-like pions as well; a factor that accounts for the difference in phase spaces (called β) between the $\pi^0\pi^-$ and the $\pi^-\pi^+$ states, due to $m_{\pi_0} \neq m_{\pi^\pm}$; finally the $\rho-\omega$ mixing phenomenon and differences in the masses and widths of the ρ^0 and ρ^- determine the strength of the isospin breaking in the two-pion charged and neutral form factors, $f_-(s)$ and $f_0(s)$ respectively. With this definition of the factor R_{IB} it is then possible to compute the correction Δa_μ to be added to the anomalous magnetic moment computed from the τ spectral functions, namely

$$\Delta a_\mu \equiv \int_{4m_\pi^2}^{m_\tau^2} ds\, K(s) \big[v_0(s) - v_-(s)\big] = \int_{4m_\pi^2}^{m_\tau^2} ds\, K(s) \big[R_{\text{IB}}(s) - 1\big] v_-(s)\,, \tag{1.1}$$

where $K(s)$ represents the muon kernel. Historically, three groups[2] have computed the isospin rotation of τ spectral functions always obtaining a value for a_μ incompatible with the corresponding determination from e^+e^- data. For this reason and also due to dramatic improvements in the

[1] In our list we omit short-distance electro-weak and radiative corrections, contained in a multiplicative factor S_{EW}.

[2] More precisely by *Cirigliano et al.* [2, 3], *Davier et al.* [6, 7, 8] and *Jegerlehner et al.* [9, 10].

PoS(LATTICE2018)135

experimental measurements of $\pi^+\pi^-$ spectral functions, this alternative theoretical determination of a_μ was abandoned, until Jegerlehner and Szafron proposed a solution [9] to the puzzle. In our work we present an alternative approach to the calculation of the correction Δa_μ based on Lattice QCD. It is worth noting that the overall size of the correction, which is about 2-3 %, combined with the precision for a_μ, $O(1\%)$, results in a required uncertainty for Δa_μ of approximately 10-20 %, a goal that with current advances in Lattice QCD+QED calculations seems achievable.

2. Lattice calculation

The progress in the last years to include QED effects in Lattice QCD calculations was significant. Based on this advances, we develop the formalism necessary to compute $R_{\rm IB}$ on the lattice and we report some preliminary results in the next section. By realizing that the electro-magnetic current is formed by an isospin 0 and 1 parts (we consider only two light flavors)

$$j_\mu^\gamma = ie\big(Q_{\rm u}\bar{u}\gamma_\mu u + Q_{\rm d}\bar{d}\gamma_\mu d\big) = \overbrace{ie\frac{Q_{\rm u}+Q_{\rm d}}{2}(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d)}^{=j_\mu^{(0)}} + \overbrace{ie\frac{Q_{\rm u}-Q_{\rm d}}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)}^{=j_\mu^{(1)}} \quad (2.1)$$

we decompose the standard vector-vector two-point function in three terms

$$G^\gamma(t) = \frac{1}{3}\sum_{k,\vec{x}}\langle j_k^\gamma(x) j_k^\gamma(0)\rangle = G_{00}^\gamma(t) + 2G_{01}^\gamma(t) + G_{11}^\gamma(t)\,, \quad G_{ij}^\gamma(t) \equiv \frac{1}{3}\sum_{k,\vec{x}}\langle j_k^{(i)}(x) j_k^{(j)}(0)\rangle\,. \quad (2.2)$$

In eq. (2.2) the term G_{01}^γ vanishes in the isospin limit and is different from zero due to QED and $O(m_{\rm u}-m_{\rm d})$ effects. Instead both G_{00}^γ and G_{11}^γ survive in the isospin limit: the former contains also disconnected contributions, while the latter dominates the signal. Then, to map our calculation to the τ^- spectral functions, we compute the expectation value of two charged isospin 1 currents

$$j_\mu^{(1,-)} = ie\frac{Q_{\rm u}-Q_{\rm d}}{\sqrt{2}}(\bar{u}\gamma_\mu d)\,, \quad G_{11}^W \equiv \frac{1}{3}\sum_{k,\vec{x}}\langle j_k^{(1,-)}(x) j_k^{(1,+)}(0)\rangle\,. \quad (2.3)$$

To compute the HVP contribution to a_μ we use the Time-Moment representation [11], $a_\mu \equiv 4\alpha^2\sum_t w_t G^\gamma(t)$, which we extend to the correction Δa_μ in our lattice calculation

$$\Delta a_\mu \equiv 4\alpha^2\sum_t w_t\big[G^\gamma(t) - G^W(t)\big]\,, \quad \Delta a_\mu[\pi\pi] = 4\alpha^2\sum_t w_t\big[2G_{01}^\gamma(t) + \overbrace{G_{11}^\gamma(t) - G_{11}^W(t)}^{=\delta G(t)}\big]\,, \quad (2.4)$$

where in the second equation we have neglected the G_{00}^γ term that contributes only to channels with an odd number of pions. Also in this case, in the isospin limit Δa_μ goes to zero; therefore to compute this quantity we perform a diagrammatic expansion where we insert a single photon in all possible ways, together with all possible insertions of the scalar operator. In Fig. (1) we draw the leading contributions and for the reader's convenience we report below, as an example, the first diagram in Fig. (1) which we call V (with D^{-1} and $\Delta^{\mu\nu}$ the quark and photon propagators[3])

$$V = \frac{1}{3}\sum_{k,\vec{x}}{\rm Tr}\,\langle\Big[\sum_{z,y}\sum_{\mu\nu}\gamma_k D^{-1}(x,z)\gamma_\mu D^{-1}(z,0)\,\gamma_k D^{-1}(0,y)\gamma_\nu D^{-1}(y,x)\cdot\Delta^{\mu\nu}(z,y)\Big]\rangle\,. \quad (2.5)$$

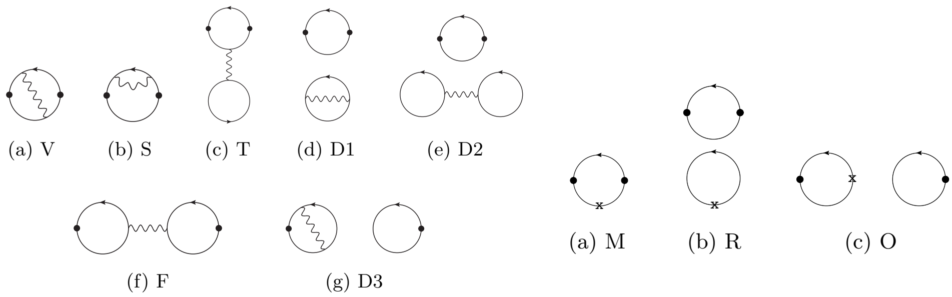

Figure 1: QED (left) and strong-isospin breaking (right) diagrams contributing to the vector-vector two-point function. The black dots represent the external photons. Note the absence of tadpole diagrams, due to local vector currents, for which the knowledge of the normalization factor Z_V is necessary.

Studying separately the pure $I=1$ correction δG has the advantage that many terms simplify and it can be directly mapped to the mass and width differences of the neutral and charged ρ mesons in the phenomenological models of the form factors. For this quantity we obtain[4]

$$\delta G \equiv G_{11}^{\gamma} - G_{11}^{W} = -(4\pi\alpha) Z_V^2 \frac{(Q_\mathrm{u}-Q_\mathrm{d})^4}{4} \big[V-F\big] \,. \tag{2.6}$$

Instead, in the contribution originating from the isospin 0 to 1 transition, which from the phenomenological point of view is very interesting due to the $\rho-\omega$ mixing, cancellations are less relevant, resulting in a dependence on many diagrams:

$$2G_{01}^{\gamma} = -\frac{(Q_\mathrm{u}^2-Q_\mathrm{d}^2)^2}{2}(4\pi\alpha)Z_V^2\big[V+2S-F+\cdots\big] - \frac{Q_\mathrm{u}^2-Q_\mathrm{d}^2}{2}(m_\mathrm{u}-m_\mathrm{d})\big[2M-2O+\cdots\big] \,. \tag{2.7}$$

Our final discussion point before turning to the numerical results, is on the connection between the previous theoretical determinations of Δa_μ and ours. The integral of Δa_μ can be compared without problems, in the continuum and infinite volume limit. However given the large amount of cancellations in the integrands in eq. (1.1) and in eq. (2.4), it is worth exploring a more detailed comparison between the two determinations. Since the lattice calculation can not be analytically continued to Minkowski space, it is natural to apply a simple Laplace transform to rotate previous determinations of $R_\mathrm{IB}(s)$ to Euclidean time

$$\Delta a_\mu[\pi\pi,\tau] = 4\alpha^2 \sum_t w_t \times \left\{ \frac{1}{12\pi^2} \int d\omega\ \omega^2 e^{-\omega t} \big[R_\mathrm{IB}(\omega^2)-1\big] v_-(\omega^2) \right\} . \tag{2.8}$$

3. Preliminary results

In our work we consider the results published by the RBC/UKQCD collaboration on the calculation of a_μ, including the leading QED and strong-isospin diagrams [12]. We perform a simple re-analysis of these results to compute the quantities of interest for this work. Firstly, when including QED in lattice calculations a precise prescription to remove the zero-modes is required and

[3] In this work we have used the Feynman gauge.

[4] The factor Z_V^2 associated to the external photons is included in our definition of the weights w_t.

in our work we adopt the QED$_\mathrm{L}$ formalism. Moreover we consider a diagrammatic expansion at $O(\alpha_\mathrm{em})$ and $O(m_\mathrm{u}-m_\mathrm{d})$ which completely defines our scheme: hence in the renormalization procedure, a new set of (hadronic) quantities must be computed (at $O(\alpha_\mathrm{em})$ and $O(m_\mathrm{u}-m_\mathrm{d})$) to tune the bare parameters of our calculation to follow a line of constant physics. In Ref. [12] the mass of the Omega baryon Ω^- has been used to re-compute the lattice spacing, while the mass difference $m_{\pi_0}-m_{\pi^+}$ and the charged pion mass were used to define m_u and m_d separately[5]. Similarly Z_V has been recomputed to include $O(\alpha_\mathrm{em})$ effects [12].

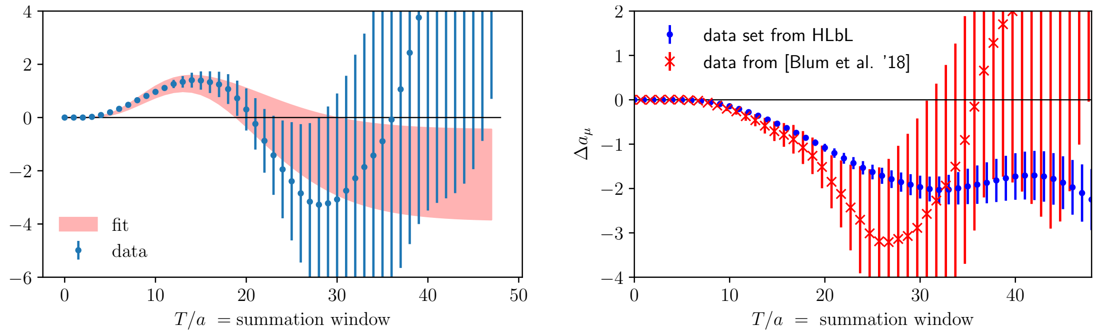

Figure 2: *Left:* pure $I=1$ part of Δa_μ (δG only) as a function of the summation window in lattice units. The fit to the lowest state with constrained energy provides a substantial reduction in the statistical noise. We estimate the systematic error by varying the energy between the two-pion and pion-photon states. *Right:* contribution of diagram F to the pure $I=1$ part of Δa_μ (i.e. contribution of diagram F to δG). The red crosses correspond to the data set published in Ref. [12] and used in the rest of this work. The blue data points correspond to a new ongoing re-analysis of the data produced to study the HLbL contribution to a_μ [13], which provides a significant statistical improvement for this observable.

In Fig. (3) we show our preliminary results obtained at fixed lattice spacing ($a^{-1}\simeq 1.73$ GeV) and fixed volume ($L\approx$ 5.4fm). The calculation is performed at physical pion masses and we refer the reader to Ref. [12] for more details on the numerical strategies used to compute every diagram. To improve the statistical error of our estimates we rely on a constrained fit with functional form $(c_0+c_1t)e^{-Et}$, also discussed in Ref. [12], where the energy is fixed to be either $E_{\pi\pi}$ or $E_{\pi\gamma}$, which are very similar due to the specific choice of L. The systematic error arising from the difference of the two fits is still much smaller than the statistical one, and the fact that the energy is constrained reduces the noise in the long tails. We demonstrate this effect in the left panel of Fig. (2) where we show the partial sum $\sum_t^T w_t \delta G(t)$ as a function of the summation window. In order to improve this calculation we are also working on a new and better determination of these diagrams where such fits can be pushed to much later (Euclidean) times, thus significantly reducing the systematics. In the right panel of Fig. (2) we present the comparison between the current and the new determination of diagram F, obtained from a re-analysis of the point-source propagators generated to study the Hadronic Light-by-Light contribution to a_μ [13].

As we can see from the comparison of the two panels of Fig. (3), the pure $I=1$ contribution δG, where we include all diagrams (see eq. (2.6)) and only QED matters at this order, has little

[5] In Ref. [12] also the strange quark mass has been properly retuned, but this does not affect our work since we consider only up-down valence contributions to a_μ.

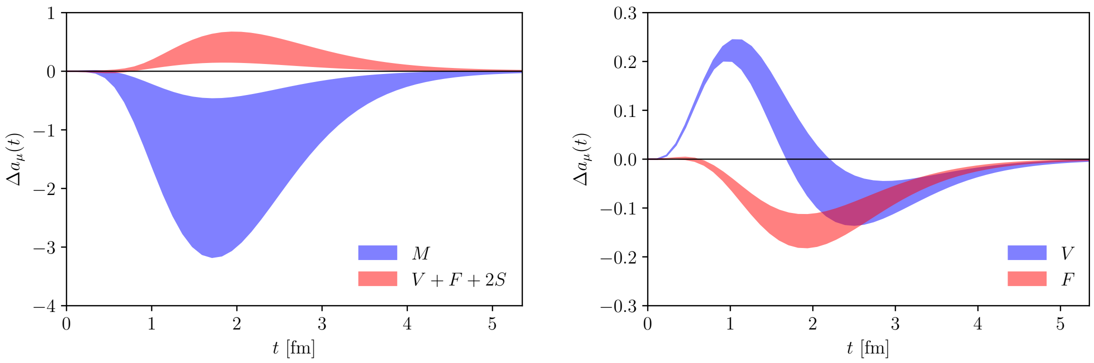

Figure 3: *Left:* contribution to Δa_μ proportional to G_{01}^{γ} where both QED and strong-isospin are present. Unfortunately the present quality of the connected strong-isospin term (M) is not sufficient to resolve it from zero. We will invest additional resources to improve this specific part and we do not proceed further now. Note also that sub-leading diagrams suppressed by at least one power of $1/N_c$ or $1/N_f$ are neglected. *Right:* pure $I=1$ term of Δa_μ. In this case all diagrams are included and only QED contributes, simplifying the calculation. A relatively good signal is already achieved with current statistics, highlighting the high degree of cancellations present in the integral.

impact compared to the isospin 0-1 channel, which in turn is completely dominated by strong-isospin breaking. This expectation matches ChPT predictions for the $\rho-\omega$ mixing parameter. However, we must note that the current determination of the connected SIB term, namely diagram M in our notation, is unable to resolve it from zero throughout the entire time extent. For this reason, with the current data we can not provide a reliable estimate of Δa_μ. In the near future we expect to obtain a significant improvement in our calculation of these diagrams, including sub-leading ones. Moreover, to control potentially large finite volume effects we plan to repeat our calculation on several lattices, differing only by the volume.

4. Conclusions

In this work we have presented some preliminary results for the isospin-breaking corrections necessary to compute a_μ from τ decays. Lattice QCD+QED calculations are finally mature enough to attack this class of problems. At first we have developed the formalism necessary to define the quantity of interest (Δa_μ) on the lattice and with the current available data we have attempted a first calculation. We have demonstrated that only two QED diagrams are needed to compute the pure $I=1$ component of the isospin-breaking rotation and we have provided some initial numerical evidence. Systematic errors such as finite volume and discretization effects have not been estimated and we refer the reader to future publications, where we plan to study both. We have also demonstrated our current progress in the calculation by showing a new determination of diagram F, which is substantially more precise, allowing us to reduce some systematics introduced by our fitting procedure. Finally we stress once more the importance of the comparison of the integrands, beyond the final results for Δa_μ, due to the high level of cancellations taking place.

Acknowledgements

We would like to thank our RBC/UKQCD collaborators for helpful discussions and support. M.B. is particularly indebeted to F. Jegerlehner for sharing private results, to K. Maltam for helpful insights on τ phenomenology and to M. Davier for stimulating discussions. This work is supported in part by US DOE Contract DESC0012704(BNL). T.I. is supported by JSPS KAKENHI grant numbers JP17H02906. C.L. is also supported by a DOE Office of Science Early Career Award.

References

[1] R. Alemany, M. Davier and A. Hocker, *Improved determination of the hadronic contribution to the muon (g-2) and to alpha (M(z)) using new data from hadronic tau decays*, Eur. Phys. J. **C2** (1998) 123 [hep-ph/9703220].

[2] V. Cirigliano, G. Ecker and H. Neufeld, *Isospin violation and the magnetic moment of the muon*, Phys. Lett. **B513** (2001) 361 [hep-ph/0104267].

[3] V. Cirigliano, G. Ecker and H. Neufeld, *Radiative tau decay and the magnetic moment of the muon*, JHEP **08** (2002) 002 [hep-ph/0207310].

[4] F. Flores-Baez, A. Flores-Tlalpa, G. Lopez Castro and G. Toledo Sanchez, *Long-distance radiative corrections to the di-pion tau lepton decay*, Phys. Rev. **D74** (2006) 071301 [hep-ph/0608084].

[5] A. Flores-Tlalpa, F. Flores-Baez, G. Lopez Castro and G. Toledo Sanchez, *Model-dependent radiative corrections to tau- —> pi- pi0 nu revisited*, Nucl. Phys. Proc. Suppl. **169** (2007) 250 [hep-ph/0611226].

[6] M. Davier, S. Eidelman, A. Hocker and Z. Zhang, *Confronting spectral functions from e+ e- annihilation and tau decays: Consequences for the muon magnetic moment*, Eur. Phys. J. **C27** (2003) 497 [hep-ph/0208177].

[7] M. Davier, S. Eidelman, A. Hocker and Z. Zhang, *Updated estimate of the muon magnetic moment using revised results from e+ e- annihilation*, Eur. Phys. J. **C31** (2003) 503 [hep-ph/0308213].

[8] M. Davier, A. Hoecker, G. Lopez Castro, B. Malaescu, X. H. Mo, G. Toledo Sanchez et al., *The Discrepancy Between tau and e+e- Spectral Functions Revisited and the Consequences for the Muon Magnetic Anomaly*, Eur. Phys. J. **C66** (2010) 127 [0906.5443].

[9] F. Jegerlehner and R. Szafron, *$\rho^0-\gamma$ mixing in the neutral channel pion form factor F^e_π and its role in comparing e^+e^- with τ spectral functions*, Eur. Phys. J. **C71** (2011) 1632 [1101.2872].

[10] F. Jegerlehner, *The anomalous magnetic moment of the muon*, Springer Tracts Mod. Phys. **226** (2008) 1.

[11] D. Bernecker and H. B. Meyer, *Vector Correlators in Lattice QCD: Methods and applications*, Eur. Phys. J. **A47** (2011) 148 [1107.4388].

[12] RBC, UKQCD collaboration, T. Blum, P. A. Boyle, V. Guelpers, T. Izubuchi, L. Jin, C. Jung et al., *Calculation of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment*, Phys. Rev. Lett. **121** (2018) 022003 [1801.07224].

[13] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung et al., *Connected and Leading Disconnected Hadronic Light-by-Light Contribution to the Muon Anomalous Magnetic Moment with a Physical Pion Mass*, Phys. Rev. Lett. **118** (2017) 022005 [1610.04603].

PoS(LATTICE2018)135